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PLANE AND SPHERICAL TRIGONOMETRY

(*WITH FIVE-PLACE TABLES*)

A TEXT-BOOK FOR TECHNICAL SCHOOLS
AND COLLEGES

BY

ROBERT E. MORITZ, PH.D., PH.N.D.

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PREFACE

In preparing the present volume the author has tried to give proper recognition to the fundamental and far-reaching changes that have been wrought during the last decade in the teaching of the elementary branches of mathematics and to bring about a more perfect adjustment of the teaching of trigonometry with the teaching of algebra and geometry on which it rests and with the progress of the arts and sciences to which it applies. Among the distinctive features of the present volume the following deserve special emphasis.

1. The subject is introduced with a chapter on the graphic solution of triangles which shows the immediate connection of trigonometry with geometry. Similarly the six cases of spherical triangles are solved by the graphic method before the analytical methods are presented.

2. The references to algebra are limited to those with which every beginner may be reasonably assumed to be familiar. Other principles are developed as they are needed.

3. A knowledge of logarithms has not been presupposed. For this reason a chapter on logarithms and their use has been incorporated in the text.

4. Many of the fundamental formulas and theorems have been derived by two or more independent methods.

5. Functions of acute, obtuse and general angles are treated in separate chapters.

6. The study of the general angle and the analytical derivation of geometrical formulas and theorems has been placed after the solution of triangles. This makes it possible for the student to complete the subject through the solution of plane triangles in a six weeks' course of four or five recitations per week.

7. The student is trained in accuracy and self-confidence in numerical calculations by a constant insistance on proper checks and verification of results.

8. The accuracy of results is strictly limited to that of the data employed. The student is constantly guarded against superfluous figures and a show of accuracy not warranted by the data or the processes of computation employed.

9. The student is guarded against disregarding figures and remainders without first determining the effect of the parts neglected on the final results. Thus, in the computation of logarithms by means of the logarithmic series, or of natural functions by means of the sine and cosine series, the effect of the neglected part of each series on the final result has been carefully considered in each case.

10. The student is led by easy stages from simple to complicated computations. For this purpose each of the cases of both plane and spherical triangles is followed by sets of problems involving successively three, four and five significant figures. This enables the student to become familiar with his formulas before he enters on the more exacting computations.

11. The best practice of the government computing offices and of practical engineers has been followed in the arrangement of the logarithmic work.

12. Trigonometric curves have received much fuller treatment than is usual. The treatment includes curves of any given amplitude and wave-length, logarithmic and exponential curves, composition of harmonic curves, the catenary and the curve of damped vibrations.

13. A special section is devoted to the angle and its functions considered as functions of time.

14. The trigonometric representation of imaginary and complex numbers is developed in connection with the theorems of Euler and Demoivre and the development of the functions in series.

15. Hyperbolic functions have received much fuller treatment than is usual. The analogies between the circular and hyperbolic functions have been developed both analytically and geometrically.

16. It is believed that a rigorous proof of Napier's Rules of Circular Parts appears here for the first time in an elementary text.

17. It is believed that the simultaneous derivation of the three fundamental formulas for the spherical triangle (11, 36) has not previously appeared in a textbook on trigonometry.

18. The lists of applied problems will be found more complete than any that have appeared in recent American texts. The problems are segregated into sets relating separately to Physics, Engi-

neering, Navigation, Geography, Astronomy and Plane and Solid Geometry.

19. Abundant historical matter has been introduced throughout the text.

ROBERT E. MORITZ.

September, 1913.

GREEK ALPHABET

α	pronounced	*alpha.*	ν	pronounced	*nu.*
β	"	*beta.*	ξ	"	*xi.*
γ	"	*gamma.*	ο	"	*omicron.*
δ	"	*delta.*	π	"	*pi.*
ε	"	*epsilon.*	ρ	"	*rho.*
ζ	"	*zeta.*	σ	"	*sigma.*
η	"	*eta.*	τ	"	*tau.*
θ	"	*theta.*	υ	"	*upsilon.*
ι	"	*iota.*	φ	"	*phi.*
κ	"	*kappa.*	χ	"	*chi.*
λ	"	*lambda.*	ψ	"	*psi.*
μ	"	*mu.*	ω	"	*omega.*

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PLANE TRIGONOMETRY

CHAPTER I

INTRODUCTION

In order to work the exercises in this chapter the student should be provided with a pair of compasses, a protractor, and a graduated ruler divided into tenths of a unit.

1. Graphic Solution of Triangles. In plane geometry it is shown that the six parts (three sides and three angles) of any plane triangle are so related that any three parts suffice to determine the shape of the triangle, and if one of the known parts is a side, the size of the triangle is also determined. Furthermore it is shown how to construct the triangle when a sufficient number of parts is given. All possible cases come under one or another of the following four cases.

To construct the triangle when there is given,—

I. One side and two angles.
II. Two sides and an angle opposite one of them.
III. Two sides and the included angle.
IV. Three sides.

Usually we have given not the actual lines and angles but their measures. From these measures lines and angles corresponding to the actual lines and angles may then be constructed by means of suitable instruments. Such instruments are,—

1. A *graduated straight-edge* for the construction and measurement of straight lines of definite lengths. The smallest divisions of the straight-edge should be decimal, either millimeters or tenths of an inch.

2. A *pair of compasses* for the construction of circles and circular arcs.

3. A *protractor* for the construction and measurement of plane angles of definite magnitudes.

EXAMPLE 1. It is required to construct a triangle which has two sides equal to 2.5 inches and 1.75 inches respectively and the included angle equal to 36°.

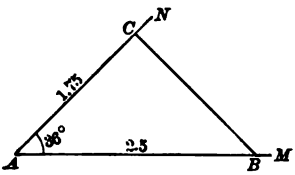

Fig. 1.

Solution. By means of the protractor construct an angle MAN (Fig. 1) equal to 36°. On AM measure off AB equal to 2.5 inches. On AN measure off AC equal to 1.75 inches. Join B and C by a straight line. ABC is the required triangle.

The numerical values of the parts which were not known at the outset may now be found by measurement. BC is thus found to be 1.49 inches, and by means of the protractor, angles B and C are found to be approximately 43.5° and 100.5° respectively.

If it is not possible or convenient to construct the triangle full size, a similar triangle may be constructed on a *reduced scale;* that is, any unit or a fraction of a unit on the scale may be taken to represent any unit occurring in the problem. Thus lines 3 and 4 inches long may be employed in the solution of a triangle whose sides are 30 and 40 miles respectively. The angles of the reduced triangle will of course be equal to the angles of the triangle represented.

Similarly, the unknown parts of a triangle which is too small for actual construction, say some microscopic triangle, may be found by measurement from a similar triangle drawn on an *enlarged scale.*

EXAMPLE 2. One side of a triangle measures 600 miles, and the angles adjacent to this side measure 23° and 100° respectively. Find the remaining parts of the triangle.

Solution. Let ½ inch represent 100 miles. Then a line AB drawn 3 inches long will represent 600 miles. At A and B draw the angles BAC and ABD 23° and 100° respectively. Let BD intersect AC at E. ABE will represent the required triangle.

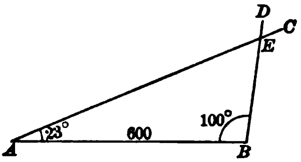

Fig. 2.

Angle E measures 57°, which of course could have been found otherwise by subtracting the sum of the angles A and B from 180°. AE and BE are found to measure approximately 3.52 and 1.40 inches respectively. Remembering that each ½ inch represents 100 miles, the actual lengths represented by AE and BE are approximately 704 and 280 miles respectively.

Solutions, like the foregoing, in which geometrical drawings to a scale are employed instead of numerical calculations, are called ***graphic solutions.***

EXERCISE 1

1. Review the following propositions in geometry. A, B, C represent the three angles of any triangle and a, b, c the sides opposite these angles.

a. Given A, B, c; to construct the triangle.
b. Given a, b, C; to construct the triangle.
c. Given a, b, c; to construct the triangle.
d. Given a, b, A; to construct the triangle.
e. Under what conditions will (*d*) give rise to two different solutions? To only one solution?

The following problems are to be solved by the graphic method.*

2. Given $a = 5$, $b = 4$, $c = 7$; find the angles to the nearest $15'$.
Ans. $A = 44° 30'$, $B = 34°$, $C = 101° 30'$.

3. Given $b = 4$, $c = 5$, $C = 90°$; find the third side and the angles to the nearest $15'$. *Ans.* $a = 3$, $A = 37°$, $B = 53°$.

4. Given $b = 270$, $c = 600$, $A = 100°$; find the third side correct to the nearest integer. *Ans.* $a = 700$.

5. Given $a = 0.029$, $B = 32° 15'$, $C = 136° 45'$; find the remaining sides. *Ans.* $b = 0.081$, $c = 0.104$.

6. Given $a = 42$, $b = 51$, $A = 55°$; find the approximate measures of the remaining parts. *Ans.* $c = 33.6$, $B = 84°$, $C = 41°$;
or $c = 24.9$, $B = 96°$, $C = 29°$.

7. Given $A = 44° 30'$, $B = 57°$, $C = 78° 30'$; find the ratios between the sides opposite these angles.
Ans. Approximately $a : b : c = 5 : 6 : 7$.

---

* In order to employ the graphic method successfully the student must practice accuracy. Two pencils of medium hardness should be used, one sharpened to a point for marking distances, the other sharpened like a chisel for drawing lines. The pencil points are easily kept sharp with the aid of a piece of fine sandpaper. The lines should be drawn sharply and they should bisect the points through which they are intended to pass. In measuring the required parts, beginners should estimate angles to quarters of a degree and lengths to quarters of the smallest division of the scale.

2. Solution of Practical Problems by the Graphic Method. Many important practical problems, in which a high degree of accuracy is not essential, can be easily solved by the graphic method. Suppose it is required to find the approximate distance *AB* across a lake or swamp, without actually measuring it. This may be accomplished in various ways, one of which is as follows:

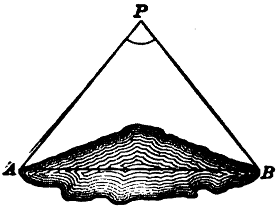

Fig. 3.

Select some point *P* from which both *A* and *B* are visible, and measure the distances *AP* and *BP* and also the angle *APB*.* This gives two sides and the included angle of the triangle *APB* from which *AB* may be found by the method of the preceding article.

Similarly the heights of towers and trees and mountains, of clouds and shooting stars, the distances through impenetrable forests across swamps and through mountains, the widths of rivers, ravines and canyons, may be determined. Even the distances between celestial objects may be approximated by the graphic method after certain other distances and angles have been measured.

Example 1. In order to determine the width of a river, the distance between two points *A* and *B* close to the bank of the river was measured and was found to be 600 feet. The angles *BAP* and *ABP*, formed with a point *P* close to the opposite bank of the river, were also measured and were found to be 50° and 36° respectively. Required the approximate width of river.

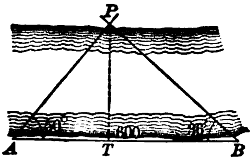

Fig. 4.

Solution. Select a suitable scale, say 1 inch to 100 feet, and construct a triangle *ABP*, having $AB = 6$ inches and the adjacent angles equal to 50° and 36° respectively. From *P* draw *PT* perpendicular to *AB*. *PT* will represent the width of the river. Measure *PT*. *PT* will be found to measure 2.7 inches, and since each inch represents 100 feet, the width of the river is 270 feet.

Definitions. Let *P* be any point and *O* the position of the observer. Through *P* draw a vertical line, and through *O* draw a horizontal line meeting the vertical line in *H*.

---

* The angle between two visible objects is readily measured by means of an instrument called a transit.

If P is above H, as in the upper figure, the angle HOP is called the *angle of elevation* of the point P as seen from O.

If P is below H, as in the lower figure, the angle HOP is called the *angle of depression* of the point P as seen from O.

It is obvious that the angle of elevation or of depression of an object depends upon the position of the observer.

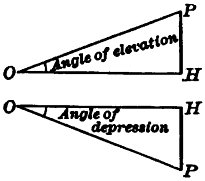

Fig. 5.

Example 2. From a point P at the foot of a mountain, the angle of elevation of the summit M is measured and is found to be 30°; after walking two miles toward the summit on an incline averaging 15°, the angle of elevation is found to measure 45°. Required the height of the mountain.

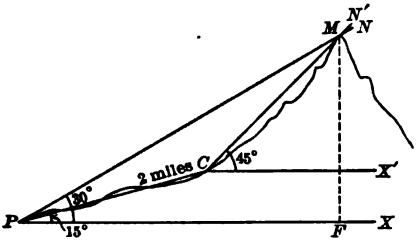

Fig. 6.

Solution. Draw a horizontal line PX. Construct an angle $XPN = 30°$; then PN represents the direction in which the summit of the mountain is seen from P. Construct angle $XPC = 15°$, and take PC two units in length. Then, if each unit represents one mile, C will represent the position from which the second observation was made.

Through C draw CX' parallel to PX, and construct an angle $X'CN' = 45°$. Then CN' represents the direction in which the summit is seen from C.

Since the summit is on each of the lines PN and CN', it must be located at their point of intersection M.

Draw MF perpendicular to PX. Then MF will represent the height of the mountain on the same scale on which PC represents two miles. Measure MF. If PC was taken equal to 6 inches, MF will measure 5.8 inches. Since 3 inches represents one mile, MF represents 1.933 miles, or 10,200 feet approximately.

Exercise 2

The following problems are to be solved graphically. The student is expected to obtain distances correct to three figures and angles correct to nearest 15'.

1. At a distance of 400 feet from the foot of a tree, the top of the tree subtends an angle of 20°. Find the height of the tree.
Ans. 145.6 ft.

2. A straight road leads from a town *A* to a town *B* 8 miles distant; another road leads from *A* to a third town *C* 10 miles distant. The angle between the roads is 65°. How far is it from *B* to *C*?
Ans. 9.82 mi.

3. What is the altitude (= angle of elevation) of the sun, when a building 75 feet high casts a shadow 190 feet long on a horizontal plane? *Ans.* 21° 30′.

4. The great pyramid of Gizeh is 762 feet square at its base and each face makes an angle of 51° 51′ with the horizontal plane. Determine the height of the pyramid, assuming that it comes to an apex. *Ans.* 485 feet.

5. As a matter of fact, the pyramid mentioned in Problem 4 does not come to a point, but terminates in a platform 32 feet square. Find the actual height of the pyramid. *Ans.* 465 feet.

6. An observer on board ship sees two headlands in a straight line N. 35° E. The ship sails northwest for 5 miles, when one of the headlands appears due east and the other due northeast. How far apart are the headlands?

7. Two observers on opposite sides of a balloon observe the balloon at the same instant and find its angles of elevation to be 56° and 42° respectively. The observers are one mile apart. Find the height of the balloon at the time the observations were taken.
Ans. 0.6 mi. nearly.

8. In order to determine the distance across a swamp, a distance *AB* was laid off 100 yards long, and at each extremity of the line *AB* the angles were measured between the other extremity of the line and each of two stakes *P* and *Q* placed at opposite ends of the swamp. At one extremity of the line the angles measured 35° and 85° respectively, at the other end the angles measured 121° and 40° respectively. Find the distance *PQ*.

9. Find the perimeter of a regular polygon of 7 sides inscribed in a circle whose radius is 10 feet. *Ans.* 60.75 ft.

10. The sides of a triangle are $a = 10$, $b = 12$, $c = 15$ respectively. Find the radii of the inscribed and of the circumscribed circles and the angles of the triangle. *Ans.* $r = 3.23$, $R = 7.52$,
$A = 41° 30′$, $B = 53°$, $C = 85° 30′$.

3. Inadequacy of the Graphic Method. The graphic method of solving triangles, though exceedingly simple and useful, is not sufficiently accurate for all purposes. For instance, in the last problem of Exercise 2 the results obtained by the graphic method are:

$$r = 3.23, \quad R = 7.52, \quad A = 41^\circ 30', \quad B = 53^\circ, \quad C = 85^\circ 30',$$

while the more accurate results, obtained by a method to be described later, are:

$$r = 3.2331, \quad R = 7.5236, \quad A = 41^\circ 38' 59'',$$
$$B = 52^\circ 53' 27'', \quad C = 85^\circ 27' 34''.$$

The causes of the inaccuracies are manifold. First of all the divisions of the scale used in measuring are not indefinitely small; if they were the eye could not distinguish them. Besides this the graphic method is subject to many other unavoidable errors. The instruments employed in the construction of lines and angles are imperfect. The straight-edge is not perfectly straight, the divisions of the scale are not exactly equidistant. The points used in the construction are not true points but dots having dimensions; likewise the lines drawn are not true lines but pencil or pen marks of unequal widths. Again, neither the hand which draws the pencil nor the eye which guides it is perfectly steady, and so on. All these sources of error are unavoidable. By employing better instruments and by using greater care, these errors may be diminished, but they cannot be entirely eliminated.

4. Definition of Trigonometry. There exists another method of solving triangles which is free from all the errors above mentioned and enables us to obtain results correct to any desired degree of accuracy. This method consists in the *computation* of the unknown parts of a triangle from the numerical values of the given parts. The development of this method has given rise to a separate branch of mathematics, called trigonometry.* Trigonometry considers the properties of angles and certain ratios associated with angles, and applies the knowledge of these properties to the solution of triangles and various other algebraic and geometric problems. Incidentally trigonometry considers also certain time-saving aids in computation such as logarithms, which are generally employed in the solution of triangles. Briefly stated,—

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* The word Trigonometry comes from two Greek words, *trigonon* = triangle, and *metron* = measure. The method was originated in the second century B.C. by Hipparchus and other early Greek astronomers in their attempts to solve certain spherical triangles. The term trigonometry was not used until the close of the sixteenth century.

Trigonometry is the science of angular magnitudes and the art of applying the principles of this science to the solution of problems.

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

5. Definition of Function. *When two variables are so related that the value of the one depends upon the value of the other, the one is said to be a function of the other.*

Examples. The area of a square is a function of its side. The volume of a sphere is a function of its radius. The velocity of a falling body is a function of the time elapsed since it began to fall. The output of a factory is a function of the number of men employed. In the expression $y = \frac{(x-1)}{(x+1)}$, y depends upon x for its value, hence y is a function of x. Similarly $x^2 - 1$, $x^3 + x - 3$, $ax + \frac{b}{x} - c$, are functions of x. $t^2 - 3t$ is a function of t, etc.

6. Definition of Reciprocal. *If the product of two quantities equals unity, each is said to be the reciprocal of the other.*

For example, if $xy = 1$, x is the reciprocal of y, and y is the reciprocal of x. $\frac{1}{2}$ is the reciprocal of 2, and 2 is the reciprocal of $\frac{1}{2}$, for $\frac{1}{2} \times 2 = 1$. In general, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals since $\frac{a}{b} \cdot \frac{b}{a} = 1$. From $xy = 1$ it follows that $x = \frac{1}{y}$, and $y = \frac{1}{x}$, that is,—

The reciprocal of any quantity is unity divided by that quantity.

7. The Six Trigonometric Functions of an Acute Angle. Let A be any acute angle, B any point on either side of the angle, and ABC the right triangle formed by drawing a perpendicular from B to the other side of the angle. Denote AC, the side adjacent to the angle A, by b (for base), BC, the side opposite the angle A, by a (for altitude), and the hypotenuse AB by h.

The three sides of the right triangle form six different ratios, namely,

$$\frac{a}{h}, \quad \frac{b}{h}, \quad \frac{a}{b},$$

and their reciprocals

$$\frac{h}{a}, \quad \frac{h}{b}, \quad \frac{b}{a}.$$

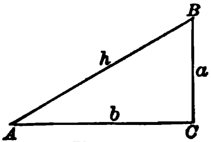

Fig. 7.

a. These ratios do not depend upon the distance of the point B from the vertex of the angle; that is, each of the six ratios will have the same value for every other point B' located on either one of the sides of the angle.

For if B' be any other point on AB or AB produced, or on AC or AC produced, and the perpendicular $B'C'$ be drawn to the other side, the triangle $AB'C'$ will be similar to triangle ABC and therefore

$$\frac{B'C'}{AB'} = \frac{BC}{AB} = \frac{a}{h},$$

and similarly for each of the other ratios.

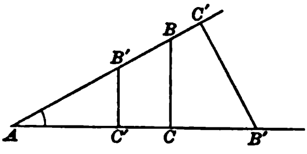

Fig. 8.

b. The ratios differ for different angles, for if the ratios were equal, the corresponding triangles would be similar (why?), and corresponding angles equal, which is contrary to the hypothesis that they are different.

Since the ratios depend upon the angle for their values, they are functions of the angle according to the general definition of a function given in 5. Each of these functions has received a special name. Referring to Fig. 7,

$\frac{a}{h}$, that is, $\frac{\textit{side opposite angle } A}{\textit{hypotenuse}}$, is called the **sine** of angle A;

$\frac{b}{h}$, that is, $\frac{\textit{side adjacent angle } A}{\textit{hypotenuse}}$, is called the **cosine** of angle A;

$\frac{a}{b}$, that is, $\frac{\textit{side opposite angle } A}{\textit{side adjacent angle } A}$, is called the **tangent** of angle A;

$\frac{h}{a}$, the *reciprocal of the sine,* is called the **cosecant** of angle A;

$\frac{h}{b}$, the *reciprocal of the cosine,* is called the **secant** of angle A;

$\frac{b}{a}$, the *reciprocal of the tangent,* is called the **cotangent** of angle A.

The six functions just defined are variously known as the *trigonometric, circular,* or *goniometric* functions: trigonometric, because they form the basis of the science of trigonometry; circular, because of their relations to the arc of a circle, as will be shown presently; goniometric, because of their use in determining angles, from *gonia,* a Greek word meaning angle. The expressions sine of angle A, cosine of angle A, etc., are abbreviated to sin A, cos A, tan A, csc A or cosec A, sec A, cot A.

Besides these six functions, two others are sometimes used,* viz.,

versed sine $A = 1 -$ cosine A, abbreviated to **vers** A;
coversed sine $A = 1 -$ sine A, abbreviated to **covers** A.

The definitions of the first six trigonometric functions must be thoroughly memorized. The first three are especially important and should be memorized in the following form:

Given an acute angle in a right triangle, —

The **sine** *of the angle is the ratio of the side opposite the angle to the hypotenuse.*

The **cosine** *of the angle is the ratio of the side adjacent the angle to the hypotenuse.*

The **tangent** *of the angle is the ratio of the side opposite the angle to the adjacent side.*

The remaining three functions may be remembered most readily by the aid of the reciprocal relations, —

$$\sin A \cdot \operatorname{cosec} A = 1,$$
$$\cos A \cdot \sec A = 1,$$
$$\tan A \cdot \cot A = 1,$$

that is,

$$\operatorname{cosec} A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}.$$

It will aid the memory to observe that only one **o** appears in each pair of reciprocal functions.

It should be noticed that while a, b, and h are lines, the ratio of any two of them is an abstract number; that is, the trigonometric functions are abstract numbers. Again, the expressions sin A, cos A, tan A, etc., are single symbols which cannot be separated. sin has no meaning except as it is associated with some angle, just as the symbol $\sqrt{\ }$ has no meaning except when used in connection with some quantity, as in $\sqrt{a}$, $\sqrt{4}$, etc.

EXAMPLE 1. The sides of a right triangle are 3, 4, 5. Find all the trigonometric functions of the angle A opposite the side 3.

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* Especially in navigation.

Solution. The hypotenuse of the triangle equals 5. Hence, applying the definitions, we have

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \tfrac{3}{5}.$$

Similarly, $\cos A = \tfrac{4}{5}$, $\tan A = \tfrac{3}{4}$.

The cotangent of A is the reciprocal of the tangent, hence

$\cot A = 1 \div \tfrac{3}{4} = \tfrac{4}{3}$, similarly, $\sec A = 1 \div \tfrac{4}{5} = \tfrac{5}{4}$, $\csc A = \tfrac{5}{3}$.

Example 2. The legs of a right triangle are a and b; find the functions of the angle A opposite the side a.

Solution. The hypotenuse $h = \sqrt{a^2 + b^2}$; hence

$$\sin A = \frac{a}{h} = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos A = \frac{b}{h} = \frac{b}{\sqrt{a^2+b^2}}, \quad \tan A = \frac{a}{b},$$

$$\csc A = \frac{h}{a} = \frac{\sqrt{a^2+b^2}}{a}, \quad \sec A = \frac{h}{b} = \frac{\sqrt{a^2+b^2}}{b}, \quad \cot A = \frac{b}{a}.$$

Exercise 3

1. A right triangle has its sides equal to 5, 12, 13. Calculate the six functions of the angle A opposite the side 5.

Ans. $\sin A = \tfrac{5}{13}$, $\cos A = \tfrac{12}{13}$, $\tan A = \tfrac{5}{12}$,
$\csc A = \tfrac{13}{5}$, $\sec A = \tfrac{13}{12}$, $\cot A = \tfrac{12}{5}$.

2. In the same triangle calculate the functions of the angle B opposite the side 12.

Ans. $\sin B = \tfrac{12}{13}$, $\cos B = \tfrac{5}{13}$, $\tan B = \tfrac{12}{5}$,
$\csc B = \tfrac{13}{12}$, $\sec B = \tfrac{13}{5}$, $\cot B = \tfrac{5}{12}$.

3. By comparing the answers in problem 2 with those in problem 1, and remembering that $A + B = 90°$, write down six equations, of which the following is the first: $\sin A = \cos B = \cos (90° - A)$.

Ans. $\cos A = \sin B = \sin (90° - A)$,
$\tan A = \cot B = \cot (90° - A)$,
$\sec A = \csc B = \csc (90° - A)$, etc.

4. Show that for any acute angle the following equations are true:

$$\sin (90° - A) = \cos A, \qquad \sec (90° - A) = \csc A,$$
$$\cos (90° - A) = \sin A, \qquad \csc (90° - A) = \sec A,$$
$$\tan (90° - A) = \cot A, \qquad \cot (90° - A) = \tan A.$$

(Suggestion. Consider A and $90° - A$ the two acute angles of a right triangle, and express the functions of each angle in terms of the sides of the triangle.)

5. The two legs of a right triangle are 8 and 15. Write down the functions of the angle A opposite the side 8; also the functions of the angle B adjacent the side 8.

Ans. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$,
$\sin B = \frac{15}{17}$, $\cos B = \frac{8}{17}$, $\tan B = \frac{15}{8}$.

6. By using the results of problem 5, show that

$$\frac{\sin A}{\cos A} = \tan A \text{ and } \frac{\cos A}{\sin A} = \cot A.$$

7. One side of a right triangle is 9 and the hypotenuse is 41. Compute the functions of the angle A included between the hypotenuse and the given side.

Ans. $\sin A = \frac{40}{41}$, $\cos A = \frac{9}{41}$, $\tan A = \frac{40}{9}$,
$\csc A = \frac{41}{40}$, $\sec A = \frac{41}{9}$, $\cot A = \frac{9}{40}$.

8. Two legs of a right triangle are $p^2 - q^2$, and $2\,pq$ respectively. Find the sine, cosine and tangent of the angle B opposite the side $2\,pq$.

$$\textit{Ans.}\quad \sin B = \frac{2\,pq}{p^2+q^2},\ \cos B = \frac{p^2-q^2}{p^2+q^2},\ \tan B = \frac{2\,pq}{p^2-q^2}.$$

9. Given $\sin A = \frac{3}{5}$, find vers A and covers A.

Ans. vers $A = \frac{1}{5}$, covers $A = \frac{2}{5}$.

Exercise 4

1. Compute all the functions of 45°.

Ans. $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2} = 0.707$, $\tan 45° = 1$,
$\csc 45° = \sec 45° = \sqrt{2} = 1.414$, $\cot 45° = 1$.

(Suggestion. Construct a right triangle having an angle 45°, and denote each of the equal sides by a.)

2. Compute all the functions of 30°.

Ans. $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\tan 30° = \frac{1}{3}\sqrt{3}$,
$\csc 30° = 2$, $\sec 30° = \frac{2}{3}\sqrt{3}$, $\cot 30° = \sqrt{3}$.

(Suggestion. If one angle of a right triangle is double the other, then the hypotenuse is double the shorter side. Call the shorter side a.)

3. By measurement find the functions of 15°.

Ans. $\sin 15° = 0.26$, $\cos 15° = 0.97$, $\tan 15° = 0.28$,
$\csc 15° = 3.87$, $\sec 15° = 1.03$, $\cot 15° = 3.73$.

(Suggestion. By means of a protractor, or otherwise, construct an angle of 15°. Draw a perpendicular to one side forming a right triangle. Measure the sides of the triangle, and compute the ratios.)

4. Given $\sin A = \frac{5}{7}$, construct and measure the angle.
Ans. $A = 45° 35'$.

5. Given $\cos A = \frac{2}{3}$, construct and measure the angle.
Ans. $A = 48° 11'$.

6. Given $\tan A = \frac{6}{5}$, construct and measure the angle.
Ans. $A = 50° 12'$.

7. Show that

$$\frac{\sin A}{\cos A} = \tan A, \quad \frac{\cos A}{\sin A} = \cot A,$$

A being any acute angle.

8. Show that

$$\sin^2 A + \cos^2 A = 1,^*$$
$$\tan^2 A + 1 = \sec^2 A,$$
$$\cot^2 A + 1 = \csc^2 A.$$

(Suggestion. Remember that $a^2 + b^2 = h^2$.)

9. Given $\sin A = \frac{7}{25}$, find all the other functions.
Ans. $\sin A = \frac{7}{25}$, $\cos A = \frac{24}{25}$, $\tan A = \frac{7}{24}$,
$\csc A = \frac{25}{7}$, $\sec A = \frac{25}{24}$, $\cot A = \frac{24}{7}$.

10. Show that as an angle increases from 0° toward 90°, its sine, tangent and secant increase, while its cosine, cotangent and cosecant decrease.

11. Show that every sine and cosine is a proper fraction, while the tangent and cotangent may have any value large or small.

8. Trigonometric Functions Determined Approximately by Measurement. There are various ways of computing the trigonometric functions of a given angle. The results of such computations for the sines, cosines, tangents and cotangents of angles between 0° and 90° have been put together into tables, known as *tables of natural functions*. We shall learn how to use such tables and later how to calculate them. It is of value to the beginner to know how approximate values of the functions may be obtained graphically.

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* $\sin^2 A$ means $(\sin A)^2$, $\tan^2 A$ means $(\tan A)^2$, etc., and generally $\sin^n A = (\sin A)^n$, $\tan^n A = (\tan A)^n$, etc., except when $n = -1$. The meaning of $\sin^{-1} A$, $\tan^{-1} A$, etc., will be explained later.

Example 1. Find graphically the functions of 25°.

Solution. By means of a protractor construct an angle $MON = 25°$. Take OB any convenient length, say 10 inches, and from O as a center and with OB as a radius describe an arc BC cutting ON in C. Draw CE and BA perpendicular to OB. By measurement we find $CE = 4.23$ inches, $OE = 9.06$ inches, $AB = 4.66$ inches, and by construction $OB = OC = 10$ inches. Hence

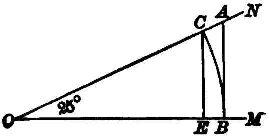

Fig. 9.

$$\sin 25° = \frac{EC}{OC} = \frac{4.23}{10} = 0.423, \quad \csc 25° = \frac{1}{\sin 25} = 2.364,$$

$$\cos 25° = \frac{OE}{OC} = \frac{9.06}{10} = 0.906, \quad \sec 25° = \frac{1}{\cos 25} = 1.104,$$

$$\tan 25° = \frac{BA}{OB} = \frac{4.66}{10} = 0.466, \quad \cot 25° = \frac{1}{\tan 25} = 2.145.$$

Observe that we have used two different triangles, the triangle COE to obtain the sine, cosine and their reciprocals, and a second triangle AOB to obtain the tangent and cotangent. This was done in order to have in each case 10 for a divisor. If for instance we had used the triangle AOB only, we would have had

$$\sin 25° = \frac{BA}{OA} = \frac{4.66}{11.03} = 0.423, \text{ instead of } \frac{4.23}{10} \text{ as above.}$$

Exercise 5

1. Obtain by measurement the sine, cosine and tangent of 40°.
Ans. $\sin 40° = 0.643$, $\cos 40° = 0.766$, $\tan 40° = 0.839$.

2. Find the sine, cosine and tangent of 35°. To avoid constructing the triangle and measuring the necessary lines, we may make use of the diagram facing page 16. $\sin 35° = \frac{AB}{OA}$. Now $OA = OR = 100$, and the measure of $AB = 57.4$ may be read off on the vertical scale. Hence $\sin 35° = \frac{57.4}{100} = 0.574$.

Similarly,

$$\cos 35° = \frac{OB}{OA} = \frac{81.9}{100} = 0.819.$$

To find the tangent it is better to use the triangle *OPR*, from which

$$\tan 35^\circ = \frac{RP}{OR} = \frac{70}{100} = 0.70.$$

3. With the aid of Fig. 10 find each of the results given in the following table.

(Suggestion. Choose your triangle so that the denominator of the fraction equals 100 or some other integer.)

NATURAL FUNCTIONS.

Angle.	sin	cos	tan	cot	
5°	0.087	0.996	0.087	11.430	85°
10°	0.174	0.985	0.174	5.671	80°
15°	0.259	0.966	0.268	3.732	75°
20°	0.342	0.940	0.364	2.747	70°
25°	0.423	0.906	0.466	2.145	65°
30°	0.500	0.866	0.577	1.732	60°
35°	0.574	0.819	0.700	1.428	55°
40°	0.643	0.766	0.839	1.192	50°
45°	0.707	0.707	1.000	1.000	45°
	cos	sin	cot	tan	Angle

Explanation of table. For angles in the left-hand column the names of the functions appear on top, for angles in the right-hand column the names of the functions appear at the bottom. Thus the number 0.423, which is in the column headed "sin" and has 25° to the left of it, is the sine of 25°; the same number being in the column which has "cos" at the bottom, and having 65° opposite it in the right column, is also the cosine of 65°.

4. By use of the table, express the numbers 0.174, 0.866, 0.643, 0.707, both as sines and cosines.

Ans. 0.174 = sin 10° = cos 80°, 0.866 = sin 60° = cos 30°, etc.

5. By use of the table, express the numbers 0.364, 2.145 and 1.000 both as tangents and cotangents.

6. Every number in the second and third columns is the sine of one angle and the cosine of another angle. What relation do you observe between each pair of angles?

Every number in the fourth and fifth columns is the tangent of one angle and the cotangent of another. What relation exists between each pair of angles?

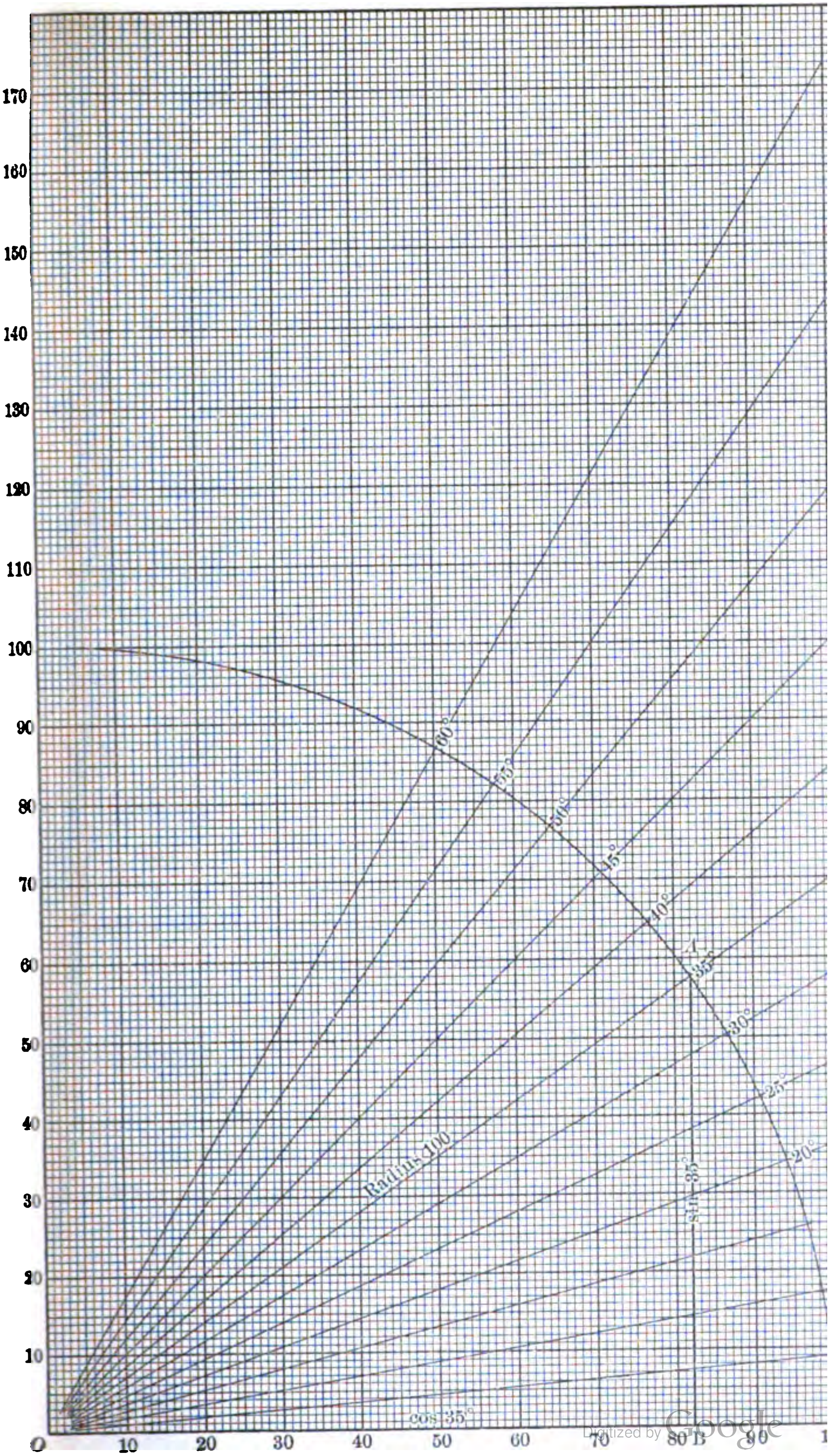
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
O
10
20
30
40
50
60
70
80 B
90
60°
55°
50°
45°
40°
A
35°
30°
25°
20°
Radius 100
sin 35°
cos 35°

Fig. 10.

7. By examining the table, verify the following statements and, if you can, give reasons for them.

a. Every sine and cosine is a fraction less than unity.

b. Every tangent of an angle less than 45° is a fraction less than 1.

c. Every tangent of an angle greater than 45° is some number greater than unity.

d. The sine and tangent of an angle increase as the angle increases.

e. The cosine and cotangent of an angle decrease as the angle increases.

f. The sine and tangent of a small angle are nearly equal.

9. Given One of Its Functions, To Construct the Angle. In the last article it was shown how to find by measurement the functions of a given angle. We will now consider the converse problem, that is, how to construct the angle when one of its functions is given.

EXAMPLE 1. To construct an angle whose tangent is $\frac{3}{4}$.

Solution. Take AC 4 units in length and at C construct a perpendicular CB 3 units in length. Join A and B. Then CAB is the required angle, for $\tan CAB = \frac{CB}{AC} = \frac{3}{4}$.

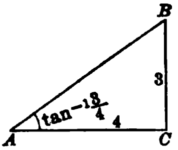

Fig. 11.

The expression

$$A \text{ is the angle whose tangent is } \tfrac{3}{4}$$

is written in short

$$A = \tan^{-1}\tfrac{3}{4},$$

and is read in either of three ways:

a. A is the angle whose tangent is $\frac{3}{4}$.

b. A is the inverse tangent $\frac{3}{4}$.

c. A is the arctangent $\frac{3}{4}$.

Similarly, if

$$y = \sin x$$

then

$$x = \sin^{-1} y,$$

in words, if y equals the sine of x,

then x equals the angle whose sine is y,

or x equals the inverse sine y,

or x equals the arcsine y.

Corresponding meanings are given to the symbols

$$y = \cos^{-1}x,\ y = \csc^{-1}x,\ y = \sec^{-1}x,\ y = \cot^{-1}x.$$

EXAMPLE 2. To construct an angle whose sine is $\frac{4}{7}$.

Solution. At a point C in a line CD of indefinite length, construct a perpendicular CB 4 units in length. From B as a center, with a radius 7 units in length, draw an arc cutting CD at A. Join A and B. Then the angle CAB is the required angle, for

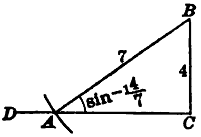

Fig. 12.

$$\sin CAB = \frac{CB}{AB} = \frac{4}{7}.$$

Another way of constructing $\sin^{-1}\frac{4}{7}$ is to construct a semicircle having a diameter AB 7 units long. From B as a center, with a radius 4 units long, describe an arc cutting the semicircle at C. Join A and C. Then BAC is the required angle. (Draw the figure and give reasons.)

EXAMPLE 3. In the preceding example, find each of the other functions of the angle A.

Solution. Two sides of the right triangle ABC being known, the third side is easily found.

$$AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{7^2 - 4^2} = \sqrt{33}.$$

Hence

$$\cos A = \frac{AC}{AB} = \frac{\sqrt{33}}{7} = 0.821, \quad \tan A = \frac{BC}{AC} = \frac{4}{\sqrt{33}} = 0.696,$$

$$\sec A = \frac{1}{\cos A} = \frac{7}{\sqrt{33}} = 1.218, \ \cot A = \frac{1}{\tan A} = \frac{\sqrt{33}}{4} = 1.436,$$

$$\csc A = \frac{1}{\sin A} = \frac{7}{4} = 1.750.$$

Observe that the expressions, —

$$\sin^{-1}\frac{4}{7},\ \cos^{-1}\frac{\sqrt{33}}{7},\ \tan^{-1}\frac{4}{\sqrt{33}},\ \csc^{-1}\frac{7}{4},\ \sec^{-1}\frac{7}{\sqrt{33}},\ \cot^{-1}\frac{\sqrt{33}}{4},$$

all represent the same angle.

EXERCISE 6

Construct an acute angle equal to A, when, —

1. $\sin A = \frac{2}{3}$. 2. $\tan A = \frac{5}{2}$. 3. $\cos A = 0.5$.
4. $\sec A = \frac{3}{2}$. 5. $\tan A = 4$. 6. $\cos A = 0.45$.

7. $\cos A = \frac{m}{n}$. 8. $\sin A = k$. 9. $\tan A = \frac{1}{k}$.

10. Read in three different ways the expressions,—

$$A = \sin^{-1}\tfrac{2}{3},\quad B = \tan^{-1} 3,\quad x = \csc^{-1} 3.5,\quad y = \cot^{-1}\sqrt{3}.$$

11. Construct and measure in degrees each of the following angles,—

$$A = \sin^{-1}\tfrac{1}{3},\quad B = \cos^{-1}\tfrac{1}{3},\quad C = \tan^{-1}\tfrac{1}{3},\quad D = \cot^{-1}\tfrac{1}{3}.$$

Ans. $A = 19.5°$; $B = 70.5°$; $C = 18.5°$; $D = 71.5°$; approximately.

12. Given $\sin A = \frac{2}{3}$; to find the other functions.

$$\textit{Ans.}\quad \cos A = \frac{\sqrt{5}}{3},\ \tan A = \frac{2}{\sqrt{5}} = \tfrac{2}{5}\sqrt{5},$$

$$\sec A = \frac{3}{\sqrt{5}} = \tfrac{3}{5}\sqrt{5},\ \cot A = \tfrac{1}{2}\sqrt{5},\ \csc A = \tfrac{3}{2}$$

13. $\tan B = \frac{15}{8}$; find the other functions of the angle B.

Ans. $\sin B = \frac{15}{17}$, $\cos B = \frac{8}{17}$, $\cot B = \frac{8}{15}$, $\sec B = \frac{17}{8}$.

14. $A = \sin^{-1}\frac{1}{2}$; find the functions of A.

Ans. $\sin A = \frac{1}{2}$, $\cos A = \frac{1}{2}\sqrt{3}$, $\tan A = \frac{1}{3}\sqrt{3}$, $\csc A = 2$, etc.

15. Show that

$$\sin^{-1}\tfrac{5}{13} = \cos^{-1}\tfrac{12}{13} = \tan^{-1}\tfrac{5}{12}.$$

16. $\sin^{-1}\frac{8}{17} = \cos^{-1} x = \tan^{-1} y$; find x and y.

Ans. $x = \frac{15}{17}$, $y = \frac{8}{15}$.

17. If $y = \sin x$, show that $x = \cos^{-1}\sqrt{1 - y^2} = \tan^{-1}\dfrac{y}{\sqrt{1 - y^2}}$.

18. Show graphically that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$. (Suggestion. Construct $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$, measure each and add the results.)

10. Functions of Complementary Angles. Let ABC be any right triangle, C the right angle, and a, b, c the three sides. Then angle A plus angle B equals 90, that is, A and B are complementary angles. Now

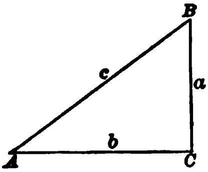

Fig. 13.

$$\sin A = \frac{a}{c},\ \text{and also}\ \cos B = \frac{a}{c},$$

therefore

$$\sin A = \frac{a}{c} = \cos B = \cos(90° - A).$$

Similarly

$$\cos A = \frac{b}{c} = \sin B = \sin (90^\circ - A),$$

$$\tan A = \frac{a}{b} = \cot B = \cot (90^\circ - A),$$

$$\cot A = \frac{b}{a} = \tan B = \tan (90^\circ - A),$$

$$\sec A = \frac{c}{b} = \csc B = \csc (90^\circ - A),$$

$$\csc A = \frac{c}{a} = \sec B = \sec (90^\circ - A).$$

These results may be put in words as follows:

The sine of an angle equals the cosine of the complement of the angle. The cosine of an angle equals the sine of the complement of the angle. The tangent of an angle equals the cotangent of the complement of the angle, and similarly for each of the remaining functions.

If we arrange the six functions in three pairs, viz., sine, cosine; tangent, cotangent; secant, cosecant; and call either function of a pair the cofunction of the other, we may express the six rules just given by a single rule, namely:

Any cofunction of an angle is equal to the corresponding function of the complement of that angle.[*]

By means of this rule, any function of an acute angle can be expressed as some function of an angle less than 45°. Thus

$$\sin 75^\circ = \cos (90^\circ - 75^\circ) = \cos 15^\circ,$$
$$\cos 80^\circ = \sin (90^\circ - 80^\circ) = \sin 10^\circ, \text{etc.}$$

EXAMPLE. Given $\cot A = \tan 8A$, to find one value of A.

Solution. $\cot A = \tan (90^\circ - A)$, and from the condition of the problem

$$\cot A = \tan 8A,$$

therefore $$\tan (90^\circ - A) = \tan 8A.$$

---

[*] The term cosine was not used until the beginning of the 17th century. Before that time the term sine of the complement (*complimenti sinus*) was used instead a contraction of which gave rise to the present name cosine. Similarly cotangent is a contraction of *complimenti tangens* and cosecant of *complimenti secans*. The abbreviations sin, cos, tan, etc., did not come into general use until the middle of the 18th century.

This last equation is satisfied when

$$90^\circ - A = 8\,A.$$

Solving $$A = 10^\circ.$$

NOTE. This is not the only value that A may have. After the definitions of the functions have been extended to angles greater than 90°, it will be seen that 30°, 50°, 70°, etc., are other values of A satisfying the equation $\cot A = \tan 8\,A$.

11. Functions of 0°, 30°, 45°, 60°, 90°. There are certain values of the angle for which the values of the functions may be easily determined exactly.

a. The functions of 0°. Let A be the angle formed by a fixed line OX and a line OP of constant length h rotating about O as a center. Let b be the base and a the altitude of the triangle formed by dropping a perpendicular from P to OX. As the angle increases, a increases and b decreases, and vice versa, as the angle decreases a decreases and b increases. As A approaches 0°, a approaches 0 and b approaches h, hence in the limit

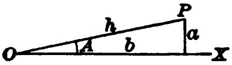

Fig. 14.

$$\sin 0^\circ = \frac{0}{h} = 0,\quad \cos 0^\circ = \frac{h}{h} = 1,\quad \tan 0^\circ = \frac{0}{h} = 0.$$

b. The functions of 30°. When the angle $A = 30^\circ$, the other acute angle of the right triangle equals 60°; the right triangle then forms one-half of an equilateral triangle. Each side of this triangle equals h, hence a, the altitude of the right triangle, equals $\frac{h}{2}$, and the base $= \sqrt{h^2 - \left(\frac{h}{2}\right)^2} = \sqrt{\frac{3}{4}h^2} = \frac{1}{2}h\sqrt{3}$, and we have

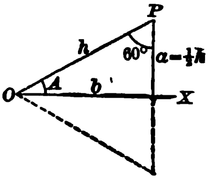

Fig. 15.

$$\sin 30^\circ = \frac{\frac{1}{2}h}{h} = \frac{1}{2},\quad \cos 30^\circ = \frac{\frac{1}{2}h\sqrt{3}}{h} = \frac{1}{2}\sqrt{3},$$

$$\tan 30^\circ = \frac{\frac{1}{2}h}{\frac{1}{2}h\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

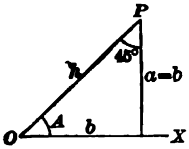

Fig. 16.

c. The functions of 45°. When $A = 45^\circ$ the right triangle is isosceles, that is, a and b are equal, and we have

$h^2 = a^2 + a^2 = 2\,a^2$, $\quad a = h\sqrt{\frac{1}{2}} = \frac{1}{2}h\sqrt{2}$, so that

$$\sin 45^\circ = \frac{\frac{1}{2}h\sqrt{2}}{h} = \tfrac{1}{2}\sqrt{2}, \quad \cos 45^\circ = \frac{\frac{1}{2}h\sqrt{2}}{h} = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45^\circ = \frac{\frac{1}{2}h\sqrt{2}}{\frac{1}{2}h\sqrt{2}} = 1.$$

d. The functions of 60°. 60° is the complement of 30°, hence by Art. 10,

$$\sin 60^\circ = \cos 30^\circ = \tfrac{1}{2}\sqrt{3}, \quad \cos 60^\circ = \sin 30^\circ = \tfrac{1}{2},$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}.$$

e. The functions of 90°. As the angle A approaches 90°, a approaches h, while b approaches 0, so that in the limit

$$\sin 90^\circ = \frac{h}{h} = 1, \quad \cos 90^\circ = \frac{0}{h} = 0, \quad \tan 90^\circ = \frac{h}{0} = \infty.^*$$

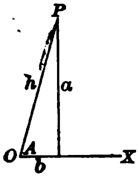

Fig. 17.

The sine, cosine, and tangent of each of the angles, 0°, 30°, 45°, 60°, 90°, must be remembered once for all. For convenience in memorizing, the results are collected in the following table.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞

$\sqrt{2} = 1.414$
$\sqrt{3} = 1.732$

A remarkable uniformity is brought out by writing the above values in another form as in the table below. The student should verify the results for himself.

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* This needs explanation. What we have is $\tan A = \frac{a}{b}$. Now as A approaches 90°, the denominator b grows smaller and smaller, while the numerator a approaches the constant length h. In consequence, the fraction $\frac{a}{b}$ grows larger and larger. When b becomes sufficiently small, $\frac{a}{b}$ becomes larger than any assignable quantity, and this is conveniently expressed by saying that as b approaches 0, $\frac{a}{b}$ approaches infinity. In symbols,

$$\tan 90^\circ = \frac{h}{0} = \infty.$$

	0°	30°	45°	60°	90°
sin	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
cos	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$
tan	0	$\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3^2}$	$\frac{1}{3}\sqrt{3^3}$	∞

EXERCISE 7

1. Express as functions of the complementary angles,
sin 40°, cos 50°, tan 48°, cot 22.5°, sec 60°, csc 56° 20′.

2. Express as functions of an angle less than 45°,
tan 75°, cos 67° 30′, sin 57° 35′, sec 73° 45′ 30″.

3. Find the functions of 30° by using a triangle whose hypotenuse is 2.

4. Find the functions of 45° by using a triangle whose base is 1.

5. Find the functions of 90° by using a triangle whose base remains constant and equal to 1.

6. 90° is the complement of 0°. Hence find the functions of 90° from those of 0°.

7. Make a table containing the values of the secant, cosecant and cotangent of each of the angles 0°, 30°, 45°, 60°, 90°.

8. Given $\cos A = \sin A$, find one value of A. *Ans.* $A = 45°$.

9. Given $\tan A = \cot (45° + A)$, find one value of A.
Ans. $A = 22° 30'$.

10. Given $\tan 2A = \cot 3A$, find one value of A.
Ans. $A = 18°$.

11. Given $\sin (45° + A) = \cos (30° + A)$, find one value of A.
Ans. $A = 7° 30'$.

12. Given $\sin 2A = \cos (45° - A)$, find one value of A.
Ans. $A = 45°$.

13. Given $\cot A = \tan (n - 1) A$, find one value of A.
Ans. $A = \frac{90°}{n}$.

14. Prove that $\text{vers } A = \text{covers } (90° - A)$,
and that $\text{covers } A = \text{vers } (90° - A)$.

15. It is shown in geometry that if the radius of a circle is divided into extreme and mean ratio, the greater segment will be equal to the chord subtending an arc of 36°; that is, if in Fig. 18 AO is divided at B so that $AO : OB = OB : BA$, then the chord AC, taken equal to OB, subtends an angle COA equal to 36°.

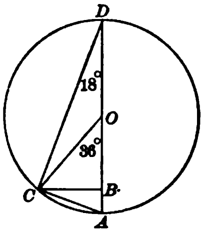

Fig. 18.

Put $AC = OB = x$, and $AO = r$, then CDA is a right triangle whose short side is x, whose hypotenuse is $2r$, and the angle $CDA = 18°$. Hence $\sin 18° = \frac{x}{2r}$, where x is the positive root of the equation

$$r : x = x : r - x.$$

Solving, we find the positive value of x, $x = r(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})$.

Hence $$\sin 18° = \frac{x}{2r} = \frac{r(-\frac{1}{2} + \frac{1}{2}\sqrt{5})}{2r} = \tfrac{1}{4}(-1 + \sqrt{5}).$$

Show now that $$\cos 18° = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}},\quad \sec 18° = \frac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5}},$$

$$\tan 18° = \frac{\sqrt{25 - 10\sqrt{5}}}{5},\quad \cot 18° = \sqrt{5 + 2\sqrt{5}}.$$

12. Fundamental Relations. In any right triangle (Fig. 19)

$$a^2 + b^2 = h^2.$$

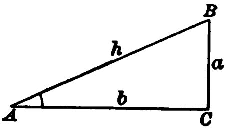

Fig. 19.

By dividing this equation first by h^2, then by b^2, and finally by a^2, we obtain in turn

$$\left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1, \quad \text{or} \quad \sin^2 A + \cos^2 A = 1, \tag{1}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{h}{b}\right)^2, \quad \text{or} \quad \tan^2 A + 1 = \sec^2 A, \tag{2}$$

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{h}{a}\right)^2, \quad \text{or} \quad 1 + \cot^2 A = \csc^2 A. \tag{3}$$

(1), (2), and (3) are called the *square relations* of the trigonometric functions. These together with the three *reciprocal relations*

$$\sin A \cdot \csc A = 1, \tag{4}$$

$$\cos A \cdot \sec A = 1, \tag{5}$$

$$\tan A \cdot \cot A = 1, \tag{6}$$

constitute the six fundamental relations of the trigonometric functions. To these, two others are usually added, viz.,

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}. \tag{7}$$

Proof.

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{h}}{\frac{b}{h}} = \frac{a}{b} = \tan A, \text{ and } \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \cot A.$$

13. To express each of the Functions in Terms of a Given One.

a. Analytic method. EXAMPLE 1. To express each of the functions in terms of the sine.

$$\cos A = \sqrt{1 - \sin^2 A} \qquad \text{(By (1), Art. 12)}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} \qquad \text{(By (7), Art. 12)}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}} \qquad \text{(By (5), Art. 12)}$$

$$\csc A = \frac{1}{\sin A} \qquad \text{(By (4), Art. 12)}$$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} \qquad \text{(By (7), Art. 12)}$$

EXAMPLE 2. To express each of the functions in terms of the tangent.

$$\sec A = \sqrt{1 + \tan^2 A} \qquad \text{(By (2), Art. 12)}$$

$$\cot A = \frac{1}{\tan A} \qquad \text{(By (6), Art. 12)}$$

$$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{1 + \tan^2 A}} \qquad \text{(By (5), Art. 12)}$$

$$\sin A = \cos A \cdot \tan A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \qquad \text{(By (7), Art. 12)}$$

$$\csc A = \frac{1}{\sin A} = \frac{\sqrt{1 + \tan^2 A}}{\tan A} \qquad \text{(By (4), Art. 12)}$$

b. Geometric method. EXAMPLE 1. To express each of the functions in terms of the sine.

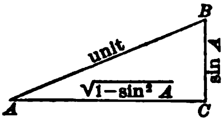

Fig. 20.

In the right triangle ABC (Fig. 20),

$$\sin A = \frac{BC}{AB}; \text{ hence if } AB \text{ is chosen for the unit of measure,}$$

$$\sin A = \frac{BC}{1} = BC, \text{ and } AC = \sqrt{AB^2 - BC^2} = \sqrt{1 - \sin^2 A}.$$

Then from the definitions of the trigonometric functions

$$\cos A = \frac{AC}{AB} = \sqrt{1 - \sin^2 A}, \quad \tan A = \frac{BC}{AC} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}, \text{ etc.}$$

EXAMPLE 2. To express each of the functions in terms of the tangent.

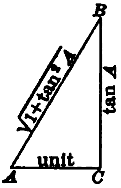

Fig. 21.

In Fig. 21,

$$\tan A = \frac{BC}{AC}; \text{ hence if } AC \text{ is chosen for the unit of measure,}$$

$$\tan A = \frac{BC}{1} = BC, \text{ and } AB = \sqrt{AC^2 + BC^2} = \sqrt{1 + \tan^2 A}.$$

Then by definition

$$\sin A = \frac{BC}{AB} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \quad \cos A = \frac{AC}{AB} = \frac{1}{\sqrt{1 + \tan^2 A}}, \text{ etc.}$$

EXERCISE 8

1. From (1) show that

$$\sin A = \sqrt{1 - \cos^2 A}, \quad \cos A = \sqrt{1 - \sin^2 A},$$
$$(1 - \sin A)(1 + \sin A) = \cos^2 A,$$
$$(1 - \cos A)(1 + \cos A) = \sin^2 A, \quad \frac{\cos A}{1 - \sin A} = \frac{1 + \sin A}{\cos A},$$
$$\frac{\sin A}{1 - \cos A} = \frac{1 + \cos A}{\sin A}.$$

2. From (2) show that

$$\sec A = \sqrt{1 + \tan^2 A}, \quad \tan A = \sqrt{\sec^2 A - 1},$$
$$(\sec A - \tan A)(\sec A + \tan A) = 1.$$

3. Show that

$$\csc A = \sqrt{1 + \cot^2 A}, \quad \cot A = \sqrt{\csc^2 A - 1},$$
$$(\csc A - \cot A)(\csc A + \cot A) = 1.$$

4. From (7) show that

$$\cos A \cdot \tan A = \sin A, \quad \sin A \cdot \cot A = \cos A,$$
$$\sin A \cdot \sec A = \tan A, \quad \cos A \cdot \csc A = \cot A.$$

5. Express in words the relations given by formulas (1) to (7).

6. Use the analytic method to express the tangent in terms of the cosecant; in terms of the cosine.

7. Use the geometric method to express each the sine and the cosine in terms of the tangent.

In the following exercises compare your results with those given in the table on page 35.

8. Express each of the functions in terms of the cosine.

9. Express each of the functions in terms of the cosecant.

10. Express each of the functions in terms of the secant.

11. Use $\sin 30° = \frac{1}{2}$ to find each of the remaining functions of 30°.

12. $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$; find the other functions of 15°.

Ans. $\cos 15° = \frac{1}{2}\sqrt{2+\sqrt{3}}$, $\tan 15° = 2 - \sqrt{3}$, $\sec 15° = 2\sqrt{2-\sqrt{3}}$,
$\csc 15° = 2\sqrt{2+\sqrt{3}}$, $\cot 15° = 2 + \sqrt{3}$.

13. If

$$\sin x = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } s = \frac{a+b+c}{2},$$

prove that $\cos x = \sqrt{\dfrac{s(s-a)}{bc}}$ and $\tan x = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

14. Reduction of Trigonometric Expressions to their Simplest Form. Like algebraic expressions, expressions involving trigonometric functions may frequently be reduced to a simpler form. As a rule the reduction is most easily effected by expressing each of the functions which occur in the expression in terms of the sine and cosine and by reducing the resulting expression like any algebraic expression, treating the sine and cosine as two separate quantities. In the end the result may again be expressed in terms of whatever function or functions give the result the simplest form. Of course one might express everything in terms of a single function, say the sine, but that would introduce radicals for every function except the cosecant. By using both the sine and cosine radicals are avoided, unless, of course, the expression involves radicals to begin with.

EXAMPLE 1. Reduce the expression $\sin A + \cot A \cos A$.

Solution. Substituting for $\cot A$ its value in terms of the sine and cosine, we have

$$\sin A + \frac{\cos A}{\sin A} \cos A.$$

Reducing to a common denominator,

$$\frac{\sin A \sin A + \cos A \cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A}.$$

Now by (1), Art. 12, $\sin^2 A + \cos^2 A = 1$; hence finally,

$$\sin A + \cot A \cos A = \frac{1}{\sin A} = \csc A.$$

EXAMPLE 2. Reduce the expression

$$\sin^2 A \tan A + \cos^2 A \cot A + 2 \sin A \cos A.$$

Solution. Substituting for $\tan A$ and $\cot A$ their values in terms of the sine and cosine, (7), Art. 12, we have

$$\sin^2 A \frac{\sin A}{\cos A} + \cos^2 A \frac{\cos A}{\sin A} + 2 \sin A \cos A$$

$$= \frac{\sin^4 A + \cos^4 A + 2 \sin^2 A \cos^2 A}{\sin A \cos A} = \frac{(\sin^2 A + \cos^2 A)^2}{\sin A \cos A}$$

$$= \frac{1}{\sin A \cos A} = \csc A \sec A.$$

EXAMPLE 3. Reduce the expression

$$(\cos x \cos y - \sin x \sin y)^2 + (\sin x \cos y + \cos x \sin y)^2.$$

Solution. Squaring the expressions enclosed in parentheses,

$$(\cos x \cos y - \sin x \sin y)^2 = \cos^2 x \cos^2 y - 2 \cos x \cos y \sin x \sin y + \sin^2 x \sin^2 y,$$

$$(\sin x \cos y + \cos x \sin y)^2 = \sin^2 x \cos y + 2 \sin x \cos y \cos x \sin y + \cos^2 x \sin^2 y.$$

Adding the right-hand members of the last two expressions, we have

$$\cos^2 y (\cos^2 x + \sin^2 x) + \sin^2 y (\sin^2 x + \cos^2 x)$$
$$= \cos^2 y + \sin^2 y = 1.$$

Complicated expressions, involving a single angle, may be most easily reduced by putting for each function its value in terms of the sides a, b, h of a right triangle. The resulting expression may then be reduced like any other algebraic expression. Of course, the relation $h^2 = a^2 + b^2$ may be made use of whenever an advantage is gained by it.

EXAMPLE 4. Reduce the expression

$$\sin^2 A \tan A + \cos^2 A \cot A + 2 \sin A \cos A.$$

Solution. Putting

$$\sin A = \frac{a}{h}, \quad \cos A = \frac{b}{h}, \quad \tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a},$$

we have

$$\frac{a^2}{h^2}\frac{a}{b} + \frac{b^2}{h^2}\frac{b}{a} + 2\frac{a}{h}\frac{b}{h} = \frac{a^4 + b^4 + 2a^2b^2}{abh^2} = \frac{(a^2+b^2)^2}{abh^2} = \frac{a^2+b^2}{ab} = \frac{a}{b} + \frac{b}{a}$$
$$= \tan A + \cot A.$$

The preceding methods are perfectly general, but frequently it is of advantage to use other expedients. Any one of the seven relations in Art. 12 may be employed in the reduction. Sometimes the denominator may be simplified or removed by multiplying both terms of the fraction by a binominal factor like $1 - \cos A$, $1 + \sin A$, $\sec A - \tan A$, $\csc A + \cot A$; in short, as in algebra, the form of the expression may be changed by any operation which does not change the value of the result. Radical expressions should be avoided whenever possible.

EXAMPLE 5. Reduce the expression

$$\frac{\sin^2 x}{1 - \cos x}.$$

Solution.
$$\frac{\sin^2 x}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} = \frac{\sin^2 x\,(1 + \cos x)}{1 - \cos^2 x}$$
$$= 1 + \cos x, \text{ since } 1 - \cos^2 x = \sin^2 x.$$

EXAMPLE 6. Reduce $\dfrac{\cos\theta}{\sec\theta + \tan\theta}$.*

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* Greek letters are frequently used to represent angles. For the benefit of those students who do not already know the Greek letters and their names, the Greek alphabet has been printed in the front part of this book. The Greek letters are written as they are printed.

Solution. Multiplying both terms of the fraction by $\sec\theta - \tan\theta$, we have

$$\frac{\cos\theta}{\sec\theta+\tan\theta} = \frac{\cos\theta}{(\sec\theta+\tan\theta)}\cdot\frac{(\sec\theta-\tan\theta)}{(\sec\theta-\tan\theta)} = \frac{\cos\theta\,(\sec\theta-\tan\theta)}{\sec^2\theta-\tan^2\theta}$$

$$= \cos\theta\,(\sec\theta-\tan\theta), \quad \text{since } \sec^2\theta-\tan^2\theta = 1$$

$$= 1-\sin\theta, \quad \text{since } \cos\theta\cdot\sec\theta = 1,\ \cos\theta\cdot\tan\theta = \sin\theta.$$

Exercise 9

Reduce the following expressions,—

1. $\dfrac{\cos A}{\sin A}\cdot\dfrac{1}{\cot^2 A}$. *Ans.* $\tan A$.
2. $\dfrac{\sin A}{\csc A}+\dfrac{\cos A}{\sec A}$. *Ans.* 1.
3. $\tan^2 x\csc^2 x - 1$. *Ans.* $\tan^2 x$.
4. $\dfrac{\cos y}{1-\tan y}-\dfrac{\sin y}{\cot y - 1}$. *Ans.* $\sin y + \cos y$.
5. $(\tan\theta+\cot\theta)\sin\theta\cos\theta$. *Ans.* 1.
6. $\cos x\tan x + \sin x\cot x$. *Ans.* $\sin x + \cos x$.
7. $\sec\alpha - \tan\alpha\sin\alpha$. *Ans.* $\cos\alpha$.
8. $\cot\theta + \dfrac{\sin\theta}{1+\cos\theta}$. *Ans.* $\csc\theta$.
9. $\sin^4 B + \cos^4 B + 2\cos^2 B\sin^2 B$. *Ans.* 1.
10. $\dfrac{\cos^2 x}{1-\sin x}$. *Ans.* $1+\sin x$.
11. $\sin A(\sec A+\csc A) - \cos A(\sec A - \csc A)$. *Ans.* $\sec A\csc A$.
12. $\cot\beta - \sec\beta\csc\beta(1-2\sin^2\beta)$. *Ans.* $\tan\beta$.
13. $(1+\sin A)(\sec A - \tan A)$. *Ans.* $\cos A$.
14. $\dfrac{\tan A+\tan B}{\cot A+\cot B}$. *Ans.* $\tan A\tan B$.
15. $(\sin x+\cos x)^2 + (\sin x - \cos x)^2$. *Ans.* 2.
16. $2(\sin^6\theta+\cos^6\theta) - 3(\sin^4\theta+\cos^4\theta)+1$. *Ans.* 0.
17. $2\,\text{vers}\,A - \text{vers}^2 A$. *Ans.* $\sin^2 A$.

18. $\dfrac{\cos A \cot A - \sin A \tan A}{\csc A - \sec A}$. *Ans.* $1 + \sin A \cos A$.

19. $\dfrac{\sin B}{1 + \cos B} + \dfrac{1 + \cos B}{\sin B}$. *Ans.* $2 \csc B$.

20. $\csc^4 x\,(1 - \cos^4 x) - 2 \cot^2 x$. *Ans.* 1.

21. $(\cos x \cos y + \sin x \sin y)^2 + (\sin x \cos y - \cos x \sin y)^2$. *Ans.* 1.

22. $(x \cos \alpha - y \sin \alpha)^2 + (x \sin \alpha + y \cos \alpha)^2$. *Ans.* $x^2 + y^2$.

23. $\dfrac{\sec^2 A \sin^2 A - \csc^2 A + \csc^2 A \cos^2 A}{\sec^2 A \sin^2 A - \csc^2 A \cos^2 A}$. *Ans.* $\sin^2 A$.

24. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}}$. *Ans.* $\sec A - \tan A$.

15. Trigonometric Identities. Equations which express general relations, that is, equations which remain true no matter what values be given to the quantities which are considered variable, are called *identities*. $(x + 1)^2 = x^2 + 2x + 1$ is an identity, because it is true no matter what value be given to x. $x^2 - 5x + 6 = 0$ is not an identity, since it is not true unless x has the value 2 or 3. Similarly, $\sin^2 x + \cos^2 x = 1$ and $\tan A = \dfrac{\sin A}{\cos A}$ are identities, for they express general relations, relations which hold true no matter what x or A may be. On the other hand, $\sin^2 x - \frac{1}{2} \sin x = -\frac{1}{4}$ is not an identity, for it is true only when $\sin x = \frac{1}{2}$, that is, when $x = \sin^{-1} \frac{1}{2}$. To distinguish equations which are not identities from those which are, the former are sometimes called equations of condition, since they express conditions to which the variables are restricted and not general relations as do identities.

The fundamental trigonometric identities are given in Art. 12. All other trigonometric identities may be derived from these by properly combining them. To prove a given identity it is sufficient to reduce both sides to the same form. If no shorter way suggests itself, this may always be accomplished by reducing each side to its simplest form.

EXAMPLE 1. Prove the identity

$$(1 - \tan A)(1 - \cot A) + \sec A \csc A = 2.$$

Solution. Putting

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \csc A = \frac{1}{\sin A},$$

the left-hand member becomes

$$\left(1-\frac{\sin A}{\cos A}\right)\left(1-\frac{\cos A}{\sin A}\right)+\frac{1}{\cos A}\cdot\frac{1}{\sin A}$$

$$=\frac{(\cos A-\sin A)(\sin A-\cos A)+1}{\cos A\sin A}$$

$$=\frac{2\sin A\cos A-\sin^2 A-\cos^2 A+1}{\cos A\sin A}.$$

$$=\frac{2\sin A\cos A}{\cos A\sin A}=2.$$

If we had put $\tan A=\frac{a}{b}$, $\cot A=\frac{b}{a}$, $\sec A=\frac{h}{b}$, $\csc A=\frac{h}{a}$, the reduction would have been as follows,

$$\left(1-\frac{a}{b}\right)\left(1-\frac{b}{a}\right)+\frac{h}{b}\frac{h}{a}=\frac{(b-a)(a-b)+h^2}{ab}$$

$$=\frac{2ab-a^2-b^2+h^2}{ab}=\frac{2ab}{ab}=2.$$

EXAMPLE 2. Show that

$$\sin x\tan^2 x+\csc x\sec^2 x=2\tan x\sec x+\csc x-\sin x.$$

Solution. Reducing each member of the equation separately, we have

$$\sin x\tan^2 x+\csc x\sec^2 x=\sin x\frac{\sin^2 x}{\cos^2 x}+\frac{1}{\sin x}\frac{1}{\cos^2 x}=\frac{\sin^4 x+1}{\sin x\cos^2 x}.$$

Likewise the second member becomes

$$2\tan x\sec x+\csc x-\sin x=\frac{2\sin x}{\cos x}\cdot\frac{1}{\cos x}+\frac{1}{\sin x}-\sin x$$

$$=\frac{2\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x}{\sin x\cos^2 x}$$

$$=\frac{2\sin^2 x+(1-\sin^2 x)\cos^2 x}{\sin x\cos^2 x}$$

$$=\frac{2\sin^2 x+(1-\sin^2 x)(1-\sin^2 x)}{\sin x\cos^2 x}$$

$$=\frac{\sin^4 x+1}{\sin x\cos^2 x}.$$

Example 3. Is $\sec^2 A \csc^2 A = \tan^2 A + \cot^2 A + 2$?

Solution.

$$\begin{aligned}\sec^2 A \csc^2 A &= (1 + \tan^2 A)(1 + \cot^2 A)\\ &= 1 + \tan^2 A + \cot^2 A + \tan^2 A \cot^2 A\\ &= 1 + \tan^2 A + \cot^2 A + 1, \text{ since } \tan A \cot A = 1.\\ &= \tan^2 A + \cot^2 A + 2.\end{aligned}$$

Hence the identity is true.

Exercise 10

Prove the following identities:

1. $\sin^2 A \sec^2 A = \sec^2 A - 1$.
2. $(\sin B + \cos B)^2 = 1 + 2 \sin B \cos B$.
3. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x = 1 - 2 \cos^2 x = 2 \sin^2 x - 1$.
4. $(\sin^2 \theta - \cos^2 \theta)^2 = 1 - 4 \sin^2 \theta \cos^2 \theta$.
5. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$.
6. $\sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x)$.
7. $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \csc A$.
8. $\dfrac{\sec\theta + \csc\theta}{\sec\theta - \csc\theta} = \dfrac{\tan\theta + 1}{\tan\theta - 1} = \dfrac{1 + \cot\theta}{1 - \cot\theta}$.
9. $\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A$.
10. $\tan\alpha + \tan\beta = \tan\alpha \tan\beta (\cot\alpha + \cot\beta)$.
11. $(1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x)$.
12. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \tan A + \sec A$.
13. $\cos^4 x - \sin^4 x = \cos^2 x (1 - \tan x)(1 + \tan x)$.
14. $\cos^2 x + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y = 1$.
15. $(\cos\alpha \cos\beta + \sin\alpha \sin\beta \cos\gamma)^2 + (\sin\alpha \cos\beta - \cos\alpha \sin\beta \cos\gamma)^2$
   $= 1 - \sin^2\beta \sin^2\gamma$.

	$\sin A$	$\cos A$	$\tan A$	$\csc A$	$\sec A$	$\cot A$
$\sin A =$		$\sqrt{1 - \cos^2 A}$	$\frac{\tan A}{\sqrt{1 + \tan^2 A}}$	$\frac{1}{\csc A}$	$\frac{\sqrt{\sec^2 A - 1}}{\sec A}$	$\frac{1}{\sqrt{1 + \cot^2 A}}$
$\cos A =$	$\sqrt{1 - \sin^2 A}$		$\frac{1}{\sqrt{1 + \tan^2 A}}$	$\frac{\sqrt{\csc^2 A - 1}}{\csc A}$	$\frac{1}{\sec A}$	$\frac{\cot A}{\sqrt{1 + \cot^2 A}}$
$\tan A =$	$\frac{\sin A}{\sqrt{1 - \sin^2 A}}$	$\frac{\sqrt{1 - \cos^2 A}}{\cos A}$		$\frac{1}{\sqrt{\csc A^2 - 1}}$	$\sqrt{\sec^2 A - 1}$	$\frac{1}{\cot A}$
$\csc A =$	$\frac{1}{\sin A}$	$\frac{1}{\sqrt{1 - \cos^2 A}}$	$\frac{\sqrt{1 + \tan^2 A}}{\tan A}$		$\frac{\sec A}{\sqrt{\sec^2 A - 1}}$	$\sqrt{1 + \cot^2 A}$
$\sec A =$	$\frac{1}{\sqrt{1 - \sin^2 A}}$	$\frac{1}{\cos A}$	$\sqrt{1 + \tan^2 A}$	$\frac{\csc A}{\sqrt{\csc^2 A - 1}}$		$\frac{\sqrt{1 + \cot^2 A}}{\cot A}$
$\cot A =$	$\frac{\sqrt{1 - \sin^2 A}}{\sin A}$	$\frac{\cos A}{\sqrt{1 - \cos^2 A}}$	$\frac{1}{\tan A}$	$\sqrt{\csc^2 A - 1}$	$\frac{1}{\sqrt{\sec^2 A - 1}}$	

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

16. Tables of Natural Functions. In the preceding chapter we computed the functions of 30°, 45° and 60°. Later we shall learn how to compute the functions of any given angle. Now we cannot remember the values of the functions for all angles, and the computation by which the values are determined is too laborious to be repeated each time the value of a particular function is needed. For this reason the values of the functions for every degree, minute and second from 0° to 90° have been computed once for all and the results tabulated in tables, known as *tables of natural functions*. Usually such tables contain the sines, cosines, tangents and cotangents of angles differing by 1 minute; in some tables the angles differ by 10 seconds, and in still others by only 1 second. From such tables, the value of the sine, cosine, tangent and cotangent may be found whenever needed, and conversely, when the value of a function is known, the corresponding angle may be found by use of the tables. Secants and cosecants are not given directly by the tables, but may be found indirectly from the cosines and sines respectively, since $\sec A = \frac{1}{\cos A}$, and $\csc A = \frac{1}{\sin A}$.

17. To Find the Natural Functions of an Angle Less than 90°.

a. When the angle is less than 45°, the degrees are found at the head of the column and the minutes in the left-hand column. The number, in the same horizontal line as the minutes and in the same column as the degrees, is that function of the angle whose name appears at the head of the column.

Example 1. To find the sine of 26° 31′.

Solution. We find the column in the table of natural sines and cosines (specimen page, p. 40) which is headed by 26° and follow down the column marked sine *on top* until we come to the number 4465 which is in the line beginning with 31′. The number 4465 considered a decimal is the required sine, that is,

$$\sin 26^\circ 31' = 0.4465.$$

b. When the angle is greater than 45°, the degrees are found at the foot of the column and the minutes in the right-hand column. The number in the same horizontal line as the minutes in the right-hand column and in the same column as the degrees at the bottom, is that function of the angle whose name appears at the foot of the column.

EXAMPLE 2. To find $\cos 63° 29'$.

Solution. We find that column in the table of natural sines and cosines (see specimen page) which has 63° at its foot, and *follow up* the column marked cos *at the bottom* until we come to the number 4465 which is in the horizontal line ending with 29′. The number 4465 considered a decimal is the required cosine, that is,

$$\cos 63° 29' = 0.4465.$$

c. When the angle consists of degrees, minutes and seconds, the result obtained for the given degrees and minutes must be corrected for the additional seconds.

EXAMPLE 3. To find $\sin 28° 46' 36''$.

Solution. $28° 46' 36'' = 28° 46.6'$.

From the table we find

$$\begin{array}{r} \sin 28° 46' = 0.4812 \\ \underline{\sin 28° 47' = 0.4815} \\ \text{difference for } 1' = 0.0003 \end{array}$$

The angle $28° 46.6'$ whose sine we seek is $\frac{6}{10}$ the way between the two angles 28° 46′ and 28° 47′, hence we take for its sine the sine of 28° 46′ increased by $\frac{6}{10}$ of the difference between $\sin 28° 46'$ and $\sin 28° 47'$. $\frac{6}{10}$ of $0.0003 = 0.00018$, or 0.0002 if we carry 4 decimal places only. Hence

$$\sin 28° 46' 36'' = 0.4812 + 0.0002 = 0.4814.$$

EXAMPLE 4. To find $\cos 61° 13' 24''$.

Solution. $61° 13' 24'' = 61° 13.4'$.

From the table we find

$$\begin{array}{r} \cos 61° 13' = 0.4815 \\ \underline{\cos 61° 14' = 0.4812} \\ \text{difference for } 1' = 0.0003 \end{array}$$

$\frac{4}{10}$ of $0.0003 = 0.00012$, or 0.0001 if we carry four places only, and since the cosine of 61° 13.4′ must be between $\cos 61° 13'$ and $\cos 61° 14'$,

$$\cos 61° 13' 24'' = 0.4815 - 0.0001 = 0.4814.$$

Observe that in Example 3 the difference was *added to the sine* of the smaller angle, while in Example 4 the difference was *subtracted from the cosine* of the smaller angle. This is because as the angle increases the *sine increases* while the *cosine decreases*. For the same reason the difference must be *added to the tangent* and *subtracted from the cotangent* of the smaller angle.

The *tangent or cotangent of an angle* is found from the table of natural tangents and cotangents in exactly the same way that the sine or cosine is found from the table of natural sines and cosines.

Example 5. To find $\tan 63° 16' 32''$.

Solution. $63° 16' 32'' = 63° 16.5\frac{1}{3}'$.

From the table of natural tangents and cotangents (see specimen page, p. 41) we find

$$\begin{array}{r}
\tan 63° 16' = 1.9854 \\
\underline{\tan 63° 17' = 1.9868} \\
\text{difference for } 1' = 0.0014
\end{array}$$

$5\frac{1}{3}$ times $0.0014 = 0.000747$, or 0.0007 to four places. Hence

$$\tan 63° 16' 32'' = 1.9854 + 0.0007 = 1.9861.$$

Example 6. To find $\cot 26° 43' 28''$.

Solution. $26° 43' 28'' = 26° 43.4\frac{2}{3}'$.

From the table (specimen page)

$$\begin{array}{r}
\cot 26° 43' = 1.9868 \\
\underline{\cot 26° 44' = 1.9854} \\
\text{difference for } 1' = 0.0014
\end{array}$$

$4\frac{2}{3}$ of $0.0014 = 0.00065$ or 0.0007 to four places. Hence

$$\cot 26° 43' 28'' = 1.9868 - 0.0007 = 1.9861.$$

The process of finding the value of a function of an angle intermediate to two consecutive angles whose functions are given directly in the table, is called *interpolation*. Thus in Example 4, $\cot 36° 43'$ and $\cot 36° 44'$ are given directly in the table, while $\cot 36° 43' 28''$, which is not given in the table, was found by interpolation.

In interpolating we assumed that the increase or decrease of the function is proportional to the increase of the angle, that is, we assumed what is known as the *principle of proportional parts*. Briefly stated it is this,—

For small changes in the angle the change in the function of an angle is nearly[*] *proportional to the change in the angle.*

EXAMPLE 7. One angle of a right triangle is 25° 48.5′ and the hypotenuse is 235.0. Find the remaining sides of the triangle.

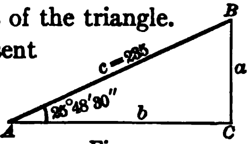

Fig. 22

Solution. In the adjacent figure, let A represent the given angle and c the hypotenuse.

To find a we have,

$$\frac{a}{c} = \sin A, \text{ or } a = c \sin A.$$

$$c = 235, \text{ and from the table we have } \sin A = 0.4354,$$

hence $$a = 235 \times 0.4354 = 102.3.$$

To find b, we have,

$$\frac{b}{c} = \cos A, \text{ or } b = c \cos A,$$

$$c = 235, \text{ and from the table we have } \cos A = 0.9003,$$

hence, $$b = 235 \times 0.9003 = 211.6.$$

To check our results, we use the relation $\frac{a}{b} = \tan A$, that is, if our results are correct, the quotient of a by b must agree with the value of $\tan A$ as found from the table.

$$\frac{a}{b} = \frac{102.3}{211.6} = 0.4835, \text{ while } \tan A \text{ as given in the table is } 0.4836.$$

The difference of 1 in the last decimal place arises from the neglected parts of the decimals. If we had carried out the work to five significant figures instead of four, the quotient of a divided by b would have been 0.4836.

EXERCISE 11

All the functions called for in this exercise are found on the specimen pages of natural functions on pp. 40, 41.

1. Find sin 26°, cos 27° 30′, tan 25° 45′, cot 29° 59′.
Ans. 0.4384, 0.8870, 0.4823, 1.7332.

2. Find cos 64°, sin 62° 30′, cot 64° 15′, tan 60° 01′.
Ans. 0.4384, 0.8870, 0.4823, 1.7332.

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[*] We say nearly, for no exact proportion exists, no matter how small the change in the angle. All we can say is, that when the change in the angle is small, the principle of proportional parts gives results which in most cases are sufficiently exact for practical purposes.

SPECIMEN PAGE

′	25°		26°		27°		28°		29°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	4226	9063	4384	8988	4540	8910	4695	8829	4848	8746	60
1	4229	9062	4386	8987	4542	8909	4697	8828	4851	8745	59
2	4231	9061	4389	8985	4545	8907	4700	8827	4853	8743	58
3	4234	9059	4392	8984	4548	8906	4702	8825	4856	8742	57
4	4237	9058	4394	8983	4550	8905	4705	8824	4858	8741	56
5	4239	9057	4397	8982	4553	8903	4708	8823	4861	8739	55
6	4242	9056	4399	8980	4555	8902	4710	8821	4863	8738	54
7	4245	9054	4402	8979	4558	8901	4713	8820	4866	8736	53
8	4247	9053	4405	8978	4561	8899	4715	8819	4868	8735	52
9	4250	9052	4407	8976	4563	8898	4718	8817	4871	8733	51
10	4253	9051	4410	8975	4566	8897	4720	8816	4874	8732	50
11	4255	9050	4412	8974	4568	8895	4723	8814	4876	8731	49
12	4258	9048	4415	8973	4571	8894	4726	8813	4879	8729	48
13	4260	9047	4418	8971	4574	8893	4728	8812	4881	8728	47
14	4263	9046	4420	8970	4576	8892	4731	8810	4884	8726	46
15	4266	9045	4423	8969	4579	8890	4733	8809	4886	8725	45
16	4268	9043	4425	8967	4581	8889	4736	8808	4889	8724	44
17	4271	9042	4428	8966	4584	8888	4738	8806	4891	8722	43
18	4274	9041	4431	8965	4586	8886	4741	8805	4894	8721	42
19	4276	9040	4433	8964	4589	8885	4743	8803	4896	8719	41
20	4279	9038	4436	8962	4592	8884	4746	8802	4899	8718	40
21	4281	9037	4439	8961	4594	8882	4749	8801	4901	8716	39
22	4284	9036	4441	8960	4597	8881	4751	8799	4904	8715	38
23	4287	9035	4444	8958	4599	8879	4754	8798	4907	8714	37
24	4289	9033	4446	8957	4602	8878	4756	8796	4909	8712	36
25	4292	9032	4449	8956	4605	8877	4759	8795	4912	8711	35
26	4295	9031	4452	8955	4607	8875	4761	8794	4914	8709	34
27	4297	9030	4454	8953	4610	8874	4764	8792	4917	8708	33
28	4300	9028	4457	8952	4612	8873	4766	8791	4919	8706	32
29	4302	9027	4459	8951	4615	8871	4769	8790	4922	8705	31
30	4305	9026	4462	8949	4617	8870	4772	8788	4924	8704	30
31	4308	9025	4465	8948	4620	8869	4774	8787	4927	8702	29
32	4310	9023	4467	8947	4623	8867	4777	8785	4929	8701	28
33	4313	9022	4470	8945	4625	8866	4779	8784	4932	8699	27
34	4316	9021	4472	8944	4628	8865	4782	8783	4934	8698	26
35	4318	9020	4475	8943	4630	8863	4784	8781	4937	8696	25
36	4321	9018	4478	8942	4633	8862	4787	8780	4939	8695	24
37	4323	9017	4480	8940	4636	8861	4789	8778	4942	8694	23
38	4326	9016	4483	8939	4638	8859	4792	8777	4944	8692	22
39	4329	9015	4485	8938	4641	8858	4795	8776	4947	8691	21
40	4331	9013	4488	8936	4643	8857	4797	8774	4950	8689	20
41	4334	9012	4491	8935	4646	8855	4800	8773	4952	8688	19
42	4337	9011	4493	8934	4648	8854	4802	8771	4955	8686	18
43	4339	9010	4496	8932	4651	8853	4805	8770	4957	8685	17
44	4342	9008	4498	8931	4654	8851	4857	8769	4960	8683	16
45	4344	9007	4501	8930	4656	8850	4810	8767	4962	8682	15
46	4347	9006	4504	8928	4659	8849	4812	8766	4965	8681	14
47	4350	9004	4506	8927	4661	8847	4815	8764	4967	8679	13
48	4352	9003	4509	8926	4664	8846	4818	8763	4970	8678	12
49	4355	9002	4511	8925	4666	8844	4820	8762	4972	8676	11
50	4358	9001	4514	8923	4669	8843	4823	8760	4975	8675	10
51	4360	8999	4517	8922	4672	8842	4825	8759	4977	8673	9
52	4363	8998	4519	8921	4674	8840	4828	8757	4980	8672	8
53	4365	8997	4522	8919	4677	8839	4830	8756	4982	8670	7
54	4368	8996	4524	8918	4679	8838	4833	8755	4985	8669	6
55	4371	8994	4527	8917	4682	8836	4835	8753	4987	8668	5
56	4373	8993	4530	8915	4684	8835	4838	8752	4990	8666	4
57	4376	8992	4532	8914	4687	8834	4840	8750	4992	8665	3
58	4378	8990	4535	8913	4690	8832	4843	8749	4995	8663	2
59	4381	8989	4537	8911	4692	8831	4846	8748	4997	8662	1
60	4384	8988	4540	8910	4695	8829	4848	8746	5000	8660	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	64°		63°		62°		61°		60°		′

SPECIMEN PAGE

′	25°		26°		27°		28°		29°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	4663	2.1445	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	60
1	4667	2.1429	4881	2.0488	5099	1.9612	5321	1.8794	5547	1.8028	59
2	4670	2.1413	4885	2.0473	5103	1.9598	5325	1.8781	5551	1.8016	58
3	4674	2.1396	4888	2.0458	5106	1.9584	5328	1.8768	5555	1.8003	57
4	4677	2.1380	4892	2.0443	5110	1.9570	5332	1.8755	5558	1.7991	56
5	4681	2.1364	4895	2.0428	5114	1.9556	5336	1.8741	5562	1.7979	55
6	4684	2.1348	4899	2.0413	5117	1.9542	5340	1.8728	5566	1.7966	54
7	4688	2.1332	4903	2.0398	5121	1.9528	5343	1.8715	5570	1.7954	53
8	4691	2.1315	4906	2.0383	5125	1.9514	5347	1.8702	5574	1.7942	52
9	4695	2.1299	4910	2.0368	5128	1.9500	5351	1.8689	5577	1.7930	51
10	4699	2.1283	4913	2.0353	5132	1.9486	5354	1.8676	5581	1.7917	50
11	4702	2.1267	4917	2.0338	5136	1.9472	5358	1.8663	5585	1.7905	49
12	4706	2.1251	4921	2.0323	5139	1.9458	5362	1.8650	5589	1.7893	48
13	4709	2.1235	4924	2.0308	5143	1.9444	5366	1.8637	5593	1.7881	47
14	4713	2.1219	4928	2.0293	5147	1.9430	5369	1.8624	5596	1.7868	46
15	4716	2.1203	4931	2.0278	5150	1.9416	5373	1.8611	5600	1.7856	45
16	4720	2.1187	4935	2.0263	5154	1.9402	5377	1.8598	5604	1.7844	44
17	4723	2.1171	4939	2.0248	5158	1.9388	5381	1.8585	5608	1.7832	43
18	4727	2.1155	4942	2.0233	5161	1.9375	5384	1.8572	5612	1.7820	42
19	4731	2.1139	4946	2.0219	5165	1.9361	5388	1.8559	5616	1.7808	41
20	4734	2.1123	4950	2.0204	5169	1.9347	5392	1.8546	5619	1.7796	40
21	4738	2.1107	4953	2.0189	5172	1.9333	5396	1.8533	5623	1.7783	39
22	4741	2.1092	4957	2.0174	5176	1.9319	5399	1.8520	5627	1.7771	38
23	4745	2.1076	4960	2.0160	5180	1.9306	5403	1.8507	5631	1.7759	37
24	4748	2.1060	4964	2.0145	5184	1.9292	5407	1.8495	5635	1.7747	36
25	4752	2.1044	4968	2.0130	5187	1.9278	5411	1.8482	5639	1.7735	35
26	4755	2.1028	4971	2.0115	5191	1.9265	5415	1.8469	5642	1.7723	34
27	4759	2.1013	4975	2.0101	5195	1.9251	5418	1.8456	5646	1.7711	33
28	4763	2.0997	4979	2.0086	5198	1.9237	5422	1.8443	5650	1.7699	32
29	4766	2.0981	4982	2.0072	5202	1.9223	5426	1.8430	5654	1.7687	31
30	4770	2.0965	4986	2.0057	5206	1.9210	5430	1.8418	5658	1.7675	30
31	4773	2.0950	4989	2.0042	5209	1.9196	5433	1.8405	5662	1.7663	29
32	4777	2.0934	4993	2.0028	5213	1.9183	5437	1.8392	5665	1.7651	28
33	4780	2.0918	4997	2.0013	5217	1.9169	5441	1.8379	5669	1.7639	27
34	4784	2.0903	5000	1.9999	5220	1.9155	5445	1.8367	5673	1.7627	26
35	4788	2.0887	5004	1.9984	5224	1.9142	5448	1.8354	5677	1.7615	25
36	4791	2.0872	5008	1.9970	5228	1.9128	5452	1.8341	5681	1.7603	24
37	4795	2.0856	5011	1.9955	5232	1.9115	5456	1.8329	5685	1.7591	23
38	4798	2.0840	5015	1.9941	5235	1.9101	5460	1.8316	5688	1.7579	22
39	4802	2.0825	5019	1.9926	5239	1.9088	5464	1.8303	5692	1.7567	21
40	4806	2.0809	5022	1.9912	5243	1.9074	5467	1.8291	5696	1.7556	20
41	4809	2.0794	5026	1.9897	5246	1.9061	5471	1.8278	5700	1.7544	19
42	4813	2.0778	5029	1.9883	5250	1.9047	5475	1.8265	5704	1.7532	18
43	4816	2.0763	5033	1.9868	5254	1.9034	5479	1.8253	5708	1.7520	17
44	4820	2.0748	5037	1.9854	5258	1.9020	5482	1.8240	5712	1.7508	16
45	4823	2.0732	5040	1.9840	5261	1.9007	5486	1.8228	5715	1.7496	15
46	4827	2.0717	5044	1.9825	5265	1.8993	5490	1.8215	5719	1.7485	14
47	4831	2.0701	5048	1.9811	5269	1.8980	5494	1.8202	5723	1.7473	13
48	4834	2.0686	5051	1.9797	5272	1.8967	5498	1.8190	5727	1.7461	12
49	4838	2.0671	5055	1.9782	5276	1.8953	5501	1.8177	5731	1.7449	11
50	4841	2.0655	5059	1.9768	5280	1.8940	5505	1.8165	5735	1.7437	10
51	4845	2.0640	5062	1.9754	5284	1.8927	5509	1.8152	5739	1.7426	9
52	4849	2.0625	5066	1.9740	5287	1.8913	5513	1.8140	5743	1.7414	8
53	4852	2.0609	5070	1.9725	5291	1.8900	5517	1.8127	5746	1.7402	7
54	4856	2.0594	5073	1.9711	5295	1.8887	5520	1.8115	5750	1.7391	6
55	4859	2.0579	5077	1.9697	5298	1.8873	5524	1.8103	5754	1.7379	5
56	4863	2.0564	5081	1.9683	5302	1.8860	5528	1.8090	5758	1.7367	4
57	4867	2.0549	5084	1.9669	5306	1.8847	5532	1.8078	5762	1.7355	3
58	4870	2.0533	5088	1.9654	5310	1.8834	5535	1.8065	5766	1.7344	2
59	4874	2.0518	5092	1.9640	5313	1.8820	5539	1.8053	5770	1.7332	1
60	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	5774	1.7321	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	64°		63°		62°		61°		60°		′

3. Find sin $28°\,30'\,24''$, tan $61°\,24'\,36''$, cos $60°\,30'\,25''$.
Ans. 0.4773, 1.8349, 0.4923.

4. Find cos $25°\,50'\,10''$, cot $28°\,25'\,50''$, tan $27°\,00'\,30''$.
Ans. 0.9001, 1.8471, 0.5097.

5. Find sec $25°\,35'$, csc $25°\,28'$. *Ans.* 1.1087, 2.3256.

Solve the following right triangles:

6. Given an acute angle $A = 26°\,15'$ and the hypotenuse $c = 35.0$; to find the remaining parts.
Ans. $B = 63°\,45'$, $a = 15.5$, $b = 31.4$.

7. Given $A = 28°\,50'$ and the side opposite, $a = 150$.
Ans. $B = 61°\,10'$, $b = 272$, $c = 311$.

8. Given $A = 63°\,40'\,30''$ and the side adjacent. $b = 363$. Find a and c and check your results.

18. To Find the Angle Less than 90° Corresponding to a Given Natural Function.

a. When the function is one of the numbers given in the table, the angle is found by taking the number of degrees which stands at the head or foot of the column according as the name of the function appears at the head or the foot of the column in which the number is found, and the number of minutes at the left or right end of the line in which the number is found according as the degrees have been taken from the top or the bottom of the column.

EXAMPLE 1. Find the angle whose sine is 0.4633.

Solution. We find the number 4633 in the column which has 27° written at its head and 62° at its foot, and in the line which begins with 36′ and ends with 24′ (see specimen page, p. 40). Since the given number is a sine, and the name "sin" appears at the head of the column in which 4633 is found, we take the degrees from the top of the column and the minutes from the left of the line in which 4633 is found, that is,

the angle whose sine is 0.4633, or $\sin^{-1} 0.4633 = 27°\,36'$.

EXAMPLE 2. Find the angle whose cosine is 0.4633.

Solution. This time the name of the function appears at the bottom of the column in which 4633 is found, hence

the angle whose cosine is 0.4633, or $\cos^{-1} 0.4633 = 62°\,24'$.

b. When the function is not given in the table, we find the corresponding angle by reversing the process by means of which we find the function when the angle is given.

EXAMPLE 3. $\tan x = 0.5492$, to find x.

Solution. The number 5492 is not found in the table of natural tangents and cotangents, but two other numbers are found (see specimen page), namely, 5490 and 5494, one of which is a little smaller, the other a little larger than the given number.

$$0.5490 = \tan 28^\circ 46'$$
$$0.5494 = \tan 28^\circ 47'.$$

It is plain, therefore, that x, the angle whose tangent is 0.5492, is somewhere between 28° 46′ and 28° 47′. Applying the principle of proportional parts we have,

$$\frac{x - 28^\circ 46'}{28^\circ 47' - 28^\circ 46'} = \frac{\tan x - \tan 28^\circ 46'}{\tan 28^\circ 47' - \tan 28^\circ 46'}$$

that is,

$$\frac{x - 28^\circ 46'}{1' \text{ or } 60''} = \frac{0.5492 - 0.5490}{0.5494 - 0.5490} = \frac{0.0002}{0.0004} = \frac{2}{4} = \frac{1}{2},$$

and solving for x,

$$x = 28^\circ 46' + \tfrac{1}{2} \text{ of } 60'' = 28^\circ 46' 30''.$$

It is not necessary to go through all this work each time. All we need to remember is that the smaller of the two angles between which x lies must be increased by

$$\frac{d}{D} \text{ of } 60'',$$

where

D is the difference (without regard to the decimal point) between the functions of the two angles between which x lies,

d is the difference (without regard to the decimal point) between the function of the smaller angle and the given function.

EXAMPLE 4. $\tan x = 1.9887$, to find x.

Solution. From the table (specimen page)

$$1.9883 = \tan 63^\circ 18'$$
$$1.9897 = \tan 63^\circ 19'$$
$$D = 19897 - 19883 = 14,$$
$$d = 19887 - 19883 = 4,$$
$$x = 63^\circ 18' + \tfrac{4}{14} \text{ of } 60'' = 63^\circ 18' 17''.$$

EXAMPLE 5. $\cos x = 0.4767$, to find x.

Solution. From the table

$$
\begin{aligned}
0.4769 &= \cos 61^\circ 31' \\
0.4766 &= \cos 61^\circ 32' \\
D &= 4769 - 4766 = 3, \\
d &= 4769 - 4767 = 2, \\
x &= 61^\circ 31' + \tfrac{2}{3} \text{ of } 60'' = 61^\circ 31' 40''.
\end{aligned}
$$

19. Accuracy of Results. When an angle has been obtained from a four-place table (a table giving four places of decimals only), the number of seconds in the angle found cannot be relied upon with certainty. This is best shown by considering a special example, as Example 4 above.

Since the tangents are given to four places only, the actual value of $\tan 63^\circ 18'$ is not necessarily 1.9883, but may have any value between 1.98825 and 1.98835.

Similarly the actual value of $\tan 63^\circ 19'$ may be any number between 1.98965 and 1.98975.

Hence D, the difference between $\tan 63^\circ 19'$ and $\tan 63^\circ 18'$, is not necessarily 0.0014, but may be any number

$$
\begin{aligned}
&\text{less than } 1.98975 - 1.98825 = 0.00150, \\
&\text{and greater than } 1.98965 - 1.98835 = 0.00130.
\end{aligned}
$$

Again, $\tan x$, four places only being given, may have any value between 1.98865 and 1.98875, so that d, the difference between $\tan x$ and $\tan 63^\circ 18'$, may be any number

$$
\begin{aligned}
&\text{less than } 1.98875 - 1.98825 = 0.00050, \\
&\text{and greater than } 1.98865 - 1.98835 = 0.00030.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{d}{D} \text{ of } 60'' \text{ may be any number less than } &\frac{50}{130} \text{ of } 60'' = 23'', \\
\text{and greater than } &\frac{30}{150} \text{ of } 60'' = 12''.
\end{aligned}
$$

The conclusion is that the result $63^\circ 18' 17''$ previously given is uncertain by as much as $6''$.

From the example just given, it appears that the amount by which the result obtained from a table is uncertain depends upon the difference D, which varies not only for different functions of the same angle, but also for the same function of different angles. No general rule can be laid down to cover the amount of uncertainty in all cases. If absolute certainty in the number of seconds is required, a seven-place table should be used, giving the values of the functions from second to second for small angles, and for intervals of 10″ for larger angles.

When a four-place table is used, and no special consideration is given to the nature of the differences involved, the number of seconds of an angle obtained by interpolation cannot be relied upon.

The following rules will be of some aid to beginners:

1. *An angle less than 45° can be obtained more accurately from a sine than from a cosine, while an angle greater than 45° can be obtained more accurately from a cosine than from a sine.*

This is because the sines of angles less than 45° vary more than the cosines, and at the same time the principle of proportional parts is more nearly true for sines of small angles than for cosines, while the opposite is true for angles greater than 45°.

2. *Very small angles can be obtained with greater accuracy by interpolation from tangents than from cotangents, while angles near 90° can be obtained with greater accuracy from cotangents than from tangents.*

The reason for this lies in the fact that the principle of proportional parts ceases to apply to cotangents of small angles and to tangents of angles near 90°.

EXERCISE 12

Find the following angles:

1. $\sin^{-1} 0.4904$, $\cos^{-1} 0.4904$, $\tan^{-1} 1.8940$, $\cot^{-1} 1.8940$.
   Ans. $29° 22'$, $60° 38'$, $62° 10'$, $27° 50'$.
2. $\sin^{-1} 0.4267$, $\cos^{-1} 0.4900$, $\tan^{-1} 2.1036$, $\cot^{-1} 0.5644$.
   Ans. $25° 15' 30''$, $60° 39' 30''$, $64° 34' 30''$, $60° 33' 30''$.
3. $\cot^{-1} 2.1441$, $\tan^{-1} 0.4737$, $\tan^{-1} 1.7611$, $\cot^{-1} 1.7611$.
   Ans. $25° 00' 15''$, $25° 20' 45''$, $60° 24' 40''$, $29° 35' 20''$.
4. Show that $\sin^{-1} 0.4250 + \sin^{-1} 0.9052 = 90°$.
5. Show that $\sin^{-1} 0.4488 + \sin^{-1} 0.4746 = 55°$.

6. If $\tan^{-1} 0.5000 + \tan^{-1} x = 87° 34'$, find x.

Ans. $x = 1.8040$.

7. In a right triangle one side $b = 4.63$ and the hypotenuse $c = 10.0$. Find the included angle A. *Ans.* $A = 62° 25'$.

(Suggestion. $\cos A = b/c$.)

8. In a right triangle the hypotenuse $c = 35.00$ and the side $a = 31.29$. Find the angle opposite a, and the third side b.

Ans. $A = 63° 23'$, $b = 15.68$.

9. Two sides of a right triangle are $a = 475.0$, $b = 237.5$. Find the angles A and B and the hypotenuse c.

Ans. $A = 63° 26'$, $B = 26° 34'$, $c = 531.1$.

10. Determine the uncertainty in the number of seconds of $\tan^{-1} 2.1211$ as given by a four-place table, assuming the principle of proportional parts. *Ans.* $\tan^{-1} 2.1211 = 64° 45' 30'' \pm 2''$.

20. Solution of Right Triangles by Natural Functions. In order to solve a right triangle, two parts must be given besides the right angle, and one of these parts must be a side.

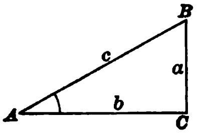

Fig. 23.

Let ABC be any right triangle, A, B, C the angles, and a, b, c the corresponding sides. Then

$$\frac{a}{c} = \sin A, \ (1), \qquad \frac{b}{c} = \cos A, \ (2), \qquad \frac{a}{b} = \tan A, \ (3).$$

These three equations are sufficient for the solution of any right triangle, for,—

If one of the given parts is an angle, we may call this angle A, and the other given part must be either a, b, or c;

I. Given A and a; then (3) gives b, and (1) gives c.
II. Given A and b; then (2) gives c, and (3) gives a.
III. Given A and c; then (1) gives a, and (2) gives b.

If the given parts are both sides, there are three more cases, —

IV. Given a and b; then (3) gives A, and c is found as in I or II.
V. Given b and c; then (2) gives A, and a is found as in II or III.
VI. Given c and a; then (1) gives A, and B is found as in III or I.

Since A and B are complementary, B may always be found from the relation

$$A + B = 90°.$$

It is not necessary to consider the various cases of right triangles separately, for the following simple rule governs all cases:

Employ that trigonometric function of the angle which involves the two sides under consideration. The two sides may both be given, or one may be given and the other required to be found.

EXAMPLE 1. Given one side of a right triangle equal to 418.5, and the hypotenuse equal to 614.0; required the other parts.

Solution. Denote the given side by a or b, let us say by b, and the hypotenuse by c. We then have

Given $b = 418.5$,
$c = 614.0$.

Required $A = 47° 02'$,
$B = 42° 58'$,
$a = 449.3$.

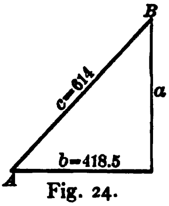

Fig. 24.

1. To find A.

We have
$$\cos A = \frac{b}{c} = \frac{418.5}{614.0} = 0.6816,$$
and from the table
$$A = \cos^{-1} 0.6816 = 47° 02'.$$

2. To find B.
$$A + B = 90°, \text{ hence } B = 90° - 47° 02' = 42° 58'.$$

3. To find a.

From (3),
$$a = b \tan A.$$
From the table
$$\tan A = \tan 47° 02' = \quad 1.0736$$
$$b = \quad \underline{418.5}$$
Multiplying,
$$a = 449.3$$

4. Check. To guard against possible mistakes, the answers should be tested *by some formula which has not already been used* in obtaining the answers. Now in solving a right triangle we never use more than two of the three formulas (1), (2), (3), hence the third may always be used as a check. In the present problem, (2) was used in finding A, and (3) in finding a, hence we may test our results by using (1), that is, if our results are correct they should satisfy the relation (1), or
$$a = c \sin A$$
$$\sin A = \sin 47° 02' = \quad 0.7318$$
$$c = \quad \underline{614}$$
Multiplying, we get
$$a = 449.3,$$
which agrees with the value of a as determined above.

NOTE. We might have used the relation $a^2 + b^2 = c^2$ as a check, but this would have required more work.

Example 2. Given one angle of a right triangle equal to $36^\circ 50'$, and the hypotenuse equal to 3.12; to solve the triangle.

Solution. Denote the angle by A, and the hypotenuse by c, then we have, —

Given $A = 36^\circ 50'$,
$c = 3.12.$

Required $a = 1.87$,
$b = 2.50$,
$B = 53^\circ 10'$.

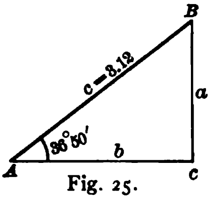

Fig. 25.

1. To find a.

From (1), $a = c \sin A.$

From the table, $\sin A = \sin 36^\circ 50' = 0.5995$
$c = 3.12$

Multiplying, $a = 1.870$

2. To find b.

From (2), $b = c \cos A.$

From the tables, $\cos A = \cos 36^\circ 50' = 0.8004$
$c = 3.12$

Multiplying, $b = 2.497$

3. To find B.

$A + B = 90^\circ$, hence $B = 90^\circ - 36^\circ 50' = 53^\circ 10'$.

4. Check. Having already used (1) and (2), we use (3).

$a = b \tan A.$

From the tables, $\tan A = \tan 36^\circ 50' = 0.7490$
$b = 2.497$

Multiplying, we get $a = 1.870$,
which agrees with the value of a as obtained above.

Instead of referring to equations (1), (2), and (3), it is better to write down from the triangle under consideration that ratio which is needed to solve for a particular side or angle. The method will be sufficiently clear from an example.

Example 3. Given $\phi = 15^\circ 25'$,
$m = 345.$

Required $\theta = 74^\circ 35'$,
$n = 1251$,
$p = 1298$.

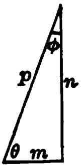

Fig. 26.

Solution.

1. To find θ. $\theta + \phi = 90^\circ$,
$\theta = 90^\circ - 15^\circ 25' = 74^\circ 35'$.

2. To find n. Since ϕ and m are given, we must use that function of ϕ which involves n and the given side m.

$$\frac{m}{n} = \tan \phi, \quad \text{or} \quad n = m \cot \phi = 345 \times 3.6264 = 1251.1.$$

3. To find p. We use that function of ϕ which involves p and the given side m.

$$\frac{m}{p} = \sin\phi, \quad \text{or} \quad p = \frac{m}{\sin\phi} = \frac{345}{0.2658} = 1298.$$

4. Check. If our results for θ, n, and p are correct, we should have

$$\frac{n}{p} = \sin\theta, \quad \text{or} \quad n = p\sin\theta = 1298 \times 0.9640 = 1251.3.$$

In this case there is a slight discrepancy between the result of the check and the value of n as found in 2. This is due to the fact that we have given the value of n to five places while the table gives but four places. The values to four places agree exactly.

EXERCISE 13

Solve the following right triangles. Check your results when the answers are not given.

1. Given $a = 10.00$, $A = 25°$; find $b = 21.45$, $c = 23.66$.
2. Given $b = 256$, $A = 36° 30'$; find a, B, and c.
3. Given $c = 350$, $A = 56° 45'$; find $B = 33° 15'$, $a = 293$, $b = 192$.
4. Given $a = 346$, $B = 50°$; find the other parts.
5. Given $c = 45.7$, $B = 44° 50'$; find $a = 32.4$, $b = 32.2$.
6. Given $b = 13.5$, $B = 28° 40'$; solve the triangle.
7. Given $a = 170$, $b = 350$; find $A = 25° 54'$, $B = 64° 06'$, $c = 389$.
8. Given $a = 0.81$, $c = 2.54$; solve the triangle.
9. Given $b = 6.57$, $c = 10.6$; find $A = 51° 42'$, $B = 38° 18'$, $a = 8.32$.

10. Find the altitude of an isosceles triangle whose base is 368, and whose equal sides make an angle of 64°. *Ans.* 294.

11. Find the perimeter and area of a regular pentagon inscribed in a circle whose radius is 10. *Ans.* 58.78, 237.76.

12. Show that the area of any right triangle is equal to either one of the expressions $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$.

13. In the right triangle ABC, Fig. 27, $AB = 338$, angle $B = 40°$; show how to find the length of the median AM, the length AS of the bisector of the angle A, and the angle included between these two.

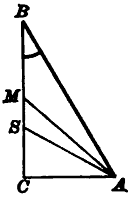

Fig. 27.

14. An oblique triangle ABC has $AB = 120$, $BC = 150$, and the angle $B = 67°30'$; solve the triangle.

Ans. $AC = 152$, $A = 65°41'$, $C = 46°49'$.

(Suggestion. Drop a perpendicular from A or C to the opposite side, dividing the triangle into two right triangles.)

15. Given one side of an oblique triangle equal to 57.3, and the adjacent angles equal to $35°45'$ and $75°30'$ respectively; find the remaining parts. *Ans.* 59.5, 35.9, $68°45'$.

(Suggestion. Draw the altitude from a vertex adjacent to the given side.)

21. Right Triangles Having a Small Angle.

Given the hypotenuse c and a side b of a right triangle, to solve the triangle in case b and c are nearly equal.

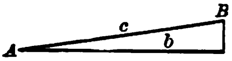

Fig. 28.

The angles A and B are given by the relation

$$\cos A = \frac{b}{c} = \sin B.$$

It is apparent from the figure that if b is nearly equal to c, angle A must be very small and angle B must be nearly equal to 90°.

By examining the table of natural sines and cosines it will be seen that for small angles the cosines are so nearly equal that there is no difference at all in the first four places, and similarly the sines of angles near 90° are nearly equal. Thus, so far as the table shows, all angles from 0° to 0° 34′ have the same cosine, and likewise all angles between 0° 34′ and 0° 60′, between 1° 00′ and 1° 16′, etc. It follows that *a small angle cannot be accurately found from its cosine nor an angle near 90° from its sine.*

To avoid using the cosine, the following formula is used whenever the given parts b and c are nearly equal, —

$$\tan \frac{A}{2} = \sqrt{\frac{c-b}{c+b}}.$$

Proof. Let ABC be any right triangle, b the base, a the altitude, c the hypotenuse. Produce CA to O, making $AO = c$. Join O and B. Now angle A $(= CAB)$ = angle AOB + angle ABO, hence since triangle AOB is isosceles, angle A = twice the angle AOB, that is, angle $AOB = \frac{A}{2}$, and

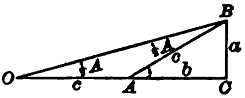

Fig. 29.

$$\tan \frac{A}{2} = \frac{a}{c+b} = \frac{\sqrt{c^2-b^2}}{c+b} = \sqrt{\frac{c^2-b^2}{(c+b)^2}} = \sqrt{\frac{c-b}{c+b}}.$$

EXAMPLE 1. Given $b = 25.7$, $c = 26.8$; to find A, B, and a.

Solution.

$$c - b = 26.8 - 25.7 = 1.1,$$

$$c + b = 26.8 + 25.7 = 52.5.$$

$$\tan\frac{A}{2} = \sqrt{\frac{1.1}{52.5}} = \sqrt{0.020952} = 0.1447,$$

$$\frac{A}{2} = 8^\circ\,14', \quad A = 16^\circ\,28',$$

$$B = 90^\circ - 16^\circ\,28' = 73^\circ\,32'.$$

$$a = b\tan A = 25.7 \times 0.2956 = 7.6.$$

Check.

$$a^2 = c^2 - b^2 = (c - b)(c + b).$$

$$a^2 = (7.6)^2 = 57.76,$$

$$(c - b)(c + b) = 1.1 \times 52.5 = 57.75.$$

EXERCISE 14

1. Given $b = 4.75$, $c = 5.25$; solve the triangle.

Ans. $A = 25^\circ\,13'$, $B = 64^\circ\,47'$, $a = 2.24$.

2. Given $a = 9.6$, $c = 10.4$; solve the triangle and check your results.

3. Show that $\tan\frac{A}{2} = \frac{c - b}{a}$.

4. From Fig. 29 show that $\sin\frac{A}{2} = \sqrt{\frac{c - b}{2c}}$, $\cos\frac{A}{2} = \sqrt{\frac{c + b}{2c}}$.

22. Historical Note. The use of tables of natural functions dates back to antiquity. In the second century B.C., Hipparchus, a Greek astronomer and mathematician, constructed tables of chords (double sines) which answered the same purpose as a table of sines. The tables of Hipparchus have been lost. The oldest table now extant is that of Ptolemaus (second century A.D.), giving double sines from minute to minute with an accuracy which in our system of numeration would be expressed by five places of decimals.

Hindu mathematicians as early as the fifth century A.D. were in possession of a small table which they memorized, very much as we memorize our multiplication table. Values not given in the table were computed from memory as occasion required, by means of a formula put in verse.

The first table which approached in extent and arrangement the tables now in use is the "Canon doctrinæ triangulorum" of Rheticus (1551). This table gives each of the six functions for intervals of 10″ from 0 to 45. Like present tables, the degrees and seconds proceed from top to bottom in the left marginal columns, while the complementary angles proceed from bottom to top in the right marginal columns. Later, Rheticus prepared a second table for which the sines of angles for intervals of 10″ were computed to 15 places of decimals, though only 10 places were retained in the "Opus Palatinum," the name under which the table was published.

Rheticus' table contained numerous errors, which were largely removed by Pitiscus, an indefatigable arithmetician of the seventeenth century. In addition to revising the existing tables he computed anew the sines of angles from 0° to 7° for intervals of 1′ to from 20 to 25 places of decimals. Pitiscus's improved tables were published in 1613 under the title "Thesaurus mathematicus." These tables formed the basis of all subsequent tables, until the discovery of improved methods of computation in recent times has made it comparatively easy to check old tables or to compute new ones.

23. Review.

1. (*a*) Define trigonometry, explain the etymology of the word and tell how the science originated. (*b*) Define in words the sine, cosine, and tangent of an acute angle. (*c*) Define the secant, cosecant, and cotangent of an angle. (*d*) What is meant by the versine and coversine of an angle?

2. (*a*) Name three pairs of functions such that in each pair either is the cofunction of the other. (*b*) Explain the origin of the terms cosine, cosecant, and cotangent. (*c*) Prove that $\sin A = \cos (90° - A)$, $\cos A = \sin (90° - A)$, $\tan A = \cot (90° - A)$. (*d*) Express $\sin 76° 40'$ as a function of an angle less than 45°.

3. (*a*) Construct the following angles: $\sin^{-1} \frac{1}{3}$, $\cos^{-1} 0.4$, $\tan^{-1} 0.5$, $\cot^{-1} 3$. (*b*) Give from memory the values of the sine, cosine, and tangent of each of the following angles: 0°, 30°, 45°, 60°, 90°. (*c*) Draw a figure and deduce the functions of 30° and 45°.

4. (*a*) Name three pairs of functions such that in each pair either function is the reciprocal of the other. (*b*) Prove the relations $\sin^2 A + \cos^2 A = 1$, $\tan^2 A + 1 = \sec^2 A$, $\cot^2 A + 1 = \csc^2 A$, $\tan A = \dfrac{\sin A}{\cos A}$. (*c*) Given $\sin A = \dfrac{3}{5}$, find each of the other functions of A. (*d*) Express each of the functions of A in terms of $\tan A$.

5. (*a*) Reduce to its simplest form the expression

$$\frac{\cos X}{1-\tan X}+\frac{\sin X}{1-\cot X}.$$

(*b*) Prove the identities, —

$$\frac{\sec A+\csc A}{\sec A-\csc A}=\frac{\tan A+1}{\tan A-1}=\frac{1+\cot A}{1-\cot A}.$$

6. (*a*) What is meant by a table of natural functions? (*b*) Which functions increase and which decrease as the angle increases from 0° to 90°? (*c*) What is meant by the term "interpolation"? by the "principle of proportional parts"? (*d*) How would you find the secant of a given angle? (*e*) Can a small angle be found more accurately from its sine or from its cosine? Why? (*f*) Can a small angle be found more accurately from its tangent or from its cotangent? Why?

7. (*a*) There are four different cases of right-triangle problems according as the given parts are: I. The hypotenuse and an angle. II. One side and an angle. III. One side and the hypotenuse. IV. Two sides. Show how to solve each case.

8. Prove the formula $\tan\frac{A}{2}=\sqrt{\frac{c-b}{c+b}}$, and write down a similar formula for $\tan\frac{B}{2}$.

9. A flagstaff AB stands on top of a tower. To determine its height, a distance OP 500 feet long was measured off in a horizontal direction from the foot of the tower. At P the angles OPA and OPB were measured, and were found to be 22° and 28° respectively. Find the length of the flagstaff by natural functions, also by the graphic method, and compare your results.

CHAPTER IV

LOGARITHMS

24. Definition of Logarithm. The numbers representing the given parts of a triangle and other numbers obtained by careful measurement usually contain three or four significant figures and sometimes five, six, or it may be even seven significant figures. Multiplication, division, the extraction of roots and raising to powers of such numbers by the ordinary methods, require a great deal of tedious labor, most of which may be avoided by using another method of computation, known as the method of logarithms. This method requires the use of tables, by the aid of which multiplication of two or more numbers is accomplished by adding certain other numbers found in the tables. Similarly division is reduced to a mere subtraction of two numbers found in the tables. To raise to a power or to extract a root of a number, the corresponding number in the table is multiplied or divided by the index of the power or root.*

The method of logarithms presupposes that when some positive number, different from unity, has been chosen, every other positive number can be expressed as some power (integral, fractional, negative, or incommensurable) of this number. Thus, if a is some positive number not equal to one, and N any positive number, we assume that a number x can always be found such that

$$a^x = N.$$

This number x is called the *logarithm of N* to the *base a*, and is usually written

$$x = \log_a N,$$

in words:

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* The usefulness of the method of logarithms may be anticipated from the testimony of Laplace, the great French astronomer, who said the method of logarithms "by reducing to a few days the labors of many months, doubles, as it were, the life of an astronomer, besides freeing him from the errors and disgust inseparable from long calculation." The advantages which the use of logarithms offers to the astronomer are shared, of course, by all others who deal much with numerical calculation.

*The logarithm** *of a number to a given base is the exponent of the power to which the base must be raised to produce the number.*

For example, since

$$
\begin{array}{rl}
10^2 = 100, & 2 = \log_{10} 100; \\
10^3 = 1000, & 3 = \log_{10} 1000; \\
10^{\frac{1}{2}} = 3.1623, & \frac{1}{2} = \log_{10} 3.1623; \\
10^{-1} = 0.1, & -1 = \log_{10} 0.1; \\
(0.5)^2 = 0.25, & 2 = \log_{0.5} 0.25; \\
27^{\frac{2}{3}} = 9, & \frac{2}{3} = \log_{27} 9;
\end{array}
$$

$$2^6 = 4^3 = 8^2 = 64, \quad 6 = \log_2 64, \quad 3 = \log_4 64, \quad 2 = \log_8 64.$$

25. Fundamental Laws Governing Logarithms. Since logarithms are exponents, the laws of logarithms are the same as the laws of exponents. Now the laws of exponents are, —

(*a*) $\mathbf{a}^x \cdot \mathbf{a}^y = \mathbf{a}^{x+y}$,

x is the logarithm (exponent) of the first factor,
y is the logarithm (exponent) of the second factor,
$x + y$ is the logarithm (exponent) of the product;

hence,

The logarithm of the product of two factors is equal to the sum of the logarithms of the factors, or

If $P = M \cdot N$, $\log P = \log M + \log N$.

Similarly,

If $P = L \cdot M \cdot N \cdots$, $\log P = \log L + \log M + \log N + \cdots$.

Thus $\log 15 = \log 3 + \log 5$, $\log 30 = \log 2 + \log 3 + \log 5$.

(*b*) $\mathbf{a}^x \div \mathbf{a}^y = \mathbf{a}^{x-y}$,

x is the logarithm (exponent) of the dividend,
y is the logarithm (exponent) of the divisor,
$x - y$ is the logarithm (exponent) of the quotient;

hence,

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* From *logos* = ratio, and *arithmos* = number, so called by the Scotch mathematician John Napier, one of the inventors of logarithms, because, as originally conceived of, logarithms were a set of numbers, as

$h, \; k, \; l, \; m, \; n,$ etc.,

corresponding to a second set $a^h, \; a^k, \; a^l, \; a^m, \; a^n,$ etc.,

the second set being so chosen that the ratio between any consecutive two is the same.

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor, or,

If $Q = M \div N$, $\log Q = \log M - \log N$.

Thus $\log \frac{5}{3} = \log 5 - \log 3$.

(*c*) $(a^x)^n = a^{nx}$,

x is the logarithm of the quantity which is to be raised to the nth power, nx is the logarithm of the resulting power; hence,

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power to which it is to be raised, or

If $P = N^n$, $\log P = n \log N$.

Thus $\log 3^5 = 5 \log 3$.

(*d*) $\sqrt[n]{a^x} = a^{\frac{x}{n}}$,

x is the logarithm of the quantity whose nth root is to be extracted, $\frac{x}{n}$ is the logarithm of the resulting root; hence,

The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root to be extracted, or

If $P = \sqrt[n]{N}$, $\log P = \frac{\log N}{n}$.

Thus $\log \sqrt[5]{17} = \frac{1}{5} \log 17$.

26. Logarithms of Special Values.

(*a*) $a^1 = a$, hence $\log a = 1$, that is,

The logarithm of the base is 1.

(*b*) Any number divided by itself is 1, but by the law of exponents $a \div a = a^0$, so that $a^0 = 1$, and therefore $\log_a 1 = 0$, that is,

The logarithm of 1 *to any base is* 0.

(*c*) By Art. 25 (*b*), $\log \frac{1}{N} = \log 1 - \log N$, and by (*b*) of this article, $\log 1 = 0$, hence

$$\log \frac{1}{N} = -\log N,$$

that is,

The logarithm of the reciprocal of any number is equal to minus the logarithm of the number.

Definition. *The logarithm of the reciprocal of a number is called* the COLOGARITHM* *of the number.*

Thus

$\log_{10} 10 = 1,$ hence $\log_{10} \frac{1}{10} = -1 = \text{colog}_{10} 10;$

$\log_{10} 100 = 2,$ hence $\log_{10} \frac{1}{100} = -2 = \text{colog}_{10} 100;$

$\log_{10} 3.1623 = \frac{1}{2},$ hence $\log_{10} \dfrac{1}{3.1623} = -\frac{1}{2} = \text{colog}_{10} 3.1623.$

Since $\dfrac{M}{N} = M \cdot \dfrac{1}{N}$, we have

$$\log \frac{M}{N} = \log M + \log \frac{1}{N} = \log M + \text{colog } N,$$

that is,

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend plus the cologarithm of the divisor.

EXERCISE 15

1. Given $5^3 = 125$, $5^2 = 25$, $5^1 = 5$; write down the logarithms to the base 5, of 125, 25, 5, $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$.

2. Given $\log_4 2 = 0.5$, $\log_3 81 = 4$, $\log_{10} 3.1623 = 0.5$, $\log_b a = c$; write down equivalent expressions free from the symbol log.

Ans. $4^{0.5} = 2$, $3^4 = 81$, etc.

3. Find $\log_3 27$, $\log_{10} 10{,}000$, $\log_a a$, $\log_3 \frac{1}{27}$, $\log_a a^{\frac{3}{2}}$.

Ans. 3, 4, 1, −3, $\frac{3}{2}$.

4. Express in terms of log 2, log 3, and log 5, the following:

$$\log 15,\ \log \frac{10}{3},\ \log \frac{5}{6},\ \log 100,\ \log \sqrt{\frac{15}{2}},\ \log \frac{\sqrt{6}\sqrt[3]{15}}{\sqrt[5]{60}}.$$

Ans. $\log 3 + \log 5$, $\log 2 + \log 5 - \log 3$, $\log 5 - \log 2 - \log 3$, $2(\log 2 + \log 5)$, $\frac{1}{2}(\log 3 + \log 5 - \log 2)$, $\frac{1}{10}\log 2 + \frac{19}{30}\log 3 + \frac{2}{15}\log 5$.

5. Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, $\log_{10} 5 = 0.69897$; find $\log_{10} 4$, $\log_{10} 0.3$, $\log_{10} 0.75$, $\log_{10} \sqrt[3]{5}$, $\log_{10}\sqrt{\dfrac{1280}{729}}$.

Ans. 0.60206, $-1 + 0.47712$,† $-1 + 0.87506$, 0.23299, 0.12224.

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* Whenever $x + y =$ constant, x and y are said to be complements (more specifically arithmetic complements) of each other. Now $\log x + \log \frac{1}{x} = \log 1$ = constant, hence $\log \frac{1}{x}$ is the complement of $\log x$, which on contraction becomes colog x.

† When 10 is the base, the logarithm is always written so that the fractional part is positive.

6. If $b = 184$, $c = 59$; find the logarithm of $\sqrt{\dfrac{b+c}{b-c}}$, and also of $b^2 - c^2$. *Ans.* 0.14434, 4.48251.

7. Show that the fractional part of a logarithm, to the base 10, is not changed if the number is multiplied by 10, or by a power of 10.

8. Compute to four places of decimals the numbers whose logarithms to the base 10 are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{6}$.

Ans. 3.1623, 1.7783, 2.1544, 1.4678.

27. The Common System of Logarithms. Each different base determines a different system of logarithms. The system in common use has 10 for its base, and is called the *common* * *system* of logarithms. Common logarithms have been carefully computed and tabulated, so that the logarithm of any number can be readily found by referring to the table, and, vice versa, if the logarithm of a number is known, the number itself can be found from the table.

The advantage of the common system over other systems of logarithms consists in this: the fractional part of any common logarithm remains unchanged when the number is multiplied or divided by 10 or a power of 10. To see this, let us consider a special case, say the number whose logarithm is 0.5.

$$10^{0.5} = 10^{\frac{1}{2}} = \sqrt{10} = 3.16228, \quad \text{therefore } \log† \, 3.16228 = 0.5.$$

Multiplying by 10,

$$10 \times 10^{0.5} = 10 \times 3.16228,$$

or $$10^{1.5} = 31.6228, \quad \text{therefore } \log 31.6228 = 1.5.$$

Similarly

$$10^{2.5} = 316.228, \quad \text{therefore } \log 316.228 = 2.5;$$
$$10^{3.5} = 3162.28, \quad \text{therefore } \log 3162.28 = 3.5;$$
$$10^{4.5} = 31622.8, \quad \text{therefore } \log 31622.8 = 4.5; \quad \text{etc.}$$

Again, dividing by 10, we have,

$$\frac{10^{0.5}}{10} = 10^{-1} \times 10^{0.5} = \frac{3.16228}{10}$$

or $$10^{-1+0.5} = 0.316228, \qquad \log 0.316228 = -1 + 0.5 = 9.5 - 10.$$

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* This system is also known as the *Briggsian system*, in honor of Henry Briggs of Oxford (1556–1630), who was the first one to compute and publish a table of logarithms to the base 10.

† In the remainder of this chapter, and always when computations are concerned, if no base is expressed, the base 10 is understood.

Similarly

$$10^{-2+0.5} = 0.0316228, \quad \log 0.0316228 = -2 + 0.5 = 8.5 - 10;$$
$$10^{-3+0.5} = 0.00316228, \quad \log 0.00316228 = -3 + 0.5 = 7.5 - 10;$$

and so on, the form 9.5 − 10 being introduced to avoid the plus sign between −1 and 0.5.

Now the logarithms of each of the numbers 3.16228, 31.6228, 316.228, 3162.28, 31622.8, etc., 0.316228, 0.0316228, 0.00316228, etc., have the same fractional part, namely, 0.5, while the integral parts 1, 2, 3, 4, etc., −1, −2, −3, etc., plainly depend upon the position of the decimal point.

For convenience the integral part of the logarithm is called the *characteristic* of the logarithm, and the fractional part is called the *mantissa*.

Thus, the logarithm of 316.228 is composed of the characteristic 2 and the mantissa 0.5.

The logarithm of 0.00316228 is composed of the characteristic −3 and the mantissa 0.5.

We have seen that the mantissa is independent of the position of the decimal point, that is, it is the same for all numbers composed of the same figures taken in the same order.

28. Rule for the Characteristic. We shall now learn how the characteristic of a logarithm of a number may be determined before the logarithm is known.

$$10^0 = 1, \quad \text{or} \quad \log 1 = 0;$$
$$10^1 = 10, \quad \text{or} \quad \log 10 = 1;$$
$$10^2 = 100, \quad \text{or} \quad \log 100 = 2;$$
$$10^3 = 1000, \quad \text{or} \quad \log 1000 = 3; \quad \text{etc.}$$

Since $$\log 1 = 0 \quad \text{and} \quad \log 10 = 1,$$

every number between 1 and 10 has a logarithm between 0 and 1,* or, every number whose integral part has *one digit* has a logarithm whose characteristic is *zero*.

Since $$\log 10 = 1 \text{ and } \log 100 = 2,$$

every number between 10 and 100 has a logarithm between 1 and 2, or, every number whose integral part has *two digits* has a logarithm whose characteristic is *one.*

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* This statement assumes that to the greater of two numbers corresponds the greater logarithm, that is, if $M > N$, $\log M > \log N$, a theorem which can be easily proven by elementary algebra.

Similarly, every number between 100 and 1000 has a logarithm between 2 and 3, or, every number whose integral part has *three digits* has a logarithm whose characteristic is *two.*

Every number whose integral part has *four digits* has a logarithm whose characteristic is *three*, and so on.

This gives us the following rule:

I. *The characteristic of the logarithm of any number greater than 1 is one less than the number of digits in the integral part of the number.*

Thus, the integral part of the number 31622.8 consists of 5 digits, hence the characteristic of its logarithm is 4.

The integral part of 3.16228 consists of 1 digit, hence the characteristic of its logarithm is 0.

Again,

$$10^0 = 1, \qquad \text{or } \log 1 = 0;$$

$$10^{-1} = \frac{1}{10} = 0.1, \qquad \text{or } \log 0.1 = -1;$$

$$10^{-2} = \frac{1}{10^2} = 0.01, \qquad \text{or } \log 0.01 = -2;$$

$$10^{-3} = \frac{1}{10^3} = 0.001, \qquad \text{or } \log 0.001 = -3;$$

$$10^{-4} = \frac{1}{10^4} = 0.0001, \qquad \text{or } \log 0.0001 = -4; \text{etc.}$$

Hence, every number between 1 and 0.1 has a logarithm between 0 and -1, or, every fraction greater than 0.1 has a logarithm whose characteristic is -1.

Similarly, every number between 0.1 and 0.01 has a logarithm between -1 and -2, or, every fraction greater than 0.01 but less than 0.1 has a logarithm whose characteristic is -2.

In like manner, every fraction greater than 0.001 but less than 0.01 has a logarithm whose characteristic is -3, and so on.

This gives us a second rule:

II. *The characteristic of the logarithm of any number less than 1 is a negative number one more than the number of ciphers between the decimal point and the first significant figure of the fraction.*

Thus, the fraction 0.00316228 has two ciphers between the decimal point and the first significant figure, hence the characteristic of its logarithm is -3 or $7 - 10$.

The fraction 0.316228 has no cipher between the decimal point and the first significant figure, hence the characteristic of its logarithm is -1 or $9 - 10$.

EXERCISE 16

1. Write down the characteristics of the common logarithms of 367, 36.7, 3670, 3.67, 0.000367, 0.367.

Ans. 2, 1, 3, 0, -4 or $6 - 10$, -1 or $9 - 10$.

2. log 635 = 2.80277; write down the logarithms of 6.35, 63500, 0.635, 0.0000635.

Ans. 0.80277, 4.80277, $9.80277 - 10$, $5.80277 - 10$.

3. How many digits are there in the integral part of the number whose logarithm is 3.1567; 1.6533; 0.6831?

4. To the number 57 corresponds the mantissa 75587; what is the number whose logarithm is 1.75587; 2.75587; 0.75587; $9.75587 - 10$?

Ans. 57; 570; 5.7; 0.57.

5. To the number 673 corresponds the mantissa 82802; find the logarithm of 673; of $(673)^2$; of $\sqrt{673}$; of $\sqrt[3]{(673)^2}$.

Ans. 2.82802; 5.65604; 1.41401; 1.88535.

6. log 3 = 0.47712; how many digits are there in 3^{25}; in 3^{100}; in 30^{15}; in 27^5?

Ans. 12; 48; 23; 8.

7. Write down the cologarithms of the numbers in problem 2.

Ans. $9.19723 - 10$; $5.19723 - 10$; 0.19723; 4.19723.

8. log 5 = 0.69897; find the logarithm of $\frac{1}{5}$; of $\frac{1}{25}$; of 0.05; of $\sqrt{5}$; of $\sqrt{\frac{1}{5}}$; of $\sqrt[3]{0.5}$; of $\sqrt[3]{\frac{1}{5}}$.

Ans. $9.30103 - 10$; $8.60206 - 10$; $8.69897 - 10$; 0.34948; $9.65052 - 10$; $9.89966 - 10$; $9.76701 - 10$.

9. To the number 3 corresponds the mantissa 47712, and to the number 7 corresponds the mantissa 84510; find the logarithm of 21; of $\frac{3}{7}$; of $\sqrt{0.3 \times 49}$; of $\sqrt[3]{\frac{7}{9}}$.

Ans. 1.32222; $9.63202 - 10$; 0.58366; $9.96362 - 10$.

10. The formula for the amount (A) of a principal (P) put out on compound interest at (R) per cent for (t) years, the interest being compounded annually, is

$$A = P\left(1 + \frac{R}{100}\right)^t,$$

whence

$$\log A = \log \mathrm{P} + t \log\left(1 + \frac{R}{100}\right).$$

Find the number of digits in the amount of $1 at compound interest at 6 per cent for 100 years, the mantissa corresponding to 106 being .02531. *Ans.* 3 digits.

29. Tables of Common Logarithms. Our next step is to learn how to use a table of logarithms in actual computation.

Common logarithms, except those belonging to numbers which are integral powers of 10, cannot be exactly expressed in decimals. We must therefore omit all the figures after a certain decimal place. Just where to stop depends upon the accuracy desired. If some of the numbers which enter a given problem are the results of measurement, as is frequently the case, their accuracy will not ordinarily exceed four or five figures, consequently it will be useless to retain more than five figures in the decimal part of their logarithms. In other words, a table of logarithms which contains the mantissas to five places will answer for the solution of most practical problems in which approximate answers are all that is necessary or possible. Such a table is known as a five-place table of logarithms.

The explanations which follow, and all the answers in this book obtained from logarithmic computation, are based upon a five-place table.

When a five-place table is used, it is not worth while to retain more than five significant figures in the answers to the problems, and even then the last figure is not always exact.

When more than five-place accuracy is required, and even when the fifth place must be known with certainty, larger tables must be used. Tables containing six places, seven places, eight places, ten places, eleven places, twenty places, and partial tables containing up to two hundred and sixty places, have been published. There are also smaller tables containing three and four places only.

Tables containing more than seven places are seldom used, for seven-place tables meet practically every demand of present-day science. The results obtained by means of a seven-place table are as exact as the most careful measurements obtained by the most skillful observers by means of the most precise instruments under the most favorable conditions.

30. To Find the Logarithm of a Given Number.

(*a*) *When the given number has four figures.* The characteristic is found by the rule in Art. 28. To find the mantissa, enter that line of the table which begins with the number made up of the first three figures of the given number and take out that number of the line which is found in the column headed by the fourth figure of the given number.

The number thus found constitutes the third, fourth and fifth figures of the required mantissa. The first and second figures are found in the column headed by 0 either in or above the line in which the third, fourth and fifth figures are found, except when these figures as given in the table are preceded by a star (*) in which case the first two figures are found in the next following line.

EXAMPLE 1. From the specimen page of logarithms, page 63, we find

mantissa log 6315 = 80037, hence log 6315 = 3.80037;
mantissa log 65.24 = 81451, hence log 65.24 = 1.81451;
mantissa log 6.608 = 82007, hence log 6.608 = 0.82007.

(*b*) *When the given number has less than four figures.* We know that the mantissa of the logarithm of a number is not changed if the number is multiplied or divided by 10 or by some power of 10. After we have determined the characteristic of a logarithm we may then annex as many ciphers to the given number as we need to make up four places and find the mantissa of this new number by case (*a*).

EXAMPLE 2. Find the logarithm of 64.

The characteristic of log 64 is 1. The mantissa of log 64 is the same as the mantissa of log 6400, which by case (*a*) is found to be 80618. Hence,

$$\log 64 = 1.80618.$$

(*c*) *When the number has more than four figures.*

EXAMPLE 3. Find the logarithm of 6425.4.

The characteristic of log 6425.4 = 3.

(SPECIMEN PAGE)

N	0	1	2	3	4	5	6	7	8	9	D
630	934	941	948	955	962	969	975	982	989	996	7
631	80 003	010	017	024	030	037	044	051	058	065	7
632	072	079	085	092	099	106	113	120	127	134	7
633	140	147	154	161	168	175	182	188	195	202	7
634	209	216	223	229	236	243	250	257	264	271	7
635	277	284	291	298	305	312	318	325	332	339	7
636	346	353	359	366	373	380	387	393	400	407	7
637	414	421	428	434	441	448	455	462	468	475	7
638	482	489	496	502	509	516	523	530	536	543	7
639	550	557	564	570	577	584	591	598	604	611	7
640	618	625	632	638	645	652	659	665	672	679	7
641	686	693	699	706	713	720	726	733	740	747	7
642	754	760	767	774	781	787	794	801	808	814	7
643	821	828	835	841	848	855	862	868	875	882	7
644	889	895	902	909	916	922	929	936	943	949	7
645	956	963	969	976	983	990	996	*003	*010	*017	7
646	81 023	030	037	043	050	057	064	070	077	084	7
647	090	097	104	111	117	124	131	137	144	151	7
648	158	164	171	178	184	191	198	204	211	218	7
649	224	231	238	245	251	258	265	271	278	285	7
650	291	298	305	311	318	325	331	338	345	351	7
651	358	365	371	378	385	391	398	405	411	418	7
652	425	431	438	445	451	458	465	471	478	485	7
653	491	498	505	511	518	525	531	538	544	551	7
654	558	564	571	578	584	591	598	604	611	617	7
655	624	631	637	644	651	657	664	671	677	684	7
656	690	697	704	710	717	723	730	737	743	750	7
657	757	763	770	776	783	790	796	803	809	816	7
658	823	829	836	842	849	856	862	869	875	882	7
659	889	895	902	908	915	921	928	935	941	948	7
660	954	961	968	974	981	987	994	*000	*007	*014	7
661	82 020	027	033	040	046	053	060	066	073	079	7
662	086	092	099	105	112	119	125	132	138	145	7
663	151	158	164	171	178	184	191	197	204	210	7
664	217	223	230	236	243	249	256	263	269	276	7
665	282	289	295	302	308	315	321	328	334	341	7
666	347	354	360	367	373	380	387	393	400	406	7
667	413	419	426	432	439	445	452	458	465	471	7
668	478	484	491	497	504	510	517	523	530	536	7
669	543	549	556	562	569	575	582	588	595	601	7
N	0	1	2	3	4	5	6	7	8	9	D

By (*a*), the mantissa of log 6425 = 80787
the mantissa of $\underline{\text{log } 6426 = 80794}$
difference for 1 = 7

Now the mantissa of log 6425.4 is evidently larger than 80787 and less than 80794, and since 6425.4 lies $\frac{4}{10}$ the way between 6425 and 6426, we will assume that the mantissa of log 6425.4 lies $\frac{4}{10}$ the way between 80787 and 80794. We must therefore increase the smaller of the two mantissas by $\frac{4}{10}$ of the difference between the two, that is, by $\frac{4}{10}$ of 7.

$$\tfrac{4}{10} \text{ of } 7 = 2.8 \text{ or } 3 \text{ to the nearest integer,}$$

and the mantissa of log 6425.4 = 80787 + 3 = 80790.
Hence

$$\log 6425.4 = 3.80790.$$

We have here assumed the ***principle of proportional parts for logarithms of numbers***, namely, that for small changes in the number, the change in the logarithm is proportional to the change in the number.

EXAMPLE 4. Find the logarithm of 6.5487.

The characteristic of log 6.5487 = 0. The mantissa of log 6.5487 is the same as the mantissa of log 6548.7.

mantissa log 6548 = 81611
mantissa $\underline{\text{log } 6549 = 81617}$
difference for 1 = 6

difference for 0.7 = 0.7 of 6 = 4.2 or 4 to the nearest integer.

Hence mantissa log 6548.7 = 81611 + 4 = 81615,
and log 6.5487 = 0.81615.

EXAMPLE 5. Find log 0.000635945.

The characteristic of log 0.000635945 = − 4 or 6 − 10.

The mantissa of log 0.000635945 is the same as the mantissa of log 6359.45.

mantissa log 6359 = 80339
mantissa $\underline{\text{log } 6360 = 80346}$
difference for 1 = 7

difference for 0.45 = 0.45 of 7 = 3.15 or 3 to the nearest integer.

Hence mantissa log 6359.45 = 80339 + 3 = 80342,
and log 6359.45 = 6.80342 − 10.

Example 6. Find the cologarithm of 65.021.

By definition, Art. 26, the cologarithm of a number is the logarithm of the reciprocal of that number, hence

$$\begin{aligned} \text{colog}\,65.021 &= \log 1 - \log 65.021. \\ \log 1 &= 0 = 10 \qquad - 10 \\ \log 65.021 &= \underline{\ \ 1.81306\ \ } \\ \text{colog}\,65.021 &= \ \ 8.18694^{*} - 10. \end{aligned}$$

31. To Find the Number Corresponding to a Given Logarithm.

(*a*) *When the given mantissa can be found in the table.* The characteristic is used only to determine the position of the decimal point after the number has been found. Find the given mantissa in the table. The first three figures of the number sought are found in the same line with the mantissa, in the column on the left, and the fourth figure is found at the top of the column containing the given mantissa.

Example 1. Find the number whose logarithm is 5.82158.

Find the given mantissa 82158 in the table (see specimen page, p. 63). The number in the left-hand column and in the same line with the given mantissa is 663, and the number at the top of the column containing the mantissa 82158 is 1, hence the significant figures of the required number are 6631. The characteristic is 5, hence the integral part of the required number has 6 places, that is, the required number is 663100.

Example 2. Find the number whose logarithm is 8.81043 − 10.

Corresponding to the mantissa 81043 we find in the table the number 6463. The characteristic is 8 − 10 or − 2, hence by the rule for the characteristic the required number is a decimal fraction with one cipher preceding the first significant figure.

Therefore the required number is 0.06463.

(*b*) *When the given mantissa cannot be found in the table*, two other consecutive mantissas can always be found in the table, one of which is a little smaller and the other a little larger than the given mantissa. The four figures corresponding to the smaller of these mantissas will be the first four figures of the required number; the fifth figure, and sometimes the sixth,* can then be found by interpolation from the principle of proportional parts.

---

* The subtraction is performed from left to right by subtracting each figure from 9 except the last one, which is subtracted from 10, thus: 1 from 9 = 8, 8 from 9 = 1, 1 from 9 = 8, 3 from 9 = 6, 0 from 9 = 9, 6 from 10 = 4.

EXAMPLE 3. $\log N = 1.80395$, to find N.

The table does not contain the mantissa 80395, but it contains the two consecutive mantissas 80393 and 80400, one of which is smaller and the other larger than the given mantissa. The numbers corresponding to these mantissas are 6367 and 6368 respectively.

$$\begin{array}{rr} \text{mantissa } \log 6367 = & 80393 \\ \text{mantissa } \log 6368 = & \underline{80400} \\ \text{difference for } 1 = & 7 \end{array}$$

Since the given mantissa lies between 80393 and 80400, we infer that the number corresponding to the given mantissa lies between 6367 and 6368; let it be denoted by $6367 + x$, we then have

$$\begin{array}{rr} \text{mantissa } \log 6367 = & 80393 \\ \text{mantissa } \log 6367 + x = & \underline{80395} \\ \text{difference for } x = & 2 \end{array}$$

and the principle of proportional parts gives

$$1 : 7 = x : 2, \text{ that is, } x = \tfrac{2}{7},$$

and the number corresponding to the given mantissa is $6367\frac{2}{7}$. It still remains to determine the decimal point. The characteristic is 1, hence the required number is

$$N = 63.67\tfrac{2}{7} = 63.673 \text{ to five figures.}$$

EXERCISE 17

(In this exercise the specimen page of logarithms may be used.)

1. Find the logarithms of the following numbers: 6315, 632.5 6.454, 0.0655, 0.0065.

Ans. 3.80037, 2.80106, 0.80983, 8.81624 − 10, 7.81291 − 10.

2. Find the cologarithms of each of the numbers in 1.

Ans. 6.19963 − 10, 7.19894 − 10, 9.19017 − 10, 1.18376, 2.18709.

---

* In the first part of the table, the sixth significant figure of a number may be found by interpolation. In the latter part of the table, where the difference in the mantissas corresponding to a difference of 1 in the numbers is much smaller than in the first part of the table, the sixth figure of the number obtained by the principle of proportional parts cannot be depended on.

3. Find log 63.454, log 65.061, log 6.6095, log 0.0064159.

Ans. 1.80246, 1.81332, 0.82017, 7.80725 − 10.

4. Find the numbers whose logarithms are: 1.80277, 2.80584, 0.81003, 9.81351 − 10, 8.80017 − 10, 3.81184.

Ans. 63.50, 639.5, 6.457, 0.6509, 0.06312, 6484.

5. Find the numbers whose logarithms are: 1.80958, 2.81922, 0.81006, 9.80002 − 10, 8.80022 − 10, 3.82506.

Ans. 64.503, 659.51, 6.4574, 0.63099, 0.063127, 6684.3.

6. log 63.275 = x, log y = 1.81864; find x and y.

Ans. $x = 1.80124$, $y = 65.863$.

7. Without multiplying or dividing the numbers, find the logarithms of 66.027×0.65034, $6301 \div 6.454$, $(6535.4)^2$, $\sqrt{63.275}$.

Ans. 1.63287, 2.98958, 7.63055, 0.90062.

8. $N = \sqrt[6]{64550}$; find log N and then N.

Ans. log $N = 0.80165$, $N = 6.3336$.

9. Find $(6.3096)^5$ by means of logarithms. *Ans.* 10000.

10. Find the logarithms of sin 40°, cos 48° 30′, cot 8° 50′.

Ans. 9.80808 − 10, 9.82125 − 10, 0.80854.

(Suggestion. First find the natural functions of the given angles and then find the logarithms of the resulting numbers.)

11. Find the first three significant figures and the number of figures in the integral part of the twenty-fifth power of 6.2.

Ans. 645, 20 places.

32. Directions for the Use of Logarithms. The following directions will aid the student in an intelligent use of logarithmic tables.

(*a*) In finding the logarithm of a given number, or in finding the number corresponding to a given logarithm, the interpolation should be performed mentally and only the complete result set down in writing.

(*b*) In writing down the cologarithm of a number, the subtraction from 10 should be performed mentally and from left to right.

(*c*) The results obtained by logarithms are approximations only. By neglecting the sixth and following significant figures of a number the inaccuracy introduced can never exceed one-half a unit in the fifth place, that is, the error cannot exceed the $\frac{1}{500}$ part of 1 per cent.

(*d*) When the sixth figure of a number is 5, and we wish to retain only five significant figures, it is immaterial whether we increase the fifth figure by 1 or leave it unchanged. In case we increase the fifth figure by 1 the resulting number will be too large by half a unit in the fifth place; if we leave the fifth figure unchanged, the number will be too small by half a unit. In order to cause the inaccuracies arising from this source to offset one another, it is customary, on dropping a final 5, to increase the preceding figure by 1 when it is odd, but to leave it unchanged when it is even.

Thus, 0.154755 becomes 0.15476,
but 0.154745 becomes 0.15474;
one-half of 4.23453 becomes 2.11726,
but one-half of 4.23463 becomes 2.11732,

when the results are abridged to five places.

(*e*) Sometimes the nature of the problem is such that a four-place table would give all the accuracy required. In that case the fifth figure of the mantissa in the table may be omitted and the fourth figure increased by 1 if the omitted figure exceeds 5. If the final figure of the mantissa given in the table is marked with a stroke, thus $\bar{5}$, on omitting it the preceding figure is left unchanged; but if the final 5 is unmarked, the preceding figure is increased by 1. The reason for this is that the final 5 of a mantissa is itself the result of approximation, that is, it is either in defect (5 plus something less than $\frac{1}{2}$), or it is in excess (4 plus something greater than $\frac{1}{2}$), and the latter case is distinguished from the former by printing a stroke over the 5.

Thus from the table,

$$\log 2.078 = 0.3176\bar{5} = 0.3176,$$

but

$$\log 2.079 = 0.31785 = 0.3179,$$

when abridged to four places.

(*f*) Every logarithm consists of two parts, — the mantissa, which is always positive, and the characteristic, which is always integral (or 0), but may be negative as well as positive. When the characteristic is negative, it is customary to change the form of the logarithm by adding and subtracting 10.

Thus, $\log 0.4562 = -1 + 0.65916$ is written $9.65916 - 10$,

$\log 0.0032 = -3 + 0.5051\bar{5}$ is written $7.5051\bar{5} - 10$.

This is done in part to avoid mistakes which might arise from confusing the positive and the negative parts of logarithms.

For similar reasons, if a logarithm whose characteristic is negative is to be divided by 2, as in extracting a square root, we first modify its form by adding and subtracting 20; if the logarithm is to be divided by 3, we first add and subtract 30; and generally, if the logarithm whose characteristic is negative is to be divided by n, we first modify its form by adding and subtracting $10n$.

$$
\begin{aligned}
\text{Thus,}\quad \log 0.03254 &= -2 + 0.51242 &&= 8.51242 - 10,\\
\tfrac{1}{2} \text{ of } \log 0.03254 &= \tfrac{1}{2} \text{ of } (18.51242 - 20) &&= 9.25621 - 10,\\
\tfrac{1}{3} \text{ of } \log 0.03254 &= \tfrac{1}{3} \text{ of } (28.51242 - 30) &&= 9.50414 - 10,\\
\tfrac{1}{4} \text{ of } \log 0.03254 &= \tfrac{1}{4} \text{ of } (38.51242 - 40) &&= 9.62811 - 10,\\
\tfrac{1}{5} \text{ of } \log 0.03254 &= \tfrac{1}{5} \text{ of } (48.51242 - 50) &&= 9.70248 - 10, \text{ etc.}
\end{aligned}
$$

(*g*) The difference between two consecutive mantissas is called the *tabular difference* and is printed under D in the last column on each page of logarithms. At the bottom of each of the first three pages of logarithms the tabular differences which occur on the page are multiplied by each of the nine digits expressed as tenths. The resulting tables, known as *tables of proportional parts*, are used as an aid in interpolation.

EXAMPLE 1. Find by logarithms the product of 37.543 by 0.85734.

Solution. Denote the product by x.

From table I, $\log 37.543 = 1.57453$

$\underline{\log 0.85734 = 9.93315 - 10}$

By Art. 25, (*a*), $\log x = 1.50768$

By Art. 31, and the table, $x = 32.187$.

EXAMPLE 2. Find by logarithms the quotient of 6.3725 divided by 82.756.

Solution. Denote the quotient by x.

From the table, $\log 6.3725 = 0.80431$

By Art 26, (*c*), and the table, $\underline{\text{colog } 82.756 = 8.08220 - 10}$

$\log x = 8.88651 - 10$

From the table, $x = 0.077003$.

EXAMPLE 3. Find the square root of 0.89355.

Solution.

$$\begin{aligned}
&& \log 0.89355 &= 9.95112 - 10\\
&& &= 19.95112 - 20\\
&\text{Dividing by 2,} & \log \sqrt{0.89355} &= 9.97556 - 10\\
&\text{Hence} & \sqrt{0.89355} &= 0.94528.
\end{aligned}$$

EXAMPLE 4. Find x, if $x^3 = (0.5824)^2$.

Solution.

$$\begin{aligned}
&& \log 0.5824 &= 9.76522 - 10\\
&\text{Multiplying by 2,} & \log x^3 &= 19.53044 - 20\\
&& &= 29.53044 - 30\\
&\text{Dividing by 3,} & \log x &= 9.84348 - 10\\
&& x &= 0.69740.
\end{aligned}$$

EXAMPLE 5. $x = \dfrac{8.35 \times (62.5)^2 \times \sqrt{5.673}}{(1.256)^3 \times 623.7 \times \sqrt[3]{5.736}}$; find x.

Solution. By the rules of Art. 25,

$$\begin{aligned}
\log x = {} & \log 8.35 + 2 \log 62.5 + \tfrac{1}{2} \log 5.673\\
& + 3 \operatorname{colog} 1.256 + \operatorname{colog} 623.7 + \tfrac{1}{3} \operatorname{colog} 5.736.
\end{aligned}$$

$$\begin{aligned}
\log 8.35 & & &= 0.92169\\
\log 62.5 &= 1.79588 & &\\
2 \log 62.5 &= 2 \times 1.79588 & &= 3.59176\\
\log 5.673 &= 0.75381 & &\\
\tfrac{1}{2} \log 5.673 &= \tfrac{1}{2} \times 0.75381 & &= 0.37690\\
\operatorname{colog} 1.256 &= 9.90101 - 10 & &\\
3 \operatorname{colog} 1.256 &= 3 \times (9.90101 - 10) & &= 9.70303 - 10\\
\operatorname{colog} 623.7 & & &= 7.20502 - 10\\
\operatorname{colog} 5.736 &= 9.24139 - 10 & &\\
\tfrac{1}{3} \operatorname{colog} 5.736 &= \tfrac{1}{3} \times (29.24139 - 30) & &= \underline{9.74713 - 10}\\
& & \log x &= 31.54553 - 30\\
& & &= 1.54553\\
& & x &= 35.118.
\end{aligned}$$

EXERCISE 18

Solve by logarithms:

1. 3.784×7.843. *Ans.* 29.678.
2. 67.845×0.03457.
3. $0.67375 \div 3.468$. *Ans.* 0.19428.
4. $92.57 \div 1.3785$.

5. $(684.7)^2 \times (0.03873)^3$. *Ans.* 27.236.

6. $(0.8003)^3 \div \sqrt{5.73}$.

7. $\sqrt[5]{3284.5}$. *Ans.* 5.05.

8. $(0.12562)^{10}$.

9. $\dfrac{1.56 \times 37.8 \times \sqrt{0.0753}}{67.574}$. *Ans.* 0.23946.

10. $\dfrac{600.45 \times 1.0025}{(37.5)^2 \times \sqrt{617.5}}$.

11. $\sqrt{\dfrac{3.1416 \times 6 \times 27.3}{1.056 \times 2.738 \times 10}}$. *Ans.* 4.2188.

12. $\dfrac{\sqrt[3]{(27.3)^2 \times 23.7}}{\sqrt{(28.92)^3 \times 16.5}}$.

13. The area of a triangle is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c represent the three sides of the triangle and s half their sum. Find A when $a = 617.34$, $b = 345.65$, $c = 467.75$.

Ans. 80127.

14. In Art. 21 it was shown that

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}},$$

where b is a side of a right triangle, c the hypotenuse, and A the angle between b and c. Find A when $c = 325.76$, $b = 324.13$.

Ans. 5° 44′.

15. Solve the equation $2^x = 3.573$.* *Ans.* 1.8371.

16. Find x in $(3.1416)^x = 9.8697$.

17. Solve the equation

$3^{2x} - 12 \times 3^x + 11 = 0$. *Ans.* $x = 0$, or 2.1827.

18. Solve

$$e^{\frac{x}{2}} + e^{-\frac{x}{2}} = 4, \quad \text{where } e = 2.7183.$$

Ans. $x = \pm 2.6339$.

---

* Such an equation as this, in which the unknown quantity appears as an exponent, is called an *exponential equation*. It is solved by first taking logarithms of both sides of the equation, thus:

$$x \log 2 = \log 3.573,$$

that is, $0.30103\,x = 0.55303,$

from which $x = 1.8371.$

33. Application of Logarithms. The solution of triangles, which furnishes the most important application of logarithms, will be fully considered later on. At present we give a few miscellaneous problems which are especially adapted to solution by logarithms.

EXERCISE 19

Many of the following list of problems require for their solution the compound interest formula, Problem 10, Exercise 16.

1. Find the amount on $100 for 100 years at 4 % compound interest. (log 1.04 = 0.0170333.) *Ans.* $5050.4.

2. In what time will a sum of money double itself at 8 % compound interest? *Ans.* 9.007 yrs.

3. A man bequeaths $500, which is to accumulate at compound interest until the interest for one year at 5 % will amount to at least $300, after which the yearly interest is to be awarded as a scholarship. How many years must elapse before the scholarship becomes available, assuming that the original bequest is made to earn 5 % compound interest? *Ans.* 51 yrs.

4. At what rate of interest must the bequest in Problem 3 be invested in order that the scholarship may become available in 40 yrs.? *Ans.* 6.4 %.

5. In 1624 the Dutch bought Manhattan Island from the Indians for about $24. Suppose that the Indians had put their money out at compound interest at 7 % and had added the interest to the principal each year, how large would be the accumulated amount in 1910? (From White's Scrap Book of Mathematics.)

Ans. In round numbers $6,000,000,000. The actual valuation of Manhattan and Bronx real and personal property in 1908 was $5,235,399,980.

6. The population of the state of Washington in 1890 was 349,400 and in 1900 it was 518,100. What was the average yearly rate of increase? Assuming the rate of increase to remain the same, what should be the population in 1910? *Ans.* 4 %; 767,970, nearly.

7. The founder of a new faith makes one convert each year, and each new convert makes another convert each year, and so on.

How long would it require to convert the whole earth to the new faith, assuming that the population of the world is 1,500,000,000?

Ans. Between 30 and 31 yrs.

8. The combined wealth of the United States and Europe was estimated (1908) to amount to about \$450,000,000,000. Let us assume that the entire wealth of the world amounts to $\$10^{12}$. How long would it take \$1 put out at compound interest at 3 % to equal or exceed this amount?

Ans. 935 yrs.

34. To Compute a Table of Common Logarithms.

A table of logarithms may be computed from the successive square-roots of 10 and multiplications.

$$10^{\frac{1}{2}} = \sqrt{10} = 3.16228, \qquad (1)$$

$$10^{\frac{1}{4}} = \sqrt{10^{\frac{1}{2}}} = \sqrt{3.16228} = 1.77828, \qquad (2)$$

$$10^{\frac{1}{8}} = \sqrt{10^{\frac{1}{4}}} = \sqrt{1.77828} = 1.33352, \qquad (3)$$

$$10^{\frac{1}{16}} = \sqrt{10^{\frac{1}{8}}} = \sqrt{1.33352} = 1.15478. \qquad (4)$$

By definition, the common logarithm of a number is the power to which 10 must be raised to produce that number, hence from

$10^{\frac{1}{16}} = 1.15478,$ $\qquad \log 1.1548^{*} = \frac{1}{16} = 0.0625;$

$10^{\frac{2}{16}} = 10^{\frac{1}{8}} = 1.33352,$ $\qquad \log 1.3335 = \frac{2}{16} = 0.1250;$

$10^{\frac{3}{16}} = 10^{\frac{2}{16}+\frac{1}{16}} = 10^{\frac{2}{16}} \cdot 10^{\frac{1}{16}} = 1.33352 \times 1.15478 = 1.53992,$

$\log 1.5399 = \frac{3}{16} = 0.1875;$

$10^{\frac{4}{16}} = 10^{\frac{3}{16}+\frac{1}{16}} = 10^{\frac{3}{16}} \cdot 10^{\frac{1}{16}} = 1.53992 \times 1.15478 = 1.77828,$

$\log 1.7783 = \frac{4}{16} = 0.2500.$

Check. $10^{\frac{4}{16}} = 10^{\frac{1}{4}} = 1.77828$ by (2).

$10^{\frac{5}{16}} = 10^{\frac{4}{16}+\frac{1}{16}} = 10^{\frac{4}{16}} \cdot 10^{\frac{1}{16}} = 1.77828 \times 1.15478 = 2.05352,$

$\log 2.0535 = \frac{5}{16} = 0.3125;$

$10^{\frac{6}{16}} = 10^{\frac{5}{16}+\frac{1}{16}} = 10^{\frac{5}{16}} \cdot 10^{\frac{1}{16}} = 2.05352 \times 1.15478 = 2.37136,$

$\log 2.3714 = \frac{6}{16} = 0.3750;$

$10^{\frac{7}{16}} = 10^{\frac{6}{16}+\frac{1}{16}} = 10^{\frac{6}{16}} \cdot 10^{\frac{1}{16}} = 2.37136 \times 1.15478 = 2.73840,$

$\log 2.7384 = \frac{7}{16} = 0.4375;$

$10^{\frac{8}{16}} = 10^{\frac{7}{16}+\frac{1}{16}} = 10^{\frac{7}{16}} \cdot 10^{\frac{1}{16}} = 2.73840 \times 1.15478 = 3.16225,$

$\log 3.1623 = \frac{8}{16} = 0.5000.$

Check. $10^{\frac{8}{16}} = 10^{\frac{1}{2}} = 3.16228$ by (1).

$10^{\frac{9}{16}} = 10^{\frac{8}{16}+\frac{1}{16}} = 10^{\frac{8}{16}} \cdot 10^{\frac{1}{16}} = 3.16228 \times 1.15478 = 3.65174,$

$\log 3.6517 = \frac{9}{16} = 0.5625;$

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* Only 5 figures are carried in order to secure accuracy in the last figure of all the numbers in the list.

$$10^{\frac{10}{16}} = 10^{\frac{9}{16}+\frac{1}{16}} = 10^{\frac{9}{16}} \cdot 10^{\frac{1}{16}} = 3.65174 \times 1.15478 = 4.21696,$$
$$\log 4.2170 = \tfrac{10}{16} = 0.6250;$$

$$10^{\frac{11}{16}} = 10^{\frac{10}{16}+\frac{1}{16}} = 10^{\frac{10}{16}} \cdot 10^{\frac{1}{16}} = 4.21696 \times 1.15478 = 4.86966,$$
$$\log 4.8697 = \tfrac{11}{16} = 0.6875;$$

$$10^{\frac{12}{16}} = 10^{\frac{11}{16}+\frac{1}{16}} = 10^{\frac{11}{16}} \cdot 10^{\frac{1}{16}} = 4.86966 \times 1.15478 = 5.62339,$$
$$\log 5.6234 = \tfrac{12}{16} = 0.7500.$$

Check. $10^{\frac{12}{16}} = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \cdot 10^{\frac{1}{4}} = 3.16228 \times 1.77828 = 5.62342$ by (1) and (2).

$$10^{\frac{13}{16}} = 10^{\frac{12}{16}+\frac{1}{16}} = 10^{\frac{12}{16}} \cdot 10^{\frac{1}{16}} = 5.62342 \times 1.15478 = 6.49382,$$
$$\log 6.4938 = \tfrac{13}{16} = 0.8125;$$

$$10^{\frac{14}{16}} = 10^{\frac{13}{16}+\frac{1}{16}} = 10^{\frac{13}{16}} \cdot 10^{\frac{1}{16}} = 6.49382 \times 1.15478 = 7.49892,$$
$$\log 7.4989 = \tfrac{14}{16} = 0.8750;$$

$$10^{\frac{15}{16}} = 10^{\frac{14}{16}+\frac{1}{16}} = 10^{\frac{14}{16}} \cdot 10^{\frac{1}{16}} = 7.49892 \times 1.15478 = 8.65960,$$
$$\log 8.6596 = \tfrac{15}{16} = 0.9375;$$

$$10^{\frac{16}{16}} = 10^{\frac{15}{16}+\frac{1}{16}} = 10^{\frac{15}{16}} \cdot 10^{\frac{1}{16}} = 8.65960 \times 1.15478 = 9.99993,$$
$$\log 10.000 = \tfrac{16}{16} = 1.0000.$$

Check. $10^{\frac{16}{16}} = 10^1 = 10.$

Each check is on the four lines just preceding, and in each case there is exact agreement to five figures between the results of the check and the result of the line just preceding. The discrepancy in the sixth figure, where it occurs, is due to the fact that all our results are approximations only.

We have thus computed the numbers whose mantissas are distributed between 0 and 1 at intervals of $\frac{1}{16}$.

Similarly, with the aid of

$$10^{\frac{1}{32}} = \sqrt{10^{\frac{1}{16}}} = \sqrt{1.15478} = 1.07468,$$
$$10^{\frac{1}{64}} = \sqrt{10^{\frac{1}{32}}} = \sqrt{1.07486} = 1.03663, \text{ etc.,}$$

longer lists of numbers could be computed whose mantissas differ by $\frac{1}{32}$, $\frac{1}{64}$, etc. By continuing this process the interval between successive mantissas can be made small at will. When the difference between successive mantissas has been made sufficiently small, the mantissa corresponding to any number, intermediate to two numbers in the list already found, may be found by interpolation. If now the numbers on the left are selected at equal intervals and tabulated, together with their mantissas, we shall have a table of common logarithms.*

The characteristics need not be tabulated, for they can always be determined from memory by the two rules of the preceding article. In most of the tables the decimal points are also omitted; for instance, we find in most tables corresponding to

the number	the mantissa
4753	67697,

meaning that .67697 is the fractional part of the logarithm of any number whose significant figures are 4753.

The method just explained, simple as it is, is not the method by which the existing tables of logarithms have actually been calculated. We shall learn later that other methods exist by means of which logarithms may be calculated with much greater ease and speed.

35. Relation between $\log_a N$ and $\log_b N$. We will now show that the logarithms of the same numbers to two different bases are proportional, so that when the logarithm of a number to a given base, as the base 10, is known, the logarithm of the same number to any other base may be obtained from it by multiplying the known logarithm by a certain constant. It is therefore unnecessary to actually construct logarithmic tables to more than one base.

To prove that

$$\log_b N = \mu \log_a N, \text{ where } \mu \text{ is the constant } \frac{1}{\log_a b}.$$

Proof. Let $\log_b N = x$, from which $b^x = N$, (1)

and $\log_a N = y$, from which $a^y = N$. (2)

From (1) and (2)

$$b^x = a^y. \tag{3}$$

Now let

$$b = a^c, \tag{4}$$

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* The student will observe that in the original arrangement the numbers on the right (mantissas) are at equal intervals. In this form the table is known as a table of *antilogarithms*. A table of antilogarithms would serve the main purposes of computation as well as a table of logarithms. Tables of antilogarithms have been published and are used by some computers.

and substitute (4) in (3), then

$$(a^c)^x = a^{cx} = a^y,$$

from which
$$cx = y,$$

or
$$x = \frac{y}{c}, \tag{5}$$

where from (4),
$$c = \log_a b.$$

Putting in (5) for x and y their values from (1) and (2), and M for $\frac{1}{c}$ we have

$$\log_b N = \mu \log_a N, \tag{6}$$

where

$$\mu = \frac{1}{\log_a b}.$$

(6) may also be written

$$\log_b N = \frac{\log_a N}{\log_a b}. \tag{7}$$

If in (7) we put $N = a$, we obtain (since $\log_a a = 1$)

$$\log_b a = \frac{1}{\log_a b},$$

that is,

$\log_b a$ and $\log_a b$ are reciprocals.

The constant multiplier μ is called the *modulus* of the system of logarithms whose base is b with reference to the system whose base is a.

36. Natural or Hyperbolic Logarithms. Theoretically any positive number different from 1 may serve as the base of a system of logarithms, but in practice only two systems are used. The first is the *common system*, used exclusively in numerical computations; the other is known as the system of *natural or hyperbolic logarithms*, which is used extensively in theoretical investigations.

The base of the natural system * is an incommensurable number, known as e, the twin sister of the number π, which like π has many remarkable properties. e is defined as the limit which $\left(1+\frac{1}{x}\right)^x$ approaches, as x increases indefinitely. It will be shown later that this limit is equivalent to the series

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* Natural or hyperbolic logarithms are known also as Napierian logarithms in honor of John Napier (1550–1617), though this name is a misnomer, since neither Napier nor any of his contemporaries had any conception of the number e or the system of logarithms which has e for its base. Napier's base is the number 0.367879, which happens to be nearly equal to $\frac{1}{e}$. The discovery of natural logarithms as well as the name is due to Nicolas Mercator (1620–1687).

$$1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots,$$

from which the approximate value of e is readily found to be

$$e = 2.71828\cdots.$$

By the preceding article we find the modulus of the natural system of logarithms with reference to the common system is

$$\mu_e=\frac{1}{\log 2.71828}=\frac{1}{0.434294}=2.30259\cdots.$$

EXERCISE 20

1. Use the results of Art. 34 to compute to four places the number whose common logarithm is $\frac{5}{32} = 0.15625$. *Ans.* 1.4330.

2. By a method similar to that of Art. 34 compute to four places the number whose common logarithm is $\frac{1}{3}$; also to four places the number whose common logarithm is $\frac{1}{9}$. *Ans.* 2.1544, 1.2913.

3. Similarly compute the number whose common logarithm is $\frac{1}{6}$. *Ans.* 1.4678.

4. Given $\log_{10} 2 = 0.30103$; compute $\log_2 10$. *Ans.* 3.3219.

5. Show that $\log_{10} N = \dfrac{\log_e N}{\log_e 10} = 0.434294 \times \log_e N$.

6. Find $\log_e 100$, $\log_e 1000$, $\log_e 0.01$, $\log_e 2$. *Ans.* 4.60518, 6.90777, -4.60518, 0.69315.

7. Use the results of Art. 34 to compute $\log_e 3.65174$. *Ans.* 1.29521.

8. Given $\log_{10} 3$; find $\log_9 10$, $\log_{27} 1000$. *Ans.* $\dfrac{1}{2\log_{10} 3}$, $\dfrac{1}{\log_{10} 3}$.

9. Show how to obtain:
   logarithms to the base 8 from logarithms to the base 2;
   logarithms to the base 3 from logarithms to the base 9.
   Ans. Divide by 3; multiply by 2.

10. Prove that

$$\log_b a \cdot \log_c b \cdot \log_a c = 1.$$

11. Given $\log 4 = 0.60206$, $\log 9 = 0.95424$; find $\log 6$.

Ans. $\log 6 = 0.77815$.

37. Tables of Logarithmic Trigonometric Functions. In the solution of triangles and elsewhere we constantly encounter expressions involving trigonometric functions. Suppose it were required to compute the value of

$$x = \frac{325.6 \sin 23^\circ 45' \tan 18^\circ 24'}{\cos 37^\circ 30'}.$$

It is plain that we might first find the values of the several trigonometric functions from the table of natural functions; thus, —

$$\sin 23^\circ 45' = 0.4027, \quad \tan 18^\circ 24' = 0.3327, \quad \cos 37^\circ 30' = 0.7934.$$

With these values the expression above written becomes

$$x = \frac{325.6 \times 0.4027 \times 0.3327}{0.7934}.$$

Solving by logarithms we have

$$\begin{aligned}
\log 325.6 & & &= 2.51268 \\
\log 0.4027 &= \log \sin 23^\circ 45' & &= 9.60498 - 10 \\
\log 0.3327 &= \log \tan 18^\circ 24' & &= 9.52205 - 10 \\
\operatorname{colog} 0.7934 &= \operatorname{colog} \cos 37^\circ 30' & &= \underline{0.10051} \\
& & \log x &= 1.74022 \\
& & x &= 54.983.
\end{aligned}$$

It will be observed that in order to obtain $\log \sin 23^\circ 45'$ we consulted two different tables: first, the table of natural sines, to find $\sin 23^\circ 45' = 0.4027$; second, the table of logarithms, to find $\log 0.4027 = 9.60498 - 10$. To avoid the necessity of referring to two tables, the logarithms of the sines, cosines, tangents and cotangents have been separately calculated and arranged in a new table, known as the table of *logarithmic trigonometric functions* *. The table contains the values of the logarithms of the sines, cosines, tangents and cotangents of angles between 0° and 90°.

We know that the sine and cosine of every angle and the tangent of every angle less than 45°, as well as the cotangent of any angle between 45° and 90°, is less than unity, consequently the characteristics of the logarithms of these functions will be negative. To avoid negative characteristics, each negative characteristic has been replaced by a positive characteristic by adding 10. For example, while the true value of

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* The choice of the term is unfortunate. Trigonometric logarithms would be much better. It is of interest to know that Napier's tables, the first tables ever published, were tables of logarithmic trigonometric functions.

$$\log \sin 23^\circ 45' = -1 + 0.60498 = 9.60498 - 10,$$

the table gives

$$\log \sin 23^\circ 45' = 9.60498.$$

This latter value is called the *tabular logarithmic sine,* hence

To find the true value of a logarithmic sine the corresponding tabular logarithmic sine must be diminished by 10.

Since the secant is the reciprocal of the cosine, and the cosecant the reciprocal of the sine, we have

The logarithmic secant or cosecant may be obtained by taking the cologarithm of the cosine or sine respectively, that is, by subtracting the tabular logarithmic cosine or sine from 10.

38. To Find the Logarithmic Trigonometric Functions of an Angle less than 90°.

(*a*) *When the angle is less than 45°*, we find the number of degrees at the top of the page and the number of minutes in the left-hand column. The values of the log sin, log tan, log cot, log cos, as indicated at the head of the column, are then found in the same line as the minutes.

If seconds are given, they may be reduced to a fractional part of a minute, and the value of the logarithmic function may be found by interpolation, just as was done in finding logarithms of numbers.

Example 1. Find the log sin, log cos, log tan and log cot of 11° 21′.

Remembering that certain of the characteristics as given in the table are too large by 10, we find from table II

$$\begin{aligned}
\log \sin 11^\circ 21' &= 9.29403 - 10,\\
\log \tan 11^\circ 21' &= 9.30261 - 10,\\
\log \cot 11^\circ 21' &= 0.69739,\\
\log \cos 11^\circ 21' &= 9.99142 - 10.
\end{aligned}$$

EXAMPLE 2. Find $\log \sin 15^\circ 24' 36''$.

From the table we find,

$$\begin{aligned} \log \sin 15^\circ 24' &= 9.42416 \\ \log \sin 15^\circ 25' &= \underline{9.42461} \\ \text{difference for } 1' &= \qquad 45 \end{aligned}$$

$36'' = 0.6'$, difference for $0.6' = 0.6 \times 45 = 27$,

hence $\log \sin 15^\circ 24' 36'' = 9.42416 + 27 = 9.42443$.

EXAMPLE 3. To find $\log \cos 27^\circ 36' 40''$.

From the table we find

$$\begin{aligned} \log \cos 27^\circ 36' &= 9.94753 \\ \log \cos 27^\circ 37' &= \underline{9.94747} \\ \text{difference for } 1' &= \qquad 6 \end{aligned}$$

$40'' = 0.6\frac{2}{3}'$, difference for $0.6\frac{2}{3}' = 4$,

hence $\log \cos 27^\circ 36' 40'' = 9.94753 - 4 = 9.94749$.

Observe that in this example the difference was subtracted from the mantissa of the logarithm of the smaller angle, because the logarithm of the cosine decreases as the angle increases.

(*b*) *When the angle is more than 45° and less than 90°*, we find the degrees at the bottom of the page and the minutes in the right-hand column. The values of the log sine, log tangent, log cotangent and log cosine as indicated at the foot of the column are then found in the same line as the minutes.

EXAMPLE 4. Find $\log \tan 61^\circ 10' 27''$.

From the table we find

$$\begin{aligned} \log \tan 61^\circ 10' &= 0.25923 \\ \log \tan 61^\circ 11' &= \underline{0.25953} \\ \text{difference for } 1' &= \qquad 30 \end{aligned}$$

$27'' = 0.4\frac{1}{2}'$, difference for $0.4\frac{1}{2}' = 4\frac{1}{2} \times 30 = 13.5$,

hence $\log \tan 61^\circ 10' 27'' = 0.25923 + 13.5 = 0.25936$.

The difference 13.5 is additive because the log tangent increases with the angle.

Example 5. Find $\log \cot 75^\circ 51' 15''$.

$$\begin{aligned} \log \cot 75^\circ 51' &= 9.40159 \\ \log \cot 75^\circ 52' &= \underline{9.40106} \\ \text{difference for } 1' &= 53 \end{aligned}$$

$15'' = 0.25'$, difference for $0.25' = 13$,

hence $\log \cot 75^\circ 51' 15'' = 9.40146 - 10$.

The difference 13 is subtracted because $\log \cot 75^\circ 52'$ is less than $\log \cot 75^\circ 51'$.

In general when interpolating:

For *sines* and *tangents* the difference must be *added* to the function of the smaller angle, because the sine and tangent of an angle *increases as the angle increases.*

For *cosines* and *cotangents* the difference must be *subtracted* from the function of the smaller angle, because the cosine and cotangent of the angle *decreases as the angle increases.*

39. To Find the Angle Corresponding to a Given Logarithmic Trigonometric Function.

When the given number can be found in the table, the number of degrees is found at the top or bottom of the page, according as the name of the function appears at the top or bottom of the column. The number of minutes is found in the same line with the given function, — in the left column if the degrees are taken from the top of the page, in the right column if from the bottom.

When the given number cannot be found in the table, two consecutive numbers can always be found, one of which is slightly smaller, the other slightly larger than the given number. The required angle can then be found by interpolation.

Example 1. Given $\log \sin x = 8.73997$; to find x.

The number 8.73997 is found in the first column on page 30 of the tables. This column has log sin written at the top and log cos at the bottom. Since the given number is to be a log sine, the degrees are taken from the top of the page and the number of minutes from the left-hand column in line with the number 8.73997. We thus find $x = 3^\circ 09'$.

Example 2. Given $\log \cos y = 8.73997$; to find y.

The given number is the same as in Example 1, but this time it represents a log cosine. Therefore we take for the required angle the number of degrees at the bottom of the page and the number of minutes found in the right-hand column, opposite the number in the table. Result, $y = 86° 51'$.

It should be observed that the given mantissa 73997 appears a second time in the table, namely, on page 60, first column. But the numbers in that column have the characteristic 9, while the given characteristic is 8. Beginners sometimes overlook the characteristic and are thus led into mistakes.

EXAMPLE 3. Given $\log \tan x = 0.08685$; to find x.

The number 0.08685 cannot be found in the table, but on page 66, third column, are found the next smaller number 0.08673 and the next larger number 0.08699. The difference between these two is 26, and to the smaller of the two corresponds the angle $50° 41'$, that is,

$$\begin{array}{lrcr} & \text{mantissa } \log \tan 50° 41' & = & 08673 \\ \text{but} & \text{mantissa } \log \tan x & = & \underline{08685} \\ & \text{difference} & = & 12. \end{array}$$

If we denote by d'' the difference between $50° 41'$ and x, the principle of proportional parts gives us

$$d'' : 12 = 60'' : 26, \text{ that is, } d'' = \tfrac{12}{26} \times 60'' = 28'',$$

and the required angle is $x = 50° 41' 28''$.

EXAMPLE 4. Given $\log \cos x = 9.87561 - 10$; to find x.

$$\begin{array}{lrcr} & \text{mantissa } \log \cos 41° 19' & = & 87568 \\ & \text{mantissa } \log \cos 41° 20' & = & \underline{87557} \\ & \text{difference for } 1' \text{ or } 60'' & = & 11. \\ \text{Also} & \text{mantissa } \log \cos 41° 19' & = & 87568 \\ & \text{mantissa } \log \cos x & = & \underline{87561} \\ & \text{difference for } d'' & = & 7 \end{array}$$

and $d'' : 7 = 60'' : 11, \qquad d'' = \tfrac{7}{11} \times 60'' = 38'',$

hence the required angle is $x = 41° 19' 38''$.

EXERCISE 21

From the table of logarithmic trigonometric functions find:

1. log sin 13° 24′, log cos 25° 12′, log tan 10° 02′, log cot 17° 00′.
   Ans. 9.36502, 9.95657, 9.24779, 0.51466.
2. log sin 72° 11′, log tan 46° 17′, log cot 65° 13′, log cos 67° 59′.
   Ans. 9.97866, 0.01946, 9.66437, 9.57389.

3. log sin $25^\circ 12' 30''$, log tan $36^\circ 30' 15''$, log sin $70^\circ 15' 40''$.
Ans. 9.62932, 9.86928, 9.97370.

4. log cos $16^\circ 26' 45''$, log cot $9^\circ 27' 42''$, log cot $54^\circ 25' 09''$.
Ans. 9.98186, 0.77818, 9.85456.

5. log sin $42^\circ 15' 10''$, log cos $17^\circ 10' 54''$, log tan $51^\circ 18' 57''$.
Ans. 9.82763, 9.98017, 0.09653.

By means of the table find x, when there is given:

6. $\log \sin x = 9.77980$, $\log \tan x = 0.40017$, $\log \cos x = 9.79558$.
Ans. $37^\circ 02'$, $68^\circ 18'$, $51^\circ 21'$.

7. $\log \sin x = 9.68931$, $\log \cos x = 9.89609$, $\log \tan x = 0.16999$.
Ans. $29^\circ 16' 30''$, $38^\circ 04' 30''$, $55^\circ 56' 15''$.

8. $\log \tan x = 0.30562$, $\log \sin x = 8.97296$, $\log \sin x = 9.97296$.
Ans. $63^\circ 40' 36''$, $5^\circ 23' 30''$, $69^\circ 59' 24''$.

9. $\log \sin x = \log 2 + \log \sin 32^\circ 10' 15'' + \log \cos 32^\circ 10' 15''$.
Ans. $x = 64^\circ 20' 30''$.

10. Verify the relation

$$\log \cos 27^\circ 10' 40'' = \log (\cos 13^\circ 35' 20'' - \sin 13^\circ 35' 20'') + \log (\cos 13^\circ 35' 20'' + \sin 13^\circ 35' 20'').$$

(Suggestion. Use the natural function table to find the quantities within the parentheses.)

11. Given $\tan x = 4.6525$; find x without using a table of natural functions.

12. Given $(\sin x)^{\frac{2}{3}} = 0.253$; find x. *Ans.* $x = 7^\circ 18' 40''$.

40. Logarithmic Functions of Angles near 0° or 90°. When the angle is very small, say less than 2°, the logarithms of the sine, tangent and cotangent vary so rapidly that the principle of proportional parts fails to apply with sufficient accuracy, causing the results obtained by interpolation to be unreliable. The same remark applies to the logarithms of the cosine, cotangent and tangent of an angle near 90°, say between 88° and 90°. In such cases tables may be used which give the required functions for sufficiently small intervals of the angle, say for each second. When such tables are not available the following rules may be employed.

If the angle is less than 2° and only tenths of minutes are considered, —

The log sin or log tan of the angle may be found by increasing the logarithm of the number of minutes in the angle by 6.46373−10.

Conversely,

The angle corresponding to a given log sin or log tan may be found by diminishing the given log sin or log tan by 6.46373−10. *The resulting number is the logarithm of the number of minutes in the required angle.*

EXAMPLE 1. To find log sin 0° 50.7′.

$$\begin{aligned} \log 50.7 &= 1.70501 \\ &\ \underline{6.46373 - 10} \\ \log \sin 50.7' &= 8.16874 - 10 \end{aligned}$$

EXAMPLE 2. Given log tan $x = 8.50724 - 10$; to find x.

$$\begin{aligned} \log \tan x &= 8.50724 - 10 \\ &\ \underline{6.46373 - 10} \\ \log x &= 2.04351 \\ x &= 110.5' = 1^\circ\ 50.5'. \end{aligned}$$

These rules are based on the theorem, which will be proved later, that the sine and tangent of a small angle are each approximately equal to the length of the arc subtended by the angle at the center of a circle whose radius is the unit of measure, so that for very small angles the measure of the arc may be substituted for either the sine or the tangent of the corresponding angle. Now the measure of an arc in terms of the radius is found by multiplying the number of minutes in the arc by 0.0002909, for since the semi-circumference (the arc subtending an angle of 180° = 10800′) measures 3.14159 . . . when the radius is 1, each minute of arc measures 3.14159 ÷ 10800 = 0.0002909. Log 0.0002909 = 6.46373 − 10, consequently if the log sin or log tan of a small angle is increased by this amount the result will be the logarithm of the number of minutes in the corresponding angle.

When the angle is less than 2° and the exact number of seconds are considered the foregoing rule fails. In such cases a special table, known as the S and T table, is commonly employed.

41. Use of the S and T Table (Table III). For small angles the ratio of either the sine or tangent to the length of the arc of unit radius is nearly constant, that is

$$\frac{\sin x}{x}, \text{ as well as } \frac{\tan x}{x}, \text{ is nearly constant,}$$

x being the ratio of the length of the arc to the radius.

On taking logarithms we have
log sin x − log x, as well as log tan x − log x, is nearly constant.
Let us put

$$\log \sin x - \log x = S, \text{ and } \log \tan x - \log x = T,$$

then on transposing log x we obtain

$$\log \sin x = \log x + S \qquad (1)$$

and

$$\log \tan x = \log x + T. \qquad (2)$$

The values of S and T, corresponding to various values of x expressed in seconds, have been carefully calculated and assembled in Table III. The given angle x is first reduced to seconds, and the corresponding value of S or T is then taken from the table. This value increased by log x gives log sin x or log tan x, as the case may be.

Conversely, if log sin x or log tan x is given and x is to be found, we take S or T from the table, subtract it from log sin x or log tan x, as the case may be, and obtain log x, from which x is found.

The log cosine or log cotangent of an angle near 90° may be found from the same table by substituting for the cosine of the angle the sine of its complement, and for the cotangent of the angle the tangent of its complement. All this will be better understood by following through the examples which are worked out below.

The characteristics of S and T are negative, so that −10 must be appended to each value taken from the table.

Example 1

Given $x = 0° 56' 26''$; to find log sin x.

$$x = 0° 56' 26'' = 3386''.$$

Applying (1), $\log x = 3.52969$
From Table III, $S = 4.68556 - 10$

$$\log \sin x = 8.21525 - 10.$$

Example 2

Given log sin x = 8.21524 − 10; to find x.

$\log \sin x = 8.21524 - 10$
From Table III, $S = 4.68555 - 10$
By (1), $\log x = 3.52969$

$$x = 3386'' = 0° 56' 26''.$$

EXAMPLE 3

To find log tan 1° 10′ 51″.

$$1° \; 10' \; 51'' = 4251''.$$

Applying (2), $\log 4251 = 3.62849$

From Table III, $T = 4.68564 - 10$

$$\log \tan 1° \; 10' \; 51'' = 8.31413 - 10.$$

EXAMPLE 4

Given $\log \tan x = 8.31413 - 10$; to find x.

$$\log \tan x = 8.31413 - 10$$

From Table III, $T = 4.68564 - 10$

Subtracting, $\log x = 3.62849$

$$x = 4251'' = 1° \; 10' \; 51''.$$

EXAMPLE 5

To find log cos 89° 25′ 11″.

$$\log \cos 89° \; 25' \; 11'' = \log \sin 0° \; 34' \; 49''$$

$$0° \; 34' \; 49'' = 2089''.$$

Applying (1), $\log 2089 = 3.31994$

From Table III, $S = 4.68557 - 10$

$$\log \sin 0° \; 34' \; 49'' = 8.00551 - 10$$

$$\log \cos 89° \; 25' \; 11'' = 8.00551 - 10.$$

EXAMPLE 6

$\log \cos x = 8.00551 - 10$; to find x.

$$\log \cos x = \log \sin (90° - x).$$

$$\log \sin (90° - x) = 8.00551 - 10$$

From Table III, $S = 4.68557 - 10$

$$\log (90° - x) = 3.31994$$

$$90° - x = 2089'' = 0° \; 34' \; 49''$$

$$x = 90° - 0° \; 34' \; 49'' = 89° \; 25' \; 11''.$$

EXAMPLE 7

To find log cot 1° 0′ 29″.

$$\log \cot 1° \; 0' \; 29'' = \frac{1}{\log \tan 1° \; 0' \; 29''}$$

$$= - \log \tan 1° \; 0' \; 29''.$$

Then, as in Example 3,

$$\log \tan 1^\circ\, 0'\, 29'' = 8.24541 - 10$$
$$\log \cot 1^\circ\, 0'\, 29'' = 10 - 8.24541$$
$$= 1.75459.$$

EXAMPLE 8

$\log \cot x = 1.75459$; to find x.

$$\log \cot x = -\log \tan x = 1.75459,$$
$$\log \tan x = 8.24541 - 10.$$

Then, as in Example 4,

$$x = 1^\circ\, 0'\, 29''.$$

EXERCISE 22

1. Find $\log \sin 1^\circ\, 20'\, 02''$, $\log \tan 0^\circ\, 45'\, 45''$, $\log \cos 88^\circ\, 54'\, 36''$.
Ans. $8.36696 - 10$, $8.12414 - 10$, $8.27928 - 10$.

2. Find $\log \sin 0^\circ\, 50'\, 50''$, $\log \cos 89^\circ\, 01'\, 55''$, $\log \tan 1^\circ\, 15'\, 17''$.

3. What would have been the error in each of the functions in 2, if their values had been found from table II by interpolation?

4. Find each of $\log \tan 88^\circ\, 05'\, 20''$, $\log \cot 89^\circ\, 16'\, 50''$ from Table III, and check your result by Table II.

5. Given $\log \sin x = 8.22925 - 10$, $\log \tan y = 8.43340 - 10$; find x and y. *Ans.* $x = 0^\circ\, 58'\, 17''$, $y = 1^\circ\, 33'\, 14''$.

42. Historical Note. With the awakening of science in the sixteenth century, measurement became more precise, the resulting numbers more complex, and computation more and more tedious and time-consuming. The demand for shorter methods of computation than were then known led to the invention of logarithms. It is therefore not very strange that the method of logarithms should have been developed independently and almost simultaneously by two mathematicians, John Napier, a Scotchman, and Jost Buergi, a German. Napier's tables were published in 1614. Buergi's tables were computed before 1611 but not published till 1620. Both Napier's and Buergi's tables were soon superseded by Briggs' tables. Briggs' tables contained fourteen place logarithms of all numbers from 1 to 20,000 and from 90,000 to 100,000. Briggs' tables were completed by Adrian Vlacq (1628), who shortened Briggs' tables to ten places, and computed the logarithms of the remaining numbers from 20,000 to 90,000.

Briggs' and Vlacq's tables are substantially the same as the tables in use to-day, though the tables have been checked and parts of them recomputed many times. The most complete check was undertaken by the French authorities in 1784. It required the work of nearly one hundred mathematicians and computers for over two years. The resulting tables, giving fourteen place logarithms of all integers from 1 to 200,000, besides natural sines and logarithmic sines and tangents, have never been published. Two manuscript copies are preserved, one at the Observatory, the other at the Institute in Paris.

CHAPTER V

LOGARITHMIC SOLUTION OF RIGHT TRIANGLES AND APPLICATIONS

43. Logarithmic Solution of Right Triangles. In Article 20 it was shown how to solve right triangles by means of natural functions. Now we shall employ logarithms, which as a rule shortens the work. We shall illustrate each of the cases which may arise by an example.

Example 1.

Given $a = 316.5$,
$c = 521.2$.

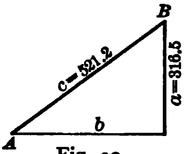

Fig. 30.

Required $A = 37^\circ\, 23.5'$,
$B = 52^\circ\, 36.5'$,
$b = 414.1$.

Solution.

(*a*) To find A and B.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} \log \sin A &= \log a + \operatorname{colog} c. \\ \log a &= 2.50037 \\ \operatorname{colog} c &= \underline{7.28300 - 10} \\ \log \sin A &= 9.78337 - 10 \\ A &= 37^\circ\, 23.5'. \\ B &= 52^\circ\, 36.5'. \end{aligned}$$

(*b*) To find b.

$$\cos A = \frac{b}{c}, \text{ or } b = c \cos A.$$

$$\begin{aligned} \log b &= \log c + \log \cos A. \\ \log c &= 2.71700 \\ \log \cos A &= \underline{9.90009 - 10} \\ \log b &= 2.61709 \\ b &= 414.1. \end{aligned}$$

(*c*) Check. If our result for b is correct, it must satisfy the relation $a^2 + b^2 = c^2$, or $c^2 - b^2 = (c + b)(c - b) = a^2$,

from which $$\log (c + b) + \log (c - b) = 2 \log a.$$

$$\begin{aligned} c + b &= 521.2 + 414.1 = 935.3, \\ c - b &= 521.2 - 414.1 = 107.1. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 2.97095 & \log a &= 2.50037 \\ \log (c - b) &= \underline{2.02979} & & \underline{2} \\ & 5.00074 & & 5.00074 \end{aligned}$$

Since b checks, A may be assumed to be correct also, for b was found from A.

EXAMPLE 2.

Given $a = 6.325$,
$b = 7.328$.

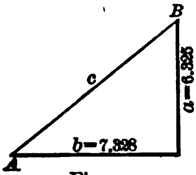

Fig. 31.

Required $A = 40° 47.9'$,
$B = 49° 12.1'$,
$c = 9.680$.

Solution.

(*a*) To find A and B.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned}
\log \tan A &= \log a + \operatorname{colog} b.\\
\log a &= 0.80106\\
\operatorname{colog} b &= \underline{9.13501 - 10}\\
\log \tan A &= 9.93607 - 10\\
A &= 40° 47.9'.\\
B &= 49° 12.1'.
\end{aligned}$$

(*b*) To find c.

$$\sin A = \frac{a}{c}, \text{ or } c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log c &= \log a + \operatorname{colog} \sin A.\\
\log a &= 0.80106\\
\operatorname{colog} \sin A &= \underline{0.18483}\\
\log c &= 0.98589\\
c &= 9.680.
\end{aligned}$$

(*c*) Check. $c^2 - b^2 = a^2$, $\log (c + b) + \log (c - b) = 2 \log a$.

$$\begin{aligned}
c + b &= 9.680 + 7.328 = 17.008,\\
c - b &= 9.680 - 7.328 = 2.352.
\end{aligned}$$

$$\begin{aligned}
\log (c + b) &= 1.23065\\
\log (c - b) &= \underline{0.37144}\\
&\ 1.60209
\end{aligned}
\qquad\qquad
\begin{aligned}
\log a &= 0.80106\\
&\underline{2}\\
&\ 1.60212
\end{aligned}$$

Here there is a slight discrepancy in the check. The discrepancy is in the last figure only; that is, if the mantissas of the final results are cut down to four figures, each side is 1.6021. Agreement in the first four figures of the mantissas of the final logarithms of the check is all that can be expected when a five-place table is used.

EXAMPLE 3.

Given $c = 35.145$,
$A = 25° 24' 30''$.

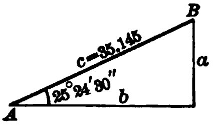

Fig. 32.

Required
$a = 15.079$,
$b = 31.745$,
$B = 64° 35' 30''$.

Solution.

(*a*) To find a.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned}
\log a &= \log c + \log \sin A\\
\log c &= 1.54586\\
\log \sin A &= \underline{9.63252 - 10}\\
\log a &= 1.17838\\
a &= 15.079.
\end{aligned}$$

(*b*) To find b.

$$\cos A = \frac{b}{c}.$$

$$\begin{aligned}
\log b &= \log c + \log \cos A.\\
\log c &= 1.54586\\
\log \cos A &= \underline{9.95582 - 10}\\
\log b &= 1.50168\\
b &= 31.745.
\end{aligned}$$

(*c*) Check.

$$\log (c + b) + \log (c - b) = 2 \log a.$$

$$c + b = 66.890, \quad c - b = 3.400.$$

$$\begin{array}{rlcrl} \log (c + b) &= 1.82536 & & \log a &= 1.17838 \\ \log (c - b) &= \underline{0.53148} & & & \underline{2} \\ & \ 2.35684 & & & \ 2.35676 \end{array}$$

Example 4.

Given $b = 25.01$,
$B = 65° 10'$.

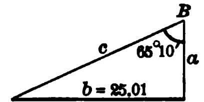

Fig. 33.

Required $a = 11.57$,
$c = 27.56$,
$A = 24° 50'$.

Solution.

(*a*) To find *a*.

$$\tan B = \frac{b}{a}.$$

$$\begin{array}{rl} \log a &= \log b + \log \cot B. \\ \log b &= 1.39811 \\ \log \cot B &= \underline{9.66537 - 10} \\ \log a &= 1.06348 \\ a &= 11.574. \end{array}$$

(*b*) To find *c*.

$$\sin B = \frac{b}{c}.$$

$$\begin{array}{rl} \log c &= \log b + \text{colog} \sin B. \\ \log b &= 1.39811 \\ \text{colog} \sin B &= \underline{0.04214} \\ \log c &= 1.44025 \\ c &= 27.56. \end{array}$$

(*c*) Check.

$$\log (c + b) + \log (c - b) = 2 \log a.$$

$$c + b = 52.568, \quad c - b = 2.548.$$

$$\begin{array}{rlcrl} \log (c + b) &= 1.72072 & & \log a &= 1.06348 \\ \log (c - b) &= \underline{0.40620} & & & \underline{2} \\ & \ 2.12692 & & & \ 2.12696 \end{array}$$

Example 5.

Given $c = 34.57$,
$b = 34.04$.

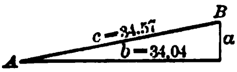

Fig. 34.

Required $A = 10° 02.7'$
$B = 79° 57.3'$
$a = 6.030$.

Solution. In this problem the given side and hypotenuse are so nearly equal that the method employed in Example 3 does not give sufficiently accurate results. We therefore use the formulas of Art. 21.

(*a*) To find A and B.

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}.$$

$$\log\tan\frac{A}{2} = \frac{1}{2}[\log(c-b) + \operatorname{colog}(c+b)].$$

$$\begin{aligned}
c - b &= 34.57 - 34.04 = 0.53,\\
c + b &= 34.57 + 34.04 = 68.61.
\end{aligned}$$

$$\begin{aligned}
\log(c-b) &= 9.72428 - 10\\
\operatorname{colog}(c+b) &= \underline{8.16361 - 10}\\
2)&\underline{17.88789 - 20}\\
\log\tan\tfrac{1}{2}A &= 8.94394 - 10
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 5^\circ\ 01'\ 22''\\
A &= 10^\circ\ 02'\ 44'',\ \text{or } 10^\circ\ 2.7' \text{ to the nearest tenth minute.}\\
B &= 79^\circ\ 57'\ 16'',\ \text{or } 79^\circ\ 57.3' \text{ to the nearest tenth minute.}
\end{aligned}$$

(*b*) To find a.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned}
\log a &= \log c + \log\sin A.\\
\log c &= 1.53870\\
\log\sin A &= \underline{9.24163 - 10}\\
\log a &= 0.78033\\
a &= 6.030.
\end{aligned}$$

(*c*) Check.

$$\log(c+b) + \log(c-b) = 2\log a.$$

$$\begin{aligned}
c + b &= 34.57 + 34.04 = 68.61,\\
c - b &= 34.57 - 34.04 = 0.53.
\end{aligned}$$

$$\begin{aligned}
\log(c+b) &= 1.83639 & \log a &= 0.78033\\
\log(c-b) &= \underline{9.72428 - 10} & &\underline{2}\\
&\ 1.56067 & &\ 1.56066
\end{aligned}$$

EXERCISE 23

Solve the following problems by logarithms, computing all angles to the nearest second and all sides to five significant figures. It is expected that the student check his results when no answers are given.

1. $a = 168.92$, $c = 289.64$.
   Find $A = 35^\circ\ 40'\ 33''$, $B = 54^\circ\ 19'\ 27''$, $b = 235.28$.
2. $a = 43.148$, $b = 84.107$. Find $A = 27^\circ\ 09'\ 29''$, $c = 94.530$.
3. $b = 2.5346$, $c = 3.7132$. Find a, A, B.
4. $a = 0.37640$, $b = 0.28634$. Find A and B.
5. $a = 547.5$, $A = 32^\circ\ 15'\ 24''$. Find $b = 867.10$, $c = 1025.4$.
6. $a = 6700$, $B = 27^\circ\ 30'$. Find b and c.

7. $c = 672.34$, $A = 35° 16' 25''$. Find $B = 54° 43' 35''$, $a = 388.26$, $b = 548.90$.

8. $c = 1.001$, $B = 45° 45' 45''$. Find a and b.

9. $c = 369.27$, $b = 235.64$. Find $A = 50° 20' 54''$, $a = 284.31$.

10. $c = 5464.35$, $a = 5452.13$. Find $B = 3° 49' 57''$.

44. Number of Significant Figures. Numerical problems are of two kinds:

(*a*) Those in which the given numbers are *exact* numbers.

(*b*) Those in which the given numbers are *approximate* only. For example, when we say that each of the sides of a hexagon inscribed in a circle with unit radius is 1, and each angle 120°, 1 and 120 are exact numbers, that is, the sides in question are to be considered neither more nor less than 1, and the angles neither more nor less than 120°. On the other hand, when we say that the side of a field measures 631.7 feet and the angle at a corner 73° 37′, the numbers 631.7 and 73° 37′ are mere approximations. So far as we know the exact length of the measured side may be any number between 631.65 and 631.75 and the measured angle may have any value between 73° 36.5′ and 73° 37.5′.

I. *When the given numbers of a problem are exact numbers*, the results asked for can be carried out to as many significant figures as the number of figures in the mantissas of the logarithms used in the solution. In this book, where the computations are based on a five-place table,* lengths must be limited to five significant figures and angles to seconds. Even then the fifth figure and the number of seconds cannot always be relied upon.

II. *When the given numbers of a problem are the results of measurement*, the answers need not contain more significant figures than the least accurate of the given parts. Thus, if one side of a triangle is measured to the nearest inch and another to the nearest tenth of an inch, the answer for the third side need only be given to the nearest inch. The following directions will assist the student to make consistent measurements and to avoid useless calculations.

1. Distances expressed to *three figures* call for angles expressed to the nearest *five minutes*, and vice versa.

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* Five-place tables answer most of the demands of applied science. The instruments ordinarily used by engineers read angles to the nearest minute only.

2. Distances expressed to *four figures* call for angles expressed to the nearest *tenth of a minute*,[*] and vice versa.

3. Distances expressed to *five figures* call for angles expressed to the nearest *second*, and vice versa.

4. Distances expressed to *six figures* call for angles expressed to the nearest *tenth of a second*, and vice versa. A six-place table must be used to obtain like accuracy in the answers.

5. Distances expressed to *seven figures* call for angles expressed to the nearest *hundredth of a second*, and vice versa. A seven-place table is necessary to obtain like accuracy in the answers.

In this connection the student should observe that, whenever a number is the result of measurement or other approximation, a cipher to the right of a decimal fraction has a distinct significance and cannot be dropped at will, as is customary in dealing with exact numbers. For example, the square root of 3 is approximately represented by each of the numbers 1.7, 1.73, 1.732, 1.7320, 1.73205, etc., the approximation being closer the more figures we write; but 1.7, when used as an approximation for $\sqrt{3}$, has not the same meaning as 1.70, for the former means that $\sqrt{3}$ has some value between 1.65 and 1.75, while the latter means that the number represented has some value between 1.695 and 1.705, which is not true. Similarly the numbers 62, 62.0, 62.00, when they represent measures of distances or other quantities, are not equivalent. The first implies that the measurement has been carried out to the nearest unit, the second to the nearest tenth, and the third, 62.00, that the measurement has been made to the nearest hundredth of a unit.

45. Applied Problems Involving Right Triangles. The following six sections deal with applied problems involving the solution of right triangles. The problems are grouped with reference to the questions dealt with, and the problems in each set are so arranged that the more difficult come last. It is not expected that any one student work every problem, but only as many as may be necessary to make him reasonably familiar with the method of solving right triangles by means of logarithms. After that an additional hour or two may profitably be spent in the analysis of the more difficult problems involving two or more right triangles. In problems where no answer is given the *result must be checked* by the student.

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[*] The nearest 10″ is somewhat closer.

46. Heights and Distances

Exercise 24

1. From a point 185 feet from the foot of a wireless telegraph mast, the top of the mast was found to form an angle of 52°. Find the height of the mast. *Ans.* 237 ft.

2. A man walking along a straight road observes a church in a direction making an angle of 50° with the road. After walking another mile, he comes to the crossroad on which the church is located. The roads cross at right angles. How far is the church from the intersection of the roads? *Ans.* 1.19 miles.

3. The summit of a mountain, known to be 14,450 feet high, is seen at an angle of elevation of 29° 15′ from a camp located at an altitude of 6935 feet. Compute the air-line distance from the camp to the summit of the mountain. *Ans.* 2.9 miles.

4. The ratio of the height of a roof to its span is one-fourth (quarter pitch), what is the inclination of the roof to the horizontal line? *Ans.* 26° 34′.

5. During a storm a tree was broken into two parts which remained connected. The broken part made an angle of 35° with the ground and its top reached a mark 165 feet from the foot of the tree. Required the height of the remaining stump and the height of the tree before it broke. *Ans.* 116 ft., 317 ft.

6. In constructing a grand-stand, timbers 28 ft. long are to be inclined at an angle of 25° and supported by four uprights, one at each end and two at equal distances between the two ends. How far apart must the uprights be placed and what are their lengths, the shortest being 6 ft. long?

7. A flagpole 25 ft. long stands on a building whose height is unknown. From a point at the same level as the foot of the building the angles of elevation of the top and bottom of the flagpole are measured and are found to be 57° and 53° respectively. Required the height of the building.

Ans. By natural functions, 156 ft.

8. From the top of a tower the angle of depression of a point in the same horizontal plane with the base of the tower is observed to be 47° 13′. What will be the angle of depression of the same point as seen from a position halfway up the tower? *Ans.* 28° 23′.

9. A spherical balloon whose radius is 10 ft. subtends an angle of 1° 46′, while from the same position and at the same time the angle of elevation of the center of the balloon is 54°. Determine the height of the center of the balloon. *Ans.* 525 ft.

10. An observer finds that the top of a spire due south of him has an angle of elevation of 25° 36′. He goes to a point 650 ft. east of his first position and now finds that the spire bears 40° 12′ southwest. Find the height of the spire.

11. It was found that the shadow of a tall factory chimney lengthened 85 ft. while the sun's elevation changed from 59° to 42°. Required the height of the chimney. *Ans.* 167 ft.

47. **Problems for Engineers.** It is suggested that the student use the graphic method in checking the problems in this set to which no answers are given.

Exercise 25

1. A branch railroad is to be constructed from a point *A* to a second point *B* which is 5.95 miles east and 9.36 miles north of the first. What will be the direction of the road, assuming that it follows a straight line? *Ans.* N. 32°27′ E.

2. To determine the width of a stream a surveyor measures a line *AB* 375 ft. long along one bank. At *B* he turns a right angle and his assistant places a stake in the line of sight at *C* on the opposite bank of the stream. The angle *BAC* measures 64° 42′. How wide is the stream? *Ans.* 793 ft.

3. On a map on which 1 inch represents 1000 ft., contour lines are drawn for differences of 100 ft. in altitude. What is the actual inclination of the surface represented by that portion of the map at which the contour lines are one-fourth inch apart? *Ans.* 21° 48′.

4. A bolt 2 inches in diameter has six threads to the inch. What is the inclination of the thread to a cross section of the bolt? *Ans.* 1° 31.2′.

5. A car track runs from *A* to *B*, a horizontal distance of 1275 ft. at an incline of 7° 45′, and then from *B* to *C* a horizontal distance of 1585 ft. *C* is known to be 509 ft. above *A*. What is the average inclination of the track from *B* to *C*?

6. Two towns A and B, of which B is 25 miles northeast of A, are to be connected by a new road. Ten miles of the road is constructed from A in the direction N. 23° E., what must be the length and direction of the remainder of the road, assuming that it follows a straight line? *Ans.* 16.17 miles, N. 58° 23.8′ E.

7. A surveyor wishes to ascertain the distance between two inaccessible objects A and B. He starts from a point C in a straight line with A and B and measures in a direction at right angles to AB a distance CD equal to 500 ft. At D he measures the angles subtended by AC and BC and finds them to be 75° 35′ and 34° 46′ respectively. Find the distance between A and B on the supposition,—

(*a*) That C is between A and B,
(*b*) That B is between A and C.

Ans. (*a*) 2292, (*b*) 1598.

8. One end of a connecting rod AB, 5 ft. long, is fastened to a crank BO, 1 ft. long, while the other end is fastened to a crosshead A which is constrained to move along AO. How far from the extreme position P of the crosshead will A be,—

(*a*) When OB is perpendicular to AB?
(*b*) When angle $BOA = 60°$?

Ans. (*a*) $\sqrt{26} - 4 = 1.099$ ft.
(*b*) $\frac{1}{2}\sqrt{97} + \frac{1}{2} - 4 = 1.324$ ft.

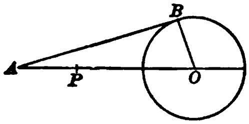

Fig. 35.

(Suggestion. In case (*b*) drop a perpendicular BC from B to AO, find OC and BC, then from the triangle ABC find AC.)

9. Two railroad tracks intersect at an angle of 54° 16′. They are to be connected by a curve AB of 100 ft. radius. Find how far from the intersection point O of the tracks the curve begins and the length of the curve.

Ans. $OB = 51.25$ ft. Arc $AB = 94.71$ ft.

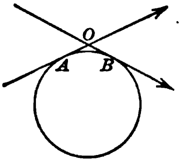

Fig. 36.

10. Two pulleys whose radii are 18 inches and 30 inches respectively are 8 ft. apart from center to center. Find the length of the belt connecting the pulleys,—

(*a*) If the pulleys are to turn in the same direction,
(*b*) If the pulleys are to turn in opposite directions.

48. Applications from Physics.

EXERCISE 26

1. The horizontal distance between the two extreme positions of a pendulum 39.1 inches long is 5.73 inches. Through what angle does it swing? *Ans.* 8° 24′.

2. Two forces of 10 and 24 lbs. respectively act at right angles to each other. Find the resultant force, and also the angle which the resultant makes with the first of the two given forces.
Ans. 26 lbs., 67° 23′.

3. What force is necessary to roll a barrel weighing 500 lbs. onto a platform 6 ft. high along an inclined ladder 12 ft. long?
Ans. 250 lbs.

4. A ball weighing 300 lbs. rests on a smooth plane inclined at an angle of 12° 30′ to the horizontal. What force is necessary to keep the ball from rolling down the plane,

(*a*) If the force acts parallel to the inclined plane?

(*b*) If the force acts in a horizontal direction?

Ans. (*a*) 64.93 lbs., (*b*) 66.51 lbs.

5. A block of wood rests on an adjustable inclined plane. As the inclination of the plane reaches 29° 37′ the block begins to slide. Find the coefficient of friction.*

6. An automobile moving at the rate of 45 miles per hour is overtaken by a shower. As seen from the automobile the raindrops seem to come down at an angle of 30° with the vertical. Find the velocity of the raindrops, assuming that their actual direction is vertical. *Ans.* 37.5 ft. per sec.

7. Through what angle must a fir log 30 ft. long and 54 inches in diameter, standing on end, be tilted before it begins to fall? The log is assumed to be cylindrical in shape. *Ans.* 7° 33′.

8. According to Wollaston the intensity of sunlight is equal to 61,000 standard candles acting at a distance of 1 meter. What is the intensity of sunlight striking a surface at an angle of 31° 08′ 27″?
Ans. 52210 c.p.

9. The fans of a windmill are inclined 25° to the plane of the wheel which is at right angles to the direction from which the wind

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* The coefficient of friction is equal to the tangent of the angle of inclination of the plane on which the block rests.

blows. What fraction of the wind's force is effective in turning the wheel? *Ans.* 0.383.

10. A weight of 437 lbs. is suspended and pushed 17° 30′ out of the vertical by a horizontal force. Required the horizontal force necessary to hold the body in this position.

11. What is the displacement *CM* of a ray of light *AB* in passing through a glass plate *PQ*, 0.215 inches thick, at an angle of 55° 47′ with the perpendicular *EB*, the index of refraction * from air to glass being approximately $\frac{3}{2}$. *Ans.* 0.098 in.

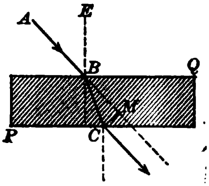

Fig. 37.

12. A sail ship sails against the wind at an angle of 60°. The sails are set so as to make an angle of 15° with the direction *OV* of the ship. What part of the wind's force is effective in producing the forward motion of the ship? *Ans.* 0.183.

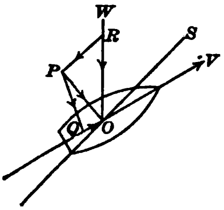

Fig. 38.

(Suggestion. Let *WO* be the direction of the wind, *OS* the direction in which the sail is set. Decompose a unit of the wind's force *RO* into two components, *RP* parallel to the sail and *PO* perpendicular to the sail. *PR* has no effect on the sail, and may therefore be disregarded. The other component *PO* may again be resolved into two components, namely, *PQ* perpendicular to the direction of the ship and *QO* in the direction of the ship. *PQ* is neutral so far as the forward motion of the ship is concerned, leaving *QO* as the only part of the wind's force effective in the direction *OV*.)

13. A person whose eye is at *E*, 10 ft. above the level of the water *PI*, observes at *I* the image of the foot of a pile driven in the water. The horizontal distance of the observer, from the place where the image is formed, is 20 ft., his distance from the pile is 65 ft. What is the length *PF* of the pile below the surface of the water, the refractive index from air to water being approximately $\frac{4}{3}$.

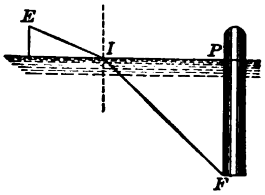

Fig. 39.

Ans. 49.7 ft.

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* Index of Refraction $= \frac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}}$.

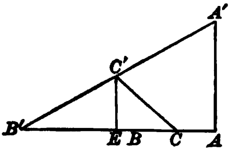

Fig. 40.

14. Two bodies A, weighing 2 lbs., and B, weighing 3 lbs., are so placed that B is exactly 10 ft. west of A. A moves north and B west, each at the rate of 12 ft. per second. What is the direction and the velocity of their common center of gravity?

Ans. 33° 41′; 8.65 ft. per sec.

(Suggestion. Locate the center of gravity, C, in two positions, as C and C'. Find EC' and EC, then solve the triangle CEC'.)

15. The arms of a lever are $FA = 2.34$ and $FB = 5.27$ respectively. At the extremity A of the first arm, a force of 5.34 units acts in a direction making an angle of $\alpha = 63° 45'$ with FA produced. What force must be applied at B, the extremity of the second arm, in a direction making an angle $\beta = 51° 15'$ with FB produced, in order that there may be equilibrium?

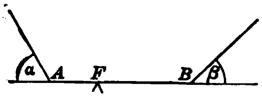

Fig. 41.

16. Six forces, $A = 15$, $B = 6$, $C = 5.7$, $D = 7.9$, $E = 12.3$, $F = 10$, act on the same point and in the same plane.
The angle between A and B is 12° 30′,
the angle between A and C is 31° 21′,
the angle between A and D is 47° 46′,
the angle between A and E is 58° 10′,
the angle between A and F is 72° 18′.
Required the resultant force and the direction it makes with A.

Ans. 50.51, 36° 34′.

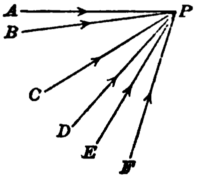

Fig. 42.

(Suggestion. Resolve each force into two components, one along PA, the other in a direction perpendicular to PA. Sum the components along each of these directions separately. The sums are the rectangular components of the required force.)

49. Problems in Navigation. In the following problems it is assumed that the student is acquainted with the divisions of the mariner's compass. On the mariner's compass the total angular space about a point is divided equally into 32 divisions, each of which is called a *point*, that is, a point is equivalent to $\frac{360°}{32} = 11° 15'$.

Each point is divided into two half-points, each half-point into two quarter-points. In the figure below, the names of the 32 points are indicated by their abbreviations. Between north and east the points read:

North by east, north northeast, northeast by north, northeast, northeast by east, east northeast, east by north, and similarly for each of the other quadrants.

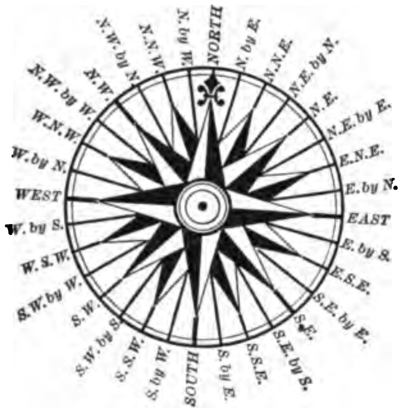

Fig. 43.

In the following problems the surface of the earth is considered a plane and the distances straight lines, not arcs. By a mile is understood a sea mile or knot, which is the length of a minute of arc measured on the earth's equator so that the earth's circumference measures exactly $360 \times 60 = 21,600$ sea miles. A sea mile is approximately $1\frac{1}{8}$ common miles.

Definitions. The east and west component of a course, or distance between two points, is called the DEPARTURE of the course or distance, the north and south component is called the difference in LATITUDE, that is, if *WN* represents any course or distance, and a right triangle *WSN* is formed, by drawing through *W* a line east and west, and through *N* a line north and south, *WS* is called the departure, and *SN* the difference in latitude of the course or distance *WN*.

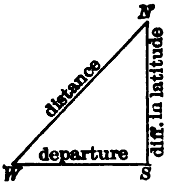

Fig. 44.

In nautical problems difference in latitude is usually expressed in degrees and minutes (1 mile = 1′).

EXERCISE 27

1. A ship sails N.E. by E. at the rate of 8 knots per hour. Find the rate at which it is moving due north.

Ans. $4\frac{4}{9}$ knots per hour.

2. A ship sails S.E. by S. a distance of 578 miles. Find its departure. *Ans.* 321.1 miles.

3. A vessel sails W. by S. until the departure is 315 miles. Find the actual distance sailed. *Ans.* 321.2 miles.

4. A ship sails from latitude 47°30′ N. on a course N.W. by N. 685 miles. Find the latitude arrived at.

Ans. Diff. in latitude $= 569.6$ miles $= 569.6' = 9° \ 29.6'$.
Required latitude $= 47° \ 30' + 9° \ 29.6' = 56° \ 59.6'$.

5. A ship sails S.W. by S. a distance of 1225 miles. Find the difference in latitude between the first and last positions of the ship and the departure made.

6. A ship sails from latitude 10° 24′ N. and after 30 hrs. reaches latitude 15° 26′ N. Its course was N.N.E. Find the average speed of the ship. *Ans.* 6.57 miles per hr.

7. A ship sails from latitude 35° 58′ N. on a course between S. and E. a distance of 359 miles to a point whose latitude is 32° 16′ N. Find the course of the ship. *Ans.* S. 51° 48′ E.

8. A vessel sails from latitude 5° 21′ S. on a course N.E. by N. a distance of 976 miles. Find the new latitude and the departure.

Ans. 8° 11′ N., 542.3 miles.

9. A steamer bearing W. by N. with a speed of 12 knots has a current setting port * broadside across her track which after 5 hours brings her to an island located 108 miles from her starting point. Find the true course of the ship. *Ans.* N.N.W.

10. One port *A* is 19 miles due N. of a second port *B*. Two vessels leave the two ports at the same time, one from *B* sailing due E. at the rate of 9 knots an hour, the other from *A*. The vessels meet 5 hours out of port. Determine the speed and the course of the second vessel. *Ans.* 9.77 knots, S. 67° 07′ E.

11. From a "crow's nest" 110 ft. above the water, the angle of depression of a rock just above the water was found to measure

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* The left-hand side of the ship as one faces ahead, — opposed to starboard.

15° 36′. Find the distance from the rock to the foot of the mast. *Ans.* 394 ft.

12. A ferry, whose speed in still water is 4 miles per hour, crosses a channel whose current is 3½ miles per hour. How much will she have to "bear up" in order to make the run straight across, and how long will it take her to cross, the channel being 7 miles wide?

13. An observer on board ship notices that the time between the flash of a gun from a fort located N.W. by W. and the report is 5 seconds. After sailing N.E. by N. the gun was heard again, and this time the interval between the flash and the report was 10 seconds. Find the distance sailed and the bearing of the fort from the second position of the ship. *Ans.* 9440 ft., S. 63° 45′ W.

(Assume the velocity of sound to be 1090 ft. per second.)

14. A ship sailing due N. observes two lighthouses in a line due W., and two hours later the bearings of the lighthouses are found to be S. by W. and S.W. by W. respectively. The distance between the lighthouses is known to be 10 miles. Find the rate at which the ship is moving.

15. A man-of-war sailing due N.E. at a uniform speed of 20 knots observes at 9.30 A.M. a fort bearing N.N.W. Twenty-four minutes later the fort is due N.W. Find the distance and bearing of the fort from the ship at 10.15 A.M. *Ans.* 20.54 miles, N. 64° 55′ W.

50. Geographical and Astronomical Problems.

EXERCISE 28

1. The shadow of a vertical pole 35 ft. high is 51 ft. long. What is the sun's altitude (angle of elevation)? *Ans.* 34° 27.6′.

2. In Fig. 45, let the circle center E represent the earth and the circle center M the moon. The angle PME, formed by the line of centers EM and a line drawn from M tangent to the earth, is known as the moon's equatorial horizontal parallax and measures 57′ 02″. EP, the earth's mean radius, is 3959 miles. Determine EM, the distance of the moon from the earth. *Ans.* By use of S and T table, 238,650 miles.

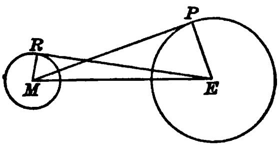

Fig. 45.

3. In Fig. 45, the angle REM, formed by the line of centers ME and a line drawn from E tangent to the moon, is known as the moon's angular semidiameter and measures $15'\ 34''$. Use the result of the last problem and determine RM, the moon's radius.

Ans. By use of S and T table, 1080.6 miles.

4. The sun's equatorial horizontal parallax (see Problem 2) is $8.8''$. The radius of the earth is 3959 miles. Find the distance of the sun from the earth. Also the sun's diameter, the angular semidiameter being $16'\ 02''$. *Ans.* 92,798,000 miles, 865,620 miles.

5. The largest angle between Venus and the sun as seen from the earth is $47^\circ\ 30'$. Using the sun's distance as given in Problem 4, find the distance of Venus from the sun, the orbit of Venus being assumed circular.

6. In Fig. 46, EO represents the earth's radius (= 3959 miles). Find AP, the radius of the arctic circle, latitude $66^\circ\ 32'$. *Ans.* 1577 miles.

7. Prove that lengths of two parallels of latitude are to each other as the cosines of the latitudes.

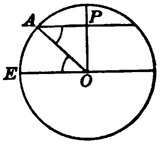

Fig. 46.

8. One degree of longitude on the equator is approximately 69.1 miles. Determine the length of a degree of longitude at Seattle, $47^\circ\ 40'$ N. latitude. *Ans.* 46.5 miles.

9. Prove that the lengths of the degrees of longitudes at different latitudes are to each other as the cosines of the latitudes.

10. If one minute of arc of longitude in latitude 60° measures 1012.7 yards, how long is the radius of the earth, assuming the earth to be a sphere?

11. A ship sails due W. 540 miles in latitude 36° N. What is the difference in longitude between the initial and final positions of the ship? *Ans.* $9^\circ\ 40'$.

12. How high above the Pole would an observer have to be to have the Arctic Circle for his horizon? (Use the data of Problem 6.)

Ans. 357 miles.

13. Beginning at latitude 40° N., two consecutive section lines run directly north for a distance of 100 miles. How far apart are they at their northern end? *Ans.* 5166 ft.

14. The shortest shadow cast by a vertical rod 25 ft. long at noon is 21 ft., the longest shadow cast by the same rod at noon is 56 ft. Find the approximate latitude of the place.

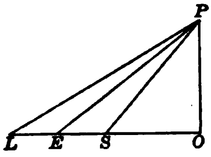

Fig. 47.

Explanation. The shortest shadow, *OS*, Fig. 47, is cast on June 21, the longest shadow, *OL*, on December 21. Halfway between these dates the sun will be on the equator; its elevation above the horizon at noon will then be the latitude of the place, that is, angle *OEP* in the figure, where *EP* bisects the angle *LPS*. *Ans.* 52° 59′.

15. A wall runs east and west; its shadow, measured at right angles to the wall, is 10 ft. wide. The altitude of the sun is 25° 30′, its azimuth (angular distance west of the south point) is 27° 45′. Determine the height of the wall.

(Suggestion. Let *WE* represent the wall, *FL* the width of the shadow measured at right angles to the wall, *MP* the direction of the sun. Then angle *FMP* is the sun's altitude and angle *MFL* its azimuth. The problem involves two right triangles, namely, *MFL* and *PFM*, in which *FL*, angle *MFL* and angle *PMF* are given and *PF* is to be found.)

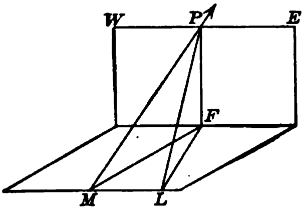

Fig. 48.

16. Each of two observers notices a bright meteor. To the first observer, A, the meteor appeared directly south and at an elevation of 54°. To the second observer, B, stationed 40 miles west of A, the meteor appeared 56° east of the South point. On comparing the times of observation it was ascertained that the same meteor had been observed. Compute the height at which the meteor was seen.

(Suggestion. In Fig. 48 consider *L* to be A's position, *M*, B's position, and *P* the position in which the meteor was observed. *PF* represents the height of the meteor.)

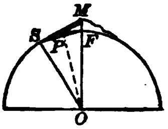

Fig. 49.

17. Find the greatest distance at sea at which a mountain 14,500 ft. high can be seen, the earth being considered a sphere, radius 3960 miles, and the distance sought being the chord joining the point at sea to the foot of the mountain.

Ans. By S and T functions, 137.4 miles.

(Suggestion. The distance sought is $SF = 2 \cdot SO \cdot \sin \frac{1}{2}(SOF)$ and $\cos SOF = \frac{OS}{OM}$, but as OS and OM are nearly equal, it is better to use the formula in Article 21 for the determination of angle SOF.)

51. Geometrical Applications. Many geometrical problems can be solved by properly dividing the given figures into right triangles and solving these. Thus, an isosceles triangle is divided into two equal right triangles by drawing a perpendicular from the vertex to the base; any rhombus is divided into four right triangles by its two diagonals; any oblique triangle is equal to the sum or difference of two right triangles, formed by drawing the perpendicular from any vertex to the opposite side; any regular ploygon is divided into as many equal isosceles triangles as the figure has sides, by the lines joining the vertices of the polygon to its center; etc.

EXAMPLE 1. Two sides of a rhombus meet at an angle. The length of one side is a. Find the lengths of the diagonals.

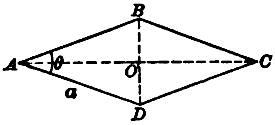

Fig. 50.

Solution. By definition the sides of a rhombus are equal and from geometry it is known that the diagonals intersect at right angles and bisect the angles of the rhombus.

Let $BAD = \theta$ be the given angle, and $AB = AD = a$, the given side.

In the right triangle ABO, two parts, namely, the hypotenuse a and the acute angle $\frac{\theta}{2}$ are known, hence we may find AO and OB.

$$AO = a \cos \tfrac{1}{2}\theta, \quad \text{and} \quad BO = a \sin \tfrac{1}{2}\theta,$$

hence

$$AC = 2\,AO = 2\,a \cos \tfrac{1}{2}\theta, \quad \text{and} \quad BD = 2\,BO = 2\,a \sin \tfrac{1}{2}\theta.$$

In particular, if the given side is 15 and the given angle 38°, we have

$$AC = 30 \cos 19^\circ = 28.365,$$

$BD = 30 \sin 19^\circ = 9.768$, by the use of natural functions.

EXAMPLE 2. The radius of a circle is r. To find the perimeter and area of a regular inscribed polygon of n sides.

Solution. By definition the sides of a regular polygon are equal.

Let O represent the center of the circle and AB one side of the inscribed polygon.

The angular magnitude about O is 360° and, since there are n sides, angle $AOB = \frac{360^\circ}{n}$.

Triangle AOB is isosceles, so that if OC is drawn from O perpendicular to AB, it will divide the triangle AOB into two equal right triangles.

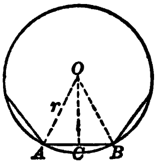

Fig. 51.

In the right triangle AOC, the hypotenuse equals r and the acute angle AOC equals $\frac{360^\circ}{2n} = \frac{180^\circ}{n}$, hence AC and OC can be found.

$$AC = r \sin \frac{180^\circ}{n}, \qquad OC = r \cos \frac{180^\circ}{n},$$

$$AB = 2\,AC = 2\,r \sin \frac{180^\circ}{n},$$

and the perimeter $= n \circ AB = 2\,nr \sin \frac{180^\circ}{n}$.

Also the area of the triangle $AOB = \frac{1}{2} \cdot AB \circ OC = r^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$, and the area of the entire polygon $= n$ times the area of triangle AOB

$$= nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}.$$

In particular, if $r = 95$ and the figure is a heptagon, $n = 7$, and we have

$$\text{Angle } AOC = \frac{180^\circ}{7} = 25^\circ\ 42'\ 51.5''.$$

$$\text{Perimeter} = 14 \cdot 95 \cdot \sin \frac{180^\circ}{7}.$$

$$\text{Area} = 7 \cdot 95^2 \cdot \sin \frac{180^\circ}{7} \cdot \cos \frac{180^\circ}{7}.$$

$\log 14 = 1.14613$

$\log 95 = 1.97772$

$\log \sin \frac{180^\circ}{7} = \underline{9.63737 - 10}$

$\log \text{perimeter} = 2.76122$

$\text{perimeter} = 577.06.$

$\log 7 = 0.84510$

$2 \log 95 = 3.95544$

$\log \sin \frac{180^\circ}{7} = 9.63737 - 10$

$\log \cos \frac{180^\circ}{7} = \underline{9.95471 - 10}$

$\log \text{area} = 4.39262$

$\text{area} = 24696.$

EXAMPLE 3. A right pyramid has for its base a square whose side is $2a$. The angle formed by a face and the base of the pyramid is θ. Determine the altitude, slant height and lateral edge of the pyramid, also the angle made by a lateral edge and the plane of the base, and the angle between the lateral edge and the edge of the base.

Solution. Let V-$ABCD$ represent the pyramid, VO its altitude and VM its slant height. Join M and O. Then $MA = MO = a$, and angle $VMO = \theta$. Further, let

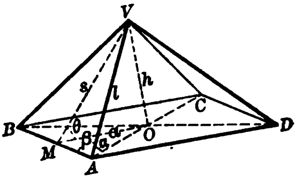

Fig. 52.

$h =$ altitude VO,
$s =$ slant height MV,
$l =$ lateral edge AV,
$\alpha =$ angle VAO, made by a lateral edge and the plane of the base,
$\beta =$ angle VAM, made by a lateral edge and an edge of the base.

In the right triangle VMO, a and θ are given, hence h and s may be found. Solving,

$$h = a \tan \theta, \qquad s = a \sec \theta.$$

In the right triangle VMA, a is given and s has just been found, hence l and β may be found. Solving,

$$\beta = \tan^{-1} \frac{s}{a}, \qquad l = a \sec \beta, \text{ or } = \sqrt{a^2 + s^2}.$$

Finally, from the right triangle VAO,

$$\alpha = \sin^{-1} \frac{h}{l}.$$

As a numerical example take the side of the base equal to 10, and the angle $\theta = 60^\circ$. Then $a = 5$,

and
$$\text{altitude, } h = 5 \tan 60^\circ = 5\sqrt{3},$$
$$\text{slant height, } s = 5 \sec 60^\circ = 10,$$
$$\text{lateral edge, } l = \sqrt{5^2 + 10^2} = 5\sqrt{5},$$
$$\alpha = \sin^{-1} \sqrt{\tfrac{3}{5}} = 50^\circ\, 46'\, 08'',$$
$$\beta = \tan^{-1} 2 = 63^\circ\, 26'\, 06''.$$

EXERCISE 29

1. The base of an isosceles triangle is 12 and the angle at the vertex is 48°. Find the altitude of the triangle. *Ans.* 13.48.

2. The chord of a circle is 20 ft. long and the angle at the center subtended by it is 42° 10′. Find the radius of the circle.

Ans. 27.80.

3. The angle between two lines is 50° 21′ 24″ and a circle whose radius is 2380 ft. is tangent to both of them. Find the distance from the intersection of the two lines to the point of tangency, —

(*a*) When the circle lies in the smaller angle,

(*b*) When the circle lies in the larger angle formed by the two lines. *Ans.* 5062.8, 1118.8.

4. The radius of the inscribed circle of an equilateral triangle is r. Find the radius of the circumscribed circle.

5. A chord of a circle subtends at the center an angle of 80° 24′. In the same circle, how large is the angle subtended by a chord half as long? *Ans.* 37° 29.4′.

(Suggestion. Call the length of the chord a.)

6. The side of a regular octagon is 7. Find the area.

Ans. 236.59.

7. The radius of a circle is r; show that a side of a circumscribed regular polygon of n sides is $2r \tan\frac{180^\circ}{n}$.

8. One side of a right triangle is 27.5 and the adjacent acute angle is 54° 38′. Compute the length of the perpendicular from the vertex of the right angle to the hypotenuse, and the segments into which the hypotenuse is divided.

9. Solve the preceding problem, using a for the given side and B for the given angle.

Ans. $p = a \sin B$, $m = a \cos B$, $n = a \sin B \tan B$, where p is the perpendicular, m and n the segments of the hypotenuse, m being the segment adjacent to B.

10. In an oblique triangle two sides and the included angle are given, namely $a = 25.37$, $b = 36.12$, $C = 35^\circ 27'$. Find the remaining parts. *Ans.* $A = 43^\circ 35.9'$, $B = 100^\circ 57.1'$, $c = 21.34$.

(Suggestion. Divide the triangle into two right triangles by drawing a perpendicular from one of the vertices, A or B, to the opposite side.)

11. In an oblique triangle one side and two adjacent angles are given, namely $c = 10$, $A = 60°$, $B = 75°$. Find the remaining parts.

Ans. $C = 45°$, $a = 5\sqrt{6}$, $b = 5(1 + \sqrt{3})$.

(Suggestion. Divide the triangle into two right triangles by drawing a perpendicular from B to the side opposite.)

12. A regular parallelopiped has for its base a rectangle whose dimensions are $AB = 8$, $AD = 10$, and its altitude $AA' = 15$. Find the angles which the diagonal AC' makes with AD, with AC and with AB.

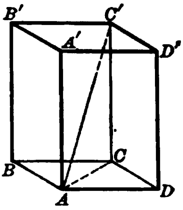

Fig. 53.

13. A right pyramid has $2a$ for an edge of the regular hexagon which forms its base and an altitude equal to a. Find the angles which a lateral edge makes with an edge of the base, with the plane of the base, and the angle which a lateral face makes with the plane of the base.

Ans. $\tan^{-1} 2 = 63° 26'$, $\tan^{-1} \frac{1}{2} = 26° 34'$, $\tan^{-1} \frac{1}{3}\sqrt{3} = 30°$.

14. Verify trigonometrically the following practical rule for inscribing a regular pentagon in a circle:

Let O be the center of the circle, OA and OC two perpendicular radii. Bisect OA in M. Take MR equal to MC. With C as center and CR as a radius, draw an arc cutting the circle in P. Join C and P. PC will be the side of the pentagon.

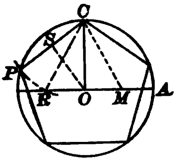

Fig. 54.

(Suggestion. Assume the radius of length r.
From triangle OCM find $CM = RM$.
From triangle COR find $CR = CP$.
From triangle COS ($SC = \frac{1}{2}PC$) find angle SOC, which should be $36°$.)

52. Oblique Triangles Solved by Right Triangles. Every oblique triangle may be solved by decomposing it into right triangles. This is done by drawing a perpendicular from one of the vertices of the triangle to the opposite side. In three of the four cases the perpendicular can be so chosen that two of the given parts become parts of one of the right triangles. This triangle having been solved, two parts of the other right triangle become known. The second triangle may now be solved, and with this all the parts of the original triangle become known. The fourth case (given the three sides) requires a somewhat different method. We will take up each case separately.

Case I. *Given one side and two adjacent angles*, as b, C, A.

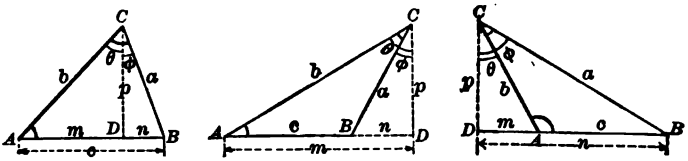

Fig. 55.

Let ABC represent the triangle. From A or C, say C, draw a perpendicular CD to the opposite side, or opposite side produced.

Let $CD = p$, $AD = m$, $DB = n$, angle $ACD = \theta$, angle $BCD = \phi$. Three different figures may arise, —

the left-hand figure, when A and B are both acute,
the middle figure, when A is acute and B obtuse,
the right-hand figure, when A is obtuse.

In the right triangle ACD, b and angle CAD are known, hence p, m, and θ may be found.

From C and θ, ϕ may be found.

Having found ϕ and p, we know two parts of the right triangle BCD, hence a and n and the angle CBD may be found.

Knowing m and n, c may be found.

To check the answers, we repeat the solution, drawing the perpendicular from A instead of from C.

Case II. *Given two sides and the angle opposite one of these sides*, as a, b, A.

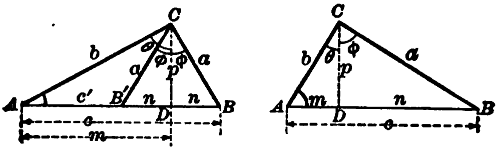

Fig. 56.

In this case draw the perpendicular from that vertex which lies between the two given sides a and b.

If $a < b$, two different triangles exist which have the given parts, as in Fig. 56, left, if $a \geqq b$, only one triangle exists.

In the right triangle CAD, b and A are given, hence p, m and θ (= angle ACD) may be found.

Next consider the right triangle BCD. p and a are known, hence n and ϕ (= angle BCD) may be found.

Finally,

$$AB = c = m + n, \quad C = \theta + \phi, \quad B = 180^\circ - (A + C).$$

The second solution, if there is one, as in the figure to the left, is given by

$$AB' = c' = m - n, \; ACB' = C' = \theta - \phi, \; AB'C = B' = 180^\circ - (A + C').$$

Check. From triangle ACD, $m = b \cos A$,

From triangle CDB, $n = a \cos B$,

and since $m + n = c$, we must have

$$b \cos A + a \cos B = c.$$

Case III. *Given two sides and an included angle*, as b, c, A.

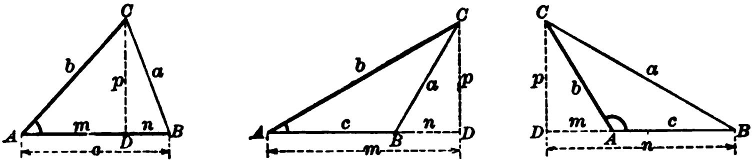

Fig. 57.

Draw the perpendicular from B or C, say C.

In the right triangle ACD, b and angle CAD are known, hence p and m may be found.

Knowing m and c, n may be found.

Having found p and n, we know two parts of the right triangle BCD, hence a and angle CBD may be found.

Finally, angle $ACB = 180^\circ - (A +$ angle $ABC)$.

Here, as in Case I, three different figures are possible, according as

A and B are both acute (left-hand figure),

A acute and B obtuse (middle figure),

A obtuse (right-hand figure),

but the above analysis applies to each figure alike.

A check is obtained by repeating the solution with the perpendicular drawn from B.

Case IV. *Given the three sides, a, b, c.*

In the first three cases we were able so to choose the vertex, from which the perpendicular was drawn, that one of the right triangles contained two of the given parts. In the present case this is not possible. The apparent difficulty is easily overcome as follows, —

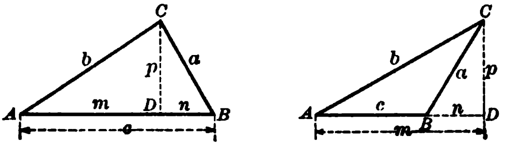

Fig. 58.

In either of the figures 58 we have

from the triangle *CAD*, $p^2 = b^2 - m^2$,

from the triangle *CDB*, $p^2 = a^2 - n^2$,

hence $b^2 - m^2 = a^2 - n^2$,

from which $b^2 - a^2 = m^2 - n^2 = (m + n)(m - n)$. (1)

We now have,

in the left figure, $m + n = c$,

in the right figure, $m - n = c$,

and from (1),

$$m - n = \frac{b^2 - a^2}{m + n} = \frac{b^2 - a^2}{c} = \frac{(b + a)(b - a)}{c},$$

$$m + n = \frac{b^2 - a^2}{m - n} = \frac{b^2 - a^2}{c} = \frac{(b + a)(b - a)}{c},$$

from which, since a, b, c are given,

$$m - n \qquad \text{or} \qquad m + n$$

may be found.

We then have in either case,

$$\frac{(m + n) + (m - n)}{2} = m,$$

$$\frac{(m + n) - (m - n)}{2} = n.$$

We now know two parts in each of the right triangles *CAD* and *CDB*, and from these the angles *A* and *B* may be computed.

As a check the solution may be repeated with the perpendicular drawn from one of the other vertices.

EXAMPLE. Given $a = 45.652$, $b = 62.735$, $c = 51.238$; to find the angles.

Solution. $$m - n = \frac{(b+a)(b-a)}{c}$$

$$\begin{aligned} b + a &= 62.735 + 45.652 = 108.387, \\ b - a &= 62.735 - 45.652 = 17.083, \\ c &= 51.238. \end{aligned}$$

$$\begin{aligned} \log(b+a) &= 2.03498 \\ \log(b-a) &= 1.23257 \\ \operatorname{colog} c &= \underline{8.29041} \\ \log(m-n) &= 1.55796 \\ m - n &= 36.138 \end{aligned}$$ [*]

$$m = \frac{(m+n)+(m-n)}{2} = \frac{51.238 + 36.138}{2} = 43.688,$$

$$n = \frac{(m+n)-(m-n)}{2} = \frac{51.238 - 36.138}{2} = 7.550.$$

From the right triangle *CAD*,

$$\frac{m}{b} = \cos A.$$

$$\begin{aligned} \log m &= 1.64036 \\ \operatorname{colog} b &= \underline{8.20249} \\ \log\cos A &= 9.84285 - 10 \\ A &= 45^\circ\, 51'\, 46'', \end{aligned}$$

From the right triangle *CDB*,

$$\frac{n}{a} = \cos B.$$

$$\begin{aligned} \log n &= 0.87795 \\ \operatorname{colog} a &= \underline{8.34054} \\ \log\cos B &= 9.21849 - 10 \\ B &= 80^\circ\, 28'\, 49'', \end{aligned}$$

$$C = 180^\circ - (A + B) = 53^\circ\, 39'\, 24''.$$

Check. Interchanging b and c in the above formula, we find

$$\begin{aligned} c + a &= 51.238 + 45.652 = 96.890, \\ c - a &= 51.338 - 45.652 = 5.586, \\ b &= 62.735. \end{aligned}$$

$$\begin{aligned} \log(c+a) &= 1.98628 \\ \log(c-a) &= 0.74710 \\ \operatorname{colog} b &= \underline{8.20249} \\ \log(m-n) &= 0.93587 \\ m - n &= 8.6272. \end{aligned}$$

$$m = \frac{(m+n)+(m-n)}{2} = \frac{62.735 + 8.6272}{2} = 35.681,$$

$$n = \frac{(m+n)-(m-n)}{2} = \frac{62.735 - 8.6272}{2} = 27.054.$$

---

[*] If this result were greater than c, we would have the right-hand figure in Fig. 58, and we should have taken $m + n = \dfrac{(b+a)(b-a)}{c}$.

$$\frac{m}{c} = \cos A. \qquad \frac{m}{a} = \cos C.$$

$$\begin{array}{rlrl} \log m &= 1.55243 & \log n &= 1.43223 \\ \text{colog}\, c &= \underline{8.29041} & \text{colog}\, a &= \underline{8.34055} \\ \log\cos A &= 9.84284 & \log\cos C &= 9.77278 \\ A &= 45^\circ\, 51'\, 51'', & C &= 53^\circ\, 39'\, 24'', \end{array}$$

$$B = 180^\circ - (A + C) = 80^\circ\, 28'\, 45''.^*$$

EXERCISE 30

Only a few triangles are given here for solution by the method of right triangles, for soon we shall study a better method, by means of which the computation can in most cases be shortened.

1. Given $a = 342.56$, $b = 125.72$, $C = 37^\circ\, 42'\, 24''$.
Ans. $A = 124^\circ\, 44'\, 28''$, $B = 17^\circ\, 33'\, 08''$, $c = 254.97$.

2. Given $b = 134.5$, $c = 235.2$, $A = 127^\circ\, 36.3'$.
Ans. $a = 334.7$, $B = 18^\circ\, 33.9'$.

3. $A = 25^\circ\, 25'\, 25''$, $B = 50^\circ\, 50'\, 50''$, $c = 278.98$.
Ans. $C = 103^\circ\, 43'\, 45''$, $a = 123.29$, $b = 222.70$.

4. $C = 127^\circ\, 36.5'$, $A = 28^\circ\, 31.3'$, $b = 312.9$.
Ans. $c = 612.55$, $a = 369.22$, $B = 23^\circ\, 52.2'$.

5. $a = 630.50$, $b = 527.39$, $A = 65^\circ\, 37'\, 12''$.
Ans. $B = 49^\circ\, 37'\, 38''$, $C = 64^\circ\, 45'\, 10''$, $c = 626.13$.

6. $b = 1825$, $c = 1563$, $B = 22^\circ\, 13.7'$. Find the remaining parts.
Ans. $C = 14^\circ\, 54.8'$, $A = 142^\circ\, 51.5'$, $a = 2913$.

7. $a = 3.537$, $b = 6.667$, $c = 5.001$. Find the remaining parts.

8. $a = 4$, $b = 5$, $c = 6$. Find the remaining parts.
Ans. $A = 41^\circ\, 24.6'$, $B = 55^\circ\, 46.3'$, $C = 82^\circ\, 49.1'$.

---

[*] When checking five-place distances and angles expressed to seconds obtained from five-place tables, the results will generally be found to agree only to four places for distances and to the nearest tenth of a minute (6″) for angles. This is because, as we have already observed, the fifth place of a number and the seconds of an angle obtained from a five-place table are not necessarily accurate. When cut down to the nearest tenth of a minute, the results of the two computations in the above example agree, each giving

$$A = 45^\circ\, 51.8', \quad B = 80^\circ\, 28.8', \quad C = 53^\circ\, 39.4'.$$

The solution may therefore be assumed to be correct.

9. Find the ratio between the sides of a triangle whose angles are, $A = 50°$, $B = 60°$, $C = 70°$.

Ans. $a : b : c = 0.7660 : 0.8660 : 0.9397$.

10. In a quadrilateral $ABCD$, Fig. 59, the following parts are known:

$AB = 673$, $BC = 589$, $CD = 223$,
angle $B = 105° 06'$, angle $C = 127° 38'$.

It is required to find the length of AD to the nearest unit. *Ans.* 1017.

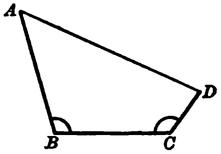

Fig. 59.

CHAPTER VI

FUNCTIONS OF AN OBTUSE ANGLE

Reason for a New Definition. In a right triangle no angle can exceed 90°, but when the triangle is oblique one of its angles may be obtuse, that is, one of its angles may have any value between 90° and 180°. In order to solve oblique triangles in the simplest way possible, we must define the trigonometric functions for obtuse angles. This is best done by means of the conception of rectangular coördinates.

53. Rectangular Coördinates. Let $X'X$ and $Y'Y$ be two lines, indefinite in length, intersecting at right angles at O. The two lines divide their plane into four parts, known as the first, second, third and fourth quadrants respectively, as indicated by the numerals I, II, III, IV, in Fig. 60.

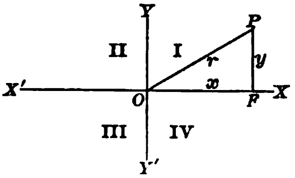

Fig. 60.

Let P be any point in the plane of the lines $X'X$ and $Y'Y$, and let PF be the perpendicular drawn from P to $X'X$. Join O and P. OP, the distance of P from O, is always positive and is designated by r.

OF, generally represented by x, is called the *abscissa* of the point P. It is considered positive if P is to the right, negative if to the left of $Y'Y$.

FP, generally represented by y, is called the *ordinate* of the point P. It is considered positive if P is above, negative if below $X'X$.

Considered together, OF or x and FP or y are known as the rectangular *coördinates* of the point P.

$X'X$ and $Y'Y$ are called the *coördinate axes*, or axes of reference; $X'X$ is the x-axis, or *axis of abscissas*; $Y'Y$ is the y-axis, or *axis of ordinates*; O is called the *origin*.

The abscissa of a point is always written first and the ordinate second. Thus, by the point (a, b), we understand the point for which $x = a$, $y = b$, that is, the point whose abscissa is a and whose ordinate is b.

From what has been said it is plain that:

In the first quadrant, x is positive, y is positive, r is positive;
in the second quadrant, x is negative, y is positive, r is positive;
in the third quadrant, x is negative, y is negative, r is positive;
in the fourth quadrant, x is positive, y is negative, r is positive.
For every point on the x-axis, $y = 0$;
for every point on the y-axis, $x = 0$;
for the origin, x and y are each 0.

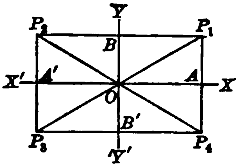

Fig. 61.

Thus, in Fig. 61,
if for P_1, $x = 4, y = 3$;

then for P_2, $x = -4, y = 3$; then for B, $x = 0, y = 3$;
for P_3, $x = -4, y = -3$; for A', $x = -4, y = 0$;
for P_4, $x = 4, y = -3$; for B', $x = 0, y = -3$;
for A, $x = 4, y = 0$; for O, $x = 0, y = 0$;

but each of the distances OP_1, OP_2, OP_3, OP_4 equals $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

54. Definition of the Trigonometric Functions of Any Angle Less than 180°.

Let angle $XOB = \theta$ represent any angle less than 180°. Take O for an origin, OX for the positive x-axis, and draw OY perpendicular to OX. Then OB will be in the first or second quadrant according as θ is acute or obtuse. Now take any point P on OB and denote by r the distance of this point from the origin, and by x and y the rectangular coördinates of this point with reference to OX and OY as axes. The trigonometric functions of θ are then defined as follows, —

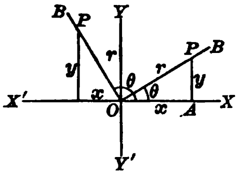

Fig. 62.

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc\theta = \frac{1}{\sin\theta},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

It will be observed that when the angle θ is acute, these definitions agree with those given in Art. 7; for x, y and r are the base, altitude and hypotenuse respectively of the right triangle which we there used in defining the trigonometric functions of an acute angle.

55. The Signs of the Functions of an Obtuse Angle. In the first quadrant, x and y, as well as r, are positive. Hence all the functions are positive for an angle in the first quadrant. In the second quadrant, x is negative, y and r are positive, hence the ratios $\frac{x}{r}$ and $\frac{y}{x}$ are negative, while $\frac{y}{r}$ remains positive, that is, the cosine and tangent and their reciprocals are negative, while the sine and its reciprocal are positive.

The functions of an obtuse angle are all negative, except the sine and its reciprocal, which are positive.

56. Fundamental Relations. Of the six fundamental relations, —

$$\sin\theta\cdot\csc\theta = 1, \qquad \sin^2\theta + \cos^2\theta = 1,$$
$$\cos\theta\cdot\sec\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta,$$
$$\tan\theta\cdot\cot\theta = 1, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

the first three rest upon the definitions of the cosecant, secant and cotangent, and hold, therefore, whether θ is acute or obtuse, and the last three depend upon the relation $x^2 + y^2 = r^2$, which is true whether x is positive or negative. These six fundamental relations hold, therefore, for the functions of obtuse angles as well as for the functions of acute angles.

Also,

$$\tan\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\theta}{\cos\theta},$$ whether x is positive or negative, hence

the two relations, —

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

also hold true when θ is an obtuse angle.

57. Functions of Supplementary Angles. Let $XOP = \theta$ be any angle less than 180°, and draw OP' so as to make angle $P'OX'$ equal to angle XOP. Then angle $XOP' = 180° - \theta$.

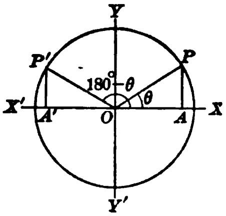

Fig. 63.

If OP and OP' are taken equal, the two triangles OAP and $OA'P'$ will be geometrically equal, and we have

$$\sin(180° - \theta) = \frac{A'P'}{OP'} = \frac{AP}{OP} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{OA'}{OP'} = \frac{-OA}{OP} = -\frac{OA}{OP} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{A'P'}{OA'} = \frac{AP}{-OA} = -\frac{AP}{OA} = -\tan\theta,$$

$$\csc(180° - \theta) = \frac{1}{\sin(180° - \theta)} = \frac{1}{\sin\theta} = \csc\theta,$$

$$\sec(180° - \theta) = \frac{1}{\cos(180° - \theta)} = \frac{1}{-\cos\theta} = -\sec\theta,$$

$$\cot(180° - \theta) = \frac{1}{\tan(180° - \theta)} = \frac{1}{-\tan\theta} = -\cot\theta.$$

By comparing our results we observe that the signs on the right are those of the functions in the second quadrant, hence it appears that, —

Any function of $(180° - \theta)$ *equals plus or minus the same function of* θ, *the sign being that of the function in the second quadrant.*

The rule just given enables us to express the functions of an obtuse angle in terms of the functions of an acute angle, thus:

$$\sin 116° = \sin(180° - 64°) = \sin 64°,$$

$$\cos 116° = \cos(180° - 64°) = -\cos 64°,$$

$$\tan 116° = \tan(180° - 64°) = -\tan 64°,$$

$$\csc 116° = \csc(180° - 64°) = \csc 64°,$$

$$\sec 116° = \sec(180° - 64°) = -\sec 64°,$$

$$\cot 116° = \cot(180° - 64°) = -\cot 64°.$$

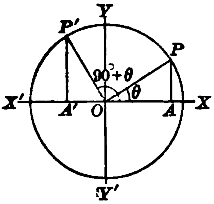

Fig. 64.

58. Functions of $(90° + \theta)$. Let angle $XOP = \theta$ be any acute angle and draw OP' perpendicular to OP. Then angle $XOP' = 90° + \theta$. If OP' is taken equal to OP, the triangles AOP and $A'OP'$ are geometrically equal, and we have

$$\sin(90° + \theta) = \frac{A'P'}{OP'} = \frac{OA}{OP} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{OA'}{OP'} = \frac{-AP}{OP} = -\frac{AP}{OP} = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{A'P'}{OA'} = \frac{OA}{-AP} = -\frac{OA}{AP} = -\cot\theta,$$

$$\csc(90° + \theta) = \frac{1}{\sin(90° + \theta)} = \frac{1}{\cos\theta} = \sec\theta,$$

$$\sec(90° + \theta) = \frac{1}{\cos(90° + \theta)} = \frac{1}{-\sin\theta} = -\csc\theta,$$

$$\cot(90° + \theta) = \frac{1}{\tan(90° + \theta)} = \frac{1}{-\cot\theta} = -\tan\theta.$$

Again the signs on the right are the signs of the functions in the second quadrant, hence,—

Any function of $(90° + \theta)$ is equal to plus or minus the corresponding cofunction of θ, the sign being that of the function in the second quadrant.

This rule, like that of the preceding article, enables us to express the functions of an obtuse angle in terms of the functions of an angle less than 90°. Thus,

$$\sin 116° = \sin(90° + 26°) = \cos 26°,$$
$$\cos 116° = \cos(90° + 26°) = -\sin 26°,$$
$$\tan 116° = \tan(90° + 26°) = -\cot 26°, \text{ etc.}$$

59. Functions of 180°. If in Fig. 62, θ is taken equal to 180°, OP must coincide with OX', the abscissa of P will be $-r$ and its ordinate zero, hence

$$\sin 180° = \frac{0}{r} = 0, \qquad \csc 180° = \frac{1}{0} = \infty,$$

$$\cos 180° = \frac{-r}{r} = -1, \qquad \sec 180° = \frac{1}{-1} = -1,$$

$$\tan 180° = \frac{0}{-r} = -0, \qquad \cot 180° = \frac{1}{-0} = -\infty.$$

The results in the last line need some explanation. The tangent and cotangent of every obtuse angle is negative, hence their limiting values, as the angle approaches 180°, is negative. Numerically these limiting values are 0 and ∞, the minus signs merely indicate that these values have been approached through a succession of negative magnitudes.

60. Angles Corresponding to a Given Function. Since the cosine, secant, tangent and cotangent of every obtuse angle is negative, we can tell whether the angle corresponding to one of these functions is obtuse or acute by noting the algebraic sign of the function. But the sine and cosecant are positive whether the angle is acute or obtuse. In fact, since the sines and cosecants of supplementary angles are equal in every respect, there will always be two angles, one acute and the other obtuse, which will correspond equally well to a given sine or cosecant. This is expressed by saying that the angle corresponding to a given sine or cosecant is ambiguous.

EXAMPLES. If $\cos\theta = \frac{1}{2}$, θ must equal 60°,

and if $\cos\theta = -\frac{1}{2}$, θ equals the supplement of 60°, or 120°.

If $\tan\theta = 1$, θ must equal 45°,

and if $\tan\theta = -1$, θ equals the supplement of 45°, or 135°.

If $\sin\theta = \frac{1}{2}$, θ may be either 30° or its supplement, 150°.

EXERCISE 31

1. Locate the points whose coördinates are $x = 8, y = 6; x = -8, y = -6; x = -8, y = 6; x = 8, y = -6$; and in each case compute the distance of the point from the origin.

2. Locate the points $x = 5, y = 0; x = 0, y = 5; x = -1, y = -1; x = 0, y = -1$.

3. The distance of a point from the origin is 2, and its abscissa is 1; find its ordinate and locate the point.

Ans. Two solutions, $y = \pm\sqrt{3}$.

4. The distance of a point from the origin is 10 and its ordinate is $\sqrt{50}$; find the abscissa and locate the point.

5. Make out a table containing the sine, cosine and tangent of each of the angles 120°, 135°, 150°.

6. Construct the angles, having given the following functions: $\cos A = -\frac{3}{5}$, $\tan B = -3$, $\sin\theta = \frac{1}{3}$.

7. Express the following functions in terms of functions of the supplementary angles: sin 115°, tan 165°, cos 125°, cot 100°, sec 170°, sin 145°, cos 136°, tan 95°.

8. Express in terms of an angle less than 45° the following:

sin 95°, cos 120°, tan 100°, sec 114°, sin 125°.

9. Express in two ways in terms of an acute angle, first by means of the rule of Art. 57, second by means of the rule of Art. 58, each of the following:

$$\sin 127^\circ\,35'\,13'',\ \cos 157^\circ\,54'\,36'',\ \tan 140^\circ\,11'\,25.3''.$$

Which of the two methods is the easier?

10. Use the table of natural functions to find the following functions: sin 112° 30′, cos 156° 25′, tan 162° 50′, sin 105°, cos 175° 10′, tan 126° 14′.

11. θ being any obtuse angle, prove the following relations:

$$\begin{aligned}
\sin(\theta - 90^\circ) &= -\cos\theta, & \csc(\theta - 90^\circ) &= -\sec\theta,\\
\cos(\theta - 90^\circ) &= \sin\theta, & \sec(\theta - 90^\circ) &= \csc\theta,\\
\tan(\theta - 90^\circ) &= -\cot\theta, & \cot(\theta - 90^\circ) &= -\tan\theta.
\end{aligned}$$

61. Review.

1. (*a*) What is meant by the logarithm of a number to a given base a? (*b*) Show that $\log ab = \log a + \log b$, $\log a^b = b \log a$. (*c*) What is the logarithm of 1? (*d*) What is meant by a cologarithm? (*e*) Given $\log 4 = 0.60206$; find log 16, log 2, $\log \frac{1}{4}$.

2. (*a*) What is meant by the common logarithm of a number? (*b*) Give from memory the common logarithms of 10, 100, 0.1, 0.01, 10^r, $\frac{1}{10^n}$, $\sqrt{10}$, $\sqrt[3]{10}$, $\sqrt[3]{100}$. (*c*) What is meant by the characteristic of a common logarithm? What by the mantissa? (*d*) Give the characteristics of the logarithms of the following numbers: 15, 153, 6.23, 0.05, 0.0105. (*e*) Give the rule for the characteristic of a number greater than 1; of a decimal fraction less than 1.

3. Explain how a table of common logarithms might be constructed by extracting square roots only. Find the number whose logarithm is $\frac{3}{8}$, without consulting the table.

4. Prove that $\log_a N = \dfrac{\log_b N}{\log_b a}$; also show that $\log_b a \cdot \log_a b = 1$.

5. What is meant by the principle of proportional parts? Read again Art. 32.

6. Work Problems 13 and 14, Exercise 18.

7. What is meant by an exponential equation? Solve Problem 18, Exercise 18.

8. (*a*) Read again Art. 44. (*b*) What accuracy is called for in the angle of a triangle, when the sides are given correct to three places? To four places? (*c*) How accurately can a number be determined with the aid of a five-place table of logarithms? An angle? (*d*) When should a six-place table be used? When a seven-place table?

9. (*a*) How many degrees constitute a point of a mariner's compass? (*b*) How many degrees in the angle between N. and N.E. by N.? Between N.E. by E. and E.N.E.? (*c*) What direction is opposite to the direction S.E. by S.?

10. Explain how to solve each of the four cases of oblique triangles by means of right triangles.

11. (*a*) What is meant by the rectangular coördinates of a point? (*b*) The line joining the origin to a point P is 5 units in length and makes an angle of 30° with the positive direction of the x-axis. What are the rectangular coördinates of the point P? (*c*) What are the coördinates of the point P, if OP makes an angle of 150° with OX?

12. (*a*) Define the sine, cosine and tangent of an obtuse angle. (*b*) Prove that $\sin(180° - A) = \sin A$, $\cos(180° - A) = -\cos A$, $\tan(90° + A) = -\cot A$. (*c*) Complete the equations

$$\tan(180° - A) = \quad, \sin(90° + A) = \quad, \cos(90° + A) = \quad.$$

13. (*a*) Find $\sin 123°$, $\cos 136°$, $\tan 105° 30'$. (*b*) Find x in each of the equations: $\sin x = 0.3423$, $\cos x = -0.9061$, $\tan x = -0.0913$, x being in each case the angle of a plane triangle.

CHAPTER VII

PROPERTIES OF TRIANGLES

In this chapter we shall develop certain properties of triangles which will enable us to compute, from a sufficient number of given parts, the remaining sides and angles, the area and other magnitudes related to the triangle. The principal applications of the results developed in this chapter will be treated in a separate chapter.

62. The Law of Sines.

(*a*) *First proof.* Let ABC be any plane triangle. Draw the perpendicular h from one of the vertices C of the triangle to the opposite side AB (Fig. 65), or AB produced (Fig. 66).

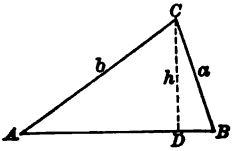

Fig. 65.

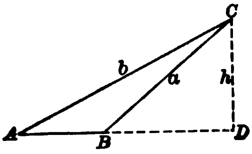

Fig. 66.

In Fig. 65,	In Fig. 66,
in the right triangle ACD	in the right triangle ACD
$h = b \sin A,$	$h = b \sin A,$
and in the right triangle BCD	and in the right triangle BCD
$h = a \sin B.$	$h = a \sin (180° - B) = a \sin B.$

Hence, whether the triangle is acute or obtuse, we have

$$h = b \sin A = a \sin B,$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, by drawing a perpendicular from B to the opposite side, or the opposite side produced, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

so that we may write

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (1)$$

Equation (1) may be otherwise written thus,—

$$a : b : c = \sin A : \sin B : \sin C. \qquad (2)$$

Equation (1) or (2) embodies what is known as the *Law of Sines*, which states that,—

In any triangle the sides are proportional to the sines of the opposite angles.

(*b*) *Second proof.* The Law of Sines may be proven in another way, which at the same time brings out the meaning of the ratios in equation (1).

Circumscribe a circle about the triangle ABC and denote by D the diameter BA', drawn through one of the vertices, as B. Join A' and C. $A'BC$ is a right triangle (Why?), and therefore

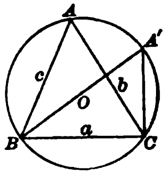

Fig. 67.

$$\frac{a}{D} = \sin A', \text{ or } D = \frac{a}{\sin A'}.$$

But angle A' = angle A (angles inscribed in the same arc are equal), hence

$$D = \frac{a}{\sin A'} = \frac{a}{\sin A},$$

and similarly

$$D = \frac{b}{\sin B}, \qquad D = \frac{c}{\sin C},$$

from which

$$D = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad (3)$$

that is,—

The ratio of any side of a triangle to the sine of the angle opposite is numerically equal to the diameter of the circumscribed circle.

63. Projection Theorem.

In Fig. 65,	In Fig. 66,
$AD = b \cos A,$	$AD = b \cos A,$
$DB = a \cos B.$	$BD = a \cos (180° - B) = -a \cos B.$

Moreover,

$c = AB = AD + DB.$	$c = AB = AD - BD.$

Substituting for AD, DB, and BD their values, we have

$$c = b\cos A + a\cos B. \qquad \begin{aligned} c &= b\cos A - (-a\cos B) \\ &= b\cos A + a\cos B. \end{aligned}$$

Hence, whether the triangle is acute or obtuse, we have

Similarly,*
$$\left.\begin{aligned} \boldsymbol{c} &= \boldsymbol{a\cos B + b\cos A}. \\ \boldsymbol{a} &= \boldsymbol{b\cos C + c\cos B}, \\ \boldsymbol{b} &= \boldsymbol{c\cos A + a\cos C}. \end{aligned}\right\} \qquad (4)$$

We may consider the line DB (Fig. 66) the negative of BD, that is, $DB = -BD$. In that case

$c = AB = AD - BD = AD - (-DB) = AD + DB$, just as in Fig. 65.

$AD = b\cos A$ is called the *projection* of AC on AB,

$DB = a\cos B$ is called the projection of CB on AB,

so that the relations (4) may be stated thus, —

In any triangle, each side is equal to the algebraic sum of the projections of the other two sides upon it.

64. The Law of Cosines.

(*a*) *First proof.*

In Fig. 65,	In Fig. 66,
$b^2 = h^2 + \overline{AD}^2$	$b^2 = h^2 + \overline{AD}^2$
$h^2 = a^2 - \overline{DB}^2$	$h^2 = a^2 - \overline{BD}^2$
$\overline{AD}^2 = (c - DB)^2$	$\overline{AD}^2 = (c + BD)^2$
$= c^2 - 2c\cdot DB + \overline{DB}^2.$	$= c^2 + 2c\cdot BD + \overline{BD}^2.$

Substituting in the first equation the values of h and AD from the second and third, we have

$$\begin{aligned} b^2 &= a^2 - \overline{DB}^2 + c^2 - 2c\cdot DB + \overline{DB}^2 \\ &= a^2 + c^2 - 2c\cdot DB. \end{aligned} \qquad \begin{aligned} b^2 &= a^2 - \overline{BD}^2 + c^2 + 2c\cdot BD + \overline{BD}^2 \\ &= a^2 + c^2 + 2c\cdot BD. \end{aligned}$$

Now from the figure

$$DB = a\cos B. \qquad BD = a\cos(180^\circ - B) = -a\cos B.$$

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[*] The second formula may be obtained from the first by replacing a by b; b by c, c by a, and A by B, B by C, and C (should it occur) by A. In the same manner the third formula may be obtained from the second, and the first from the third. That is, if any one of these formulas is given, the other two can be supplied by *cyclic substitution*.

Substituting these values in the equation just preceding, we obtain in either case

Similarly,

$$\left.\begin{aligned} b^2 &= c^2 + a^2 - 2\,ca\cos B.\\ c^2 &= a^2 + b^2 - 2\,ab\cos C,\\ a^2 &= b^2 + c^2 - 2\,bc\cos A.\end{aligned}\right\}\qquad(5)$$

These formulas embody the so-called

Law of Cosines: In any triangle, the square on any side is equal to the sum of the squares on the other two sides diminished by twice the product of those two sides times the cosine of the included angle.

(*b*) *Second proof.* The law of cosines may be proved even more easily than above by the aid of the projection formulas. We need only to multiply the first of the formulas (4) by c, to add a times the second and to subtract b times the third. The result is

$$\begin{aligned} c^2 + a^2 - b^2 &= c\,(a\cos B + b\cos A) + a\,(b\cos C + c\cos B)\\ &\quad - b\,(c\cos A + a\cos C)\\ &= 2\,ac\cos B,\end{aligned}$$

from which

$$b^2 = c^2 + a^2 - 2\,ca\cos B.$$

65. Arithmetic Solution of Triangles. The law of sines and the law of cosines are sufficient to solve each of the four cases of oblique triangles.

I. If one side a and two angles are given, the third angle is found immediately from the relation $A + B + C = 180°$. The remaining sides may then be found from the law of sines, thus, —

$$b = \frac{a\sin B}{\sin A},\qquad c = \frac{a\sin C}{\sin A}.$$

II. If two sides b and c and the angle opposite one of them, say B, is given, the third side a may be found by the law of cosines; for, solving the first of equations (5), considering a as the unknown quantity, we find

$$a = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

This gives two values for a, as it should, for we know that this case has in general two solutions.

Having found the third side, the angles A and C may now be found from the law of sines, thus:

$$\sin A = \frac{a}{b} \sin B, \qquad \sin C = \frac{c}{b} \sin B.$$

III. If two sides b and c and the included angle A are given, the third side a may be found from the law of cosines, for the third equation (5) gives

$$a = \sqrt{b^2 + c^2 - 2\,bc \cos A}.$$

The sides and one angle being known, the law of sines will give the other angles.

IV. If the three sides are given, the three angles may be found from the law of cosines. Thus, to find A, we have from the third of the equations (5),

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

While the law of sines and the law of cosines are theoretically all that is necessary to solve triangles, the law of cosines, which would have to be used in three out of the four cases, is not adapted to logarithmetic computation. The numerical work necessary to solve triangles will be greatly shortened by the use of other formulas which we will develop in the following articles.

Exercise 32

1. Apply the law of sines to a right triangle and reduce the resulting equations to their simplest form.

2. Apply the law of cosines, $a^2 = b^2 + c^2 - 2\,bc \cos A$, to the cases when the angle $A = 0°, 90°, 180°$.

Ans. $A = 0°$, $a^2 = b^2 + c^2 - 2\,bc = (b - c)^2$.
$A = 90°$, $a^2 = b^2 + c^2$.
$A = 180°$, $a^2 = b^2 + c^2 + 2\,bc = (b + c)^2$.

3. Apply the projection theorem to the case $A = 90°$; to the case $A = B$.

Solve the following problems without the aid of logarithms, —

4. Given $A = 35°$, $B = 75°$, $a = 7$; find b and c.

Ans. $b = 11.8$, $c = 11.5$.

5. Given $A = 65°$, $b = 10$, $a = 15$; find B and c.

Ans. $B = 37° \, 10'$, $c = 16.2$.

6. Given $A = 16°$, $b = 15$, $a = 6$; find the remaining parts.

Ans. $B = 43° \, 33.5'$, $C = 120° \, 26.5'$, $c = 18.76$.

or $B' = 136° \, 26.5'$, $C' = 27° \, 33.5'$, $c' = 10.07$.

7. Given $a = 150$, $b = 200$, $C = 27° \, 30'$; find c.

Ans. $c = 96.3$.

8. Given $a = 2$, $b = 3$, $c = 4$; find the angles.

Ans. $A = 28° \, 57.3'$, $B = 46° \, 34.1'$, $c = 104° \, 28.6'$.

9. By means of the law of sines prove that the bisector of an angle of any triangle divides the opposite side into segments proportional to the adjacent sides.

10. Derive the law of sines from the law of cosines.

(Suggestion. Form the ratio $\dfrac{\sqrt{1 - \cos^2 A}}{\sqrt{1 - \cos^2 B}} = \dfrac{\sin A}{\sin B}$ and show that it is equal to $\dfrac{a}{b}$.)

66. The Law of Tangents.

(*a*) *First proof.* Let ABC be any triangle. With a vertex C as a center and b, the shorter of the sides adjacent to C, as a radius, draw a circle cutting BC in P and BC produced in Q. Draw AP and AQ. Triangles ACP and ACQ are isosceles and QAP is a right angle (Why?). Denote the whole angle at A by w, and the three parts by x, y, z, as indicated in the figure, then angle $APC = x$ and angle $AQC = z$ (Why ?). Also

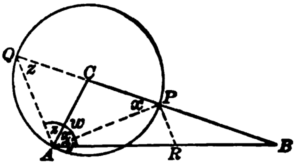

Fig. 68.

$$x + y = A, \quad x - y = B, \quad x + z = 90°, \quad x + y + z = w.$$

Solving these equations for x, y, z and w, we obtain

$$x = \tfrac{1}{2}(A + B), \quad y = \tfrac{1}{2}(A - B), \quad z = 90° - \tfrac{1}{2}(A + B),$$
$$w = 90° + \tfrac{1}{2}(A - B).$$

Now apply the law of sines to each of the triangles APB and AQB, thus, —

$$\frac{AB}{BP} = \frac{\sin(180 - x)}{\sin y} = \frac{\sin x}{\sin y}, \quad \text{and} \quad \frac{AB}{BQ} = \frac{\sin z}{\sin w},$$

or

$$\left.\begin{aligned}
&\frac{c}{a-b}=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}, \qquad \frac{c}{a+b}=\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}.\\
&\text{Similarly,}\\
&\frac{a}{b-c}=\frac{\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B-C)}, \qquad \frac{a}{b+c}=\frac{\cos\frac{1}{2}(B+C)}{\cos\frac{1}{2}(B-C)},\\
&\frac{b}{c-a}=\frac{\sin\frac{1}{2}(C+A)}{\sin\frac{1}{2}(C-A)}, \qquad \frac{b}{c+a}=\frac{\cos\frac{1}{2}(C+A)}{\cos\frac{1}{2}(C-A)}.
\end{aligned}\right\} \quad (6)$$

Dividing the first of each pair of equations by the second gives

$$\left.\begin{aligned}
&\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \qquad \frac{b+c}{b-c}=\frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)},\\
&\qquad\qquad \frac{c+a}{c-a}=\frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}.
\end{aligned}\right\} \quad (7)$$

Formulas (7) embody the

Law of tangents: In any triangle, the sum of two sides is to their difference as the tangent of half the sum of the angles opposite is to the tangent of half their difference.

The formulas (6), which we shall have occasion to use hereafter, we shall refer to by the name of *Double Formulas.*[*]

(*b*) *Second Proof.* The law of tangents can be proven more easily without the intervention of the double formulas. In Fig. 68 draw PR parallel to QA, then angle APR = angle $QAP = 90°$. From the similar triangles BQA and BPR, we have

$$\frac{BQ}{BP}=\frac{AQ}{RP}=\frac{\frac{AQ}{AP}}{\frac{RP}{AP}},$$

but $\quad BQ = a + b, \quad BP = a - b,$

and $\quad \dfrac{AQ}{AP} = \tan APQ = \tan x = \tan \frac{1}{2}(A + B),$

$$\frac{RP}{AP} = \tan RAP = \tan y = \tan \tfrac{1}{2}(A - B),$$

---

[*] Also called Mollweide's formulas, after the German astronomer (1774–1825) who introduced their use. The cosine form of these formulas appears in Newton's *Arithmetica universalis* (1707).

hence

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

The law of tangents and double formulas is adapted to the logarithmic solution of the third case of oblique triangles, that is, when two sides and the included angle are given. Suppose the given parts are a, b and C. The different steps in the solution are as follows:

1. $\frac{1}{2}(A+B)$ is found from the relation $A+B=180^\circ-C$.
2. $\frac{1}{2}(A-B)$ is found from the law of tangents, formula (7), 1st. equation.
3. Adding and subtracting the results of 1 and 2 we have
   $\frac{1}{2}(A+B)+\frac{1}{2}(A-B)=A$, $\quad\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B$.
4. c is found from the double formula (6).

Having found A and B, c could have been determined from the law of sines, thus, —

$$c=\frac{a\sin C}{\sin A},\quad\text{or}\quad c=\frac{b\sin C}{\sin B},$$

but this would require us to look up three new logarithms, namely, those of

$$a, \sin C, \sin A,\quad\text{or}\quad b, \sin C, \sin B.$$

while the double formula requires but two new logarithms, those of

$$\sin\tfrac{1}{2}(A+B), \sin\tfrac{1}{2}(A-B),\quad\text{or}\quad\cos\tfrac{1}{2}(A+B), \cos\tfrac{1}{2}(A-B),$$

and these may be taken out at the same time and the same opening of the table with the logarithms of $\tan\frac{1}{2}(A+B)$ and $\tan\frac{1}{2}(A-B)$.

67. Formulas for the Area of a Triangle.

(*a*) *In terms of the base c and the altitude h.* If c represents the base of the triangle, Figs. 65, 66, h its altitude and T its area, then by elementary geometry

$$T=\tfrac{1}{2}ch. \tag{1}$$

(*b*) *In terms of two sides b and c and the included angle A.* From the right triangle, Fig. 65 or 66, we have $h=b\sin A$. This value of h substituted in (1) gives

$$T=\tfrac{1}{2}bc\sin A. \tag{2}$$

(*c*) *In terms of one side c and the angles A, B, C.* By the law of sines,

$$b = \frac{c \sin B}{\sin C}.$$

Substituting this value of b in (2) we obtain

$$T = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C}. \tag{3}$$

(*d*) *In terms of the three sides a, b, c.* It is shown in plane geometry that

$$T = \sqrt{s(s-a)(s-b)(s-c)},^* \tag{4}$$

where $s = \frac{1}{2}(a + b + c)$, that is, s equals half the sum of the three sides.

(*e*) *In terms of s and the radius k of the inscribed circle.*

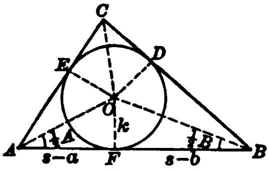

Fig. 69.

Let O be the center of the inscribed circle, Fig. 69. Join O to the vertices of the triangle, and draw the radii of the inscribed circle to the points of tangency. These radii will be perpendicular to the respective sides.

Area of triangle $BOC = \frac{1}{2} ka$,
Area of triangle $COA = \frac{1}{2} kb$,
Area of triangle $AOB = \frac{1}{2} kc$.

Adding,

$$T = \tfrac{1}{2} ka + \tfrac{1}{2} kb + \tfrac{1}{2} kc = \tfrac{1}{2} k(a + b + c),$$

or

$$T = ks, \text{ where } s = \tfrac{1}{2}(a + b + c). \tag{5}$$

If the three sides are known separately, (5) enables us to find the radius of the inscribed circle of a triangle, for we have

$$k = \frac{T}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{6}$$

on substituting the value of T from (4).

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[*] This formula is known as Hero's formula for the area of a triangle, after Hero of Alexandria (1st century B.C.), who, so far as we know, was the first to prove and apply this remarkable formula.

68. Functions of Half the Angles in Terms of the Sides.

In Fig. 69,

$$\begin{aligned} AF &= AE \\ BD &= BF \\ DC &= EC \end{aligned}$$

Adding, $AF + (BD + DC) = BF + (AE + EC)$,

and since the sum of the six segments equals $a + b + c = 2s$, we have

$$AF + (BD + DC) = s, \quad BF + (AE + EC) = s,$$

or

$$AF + a = s, \qquad BF + b = s,$$

from which

$$AF = s - a, \qquad BF = s - b. \tag{1}$$

Also, since the lines AO and BO bisect the angles A and B respectively, we have

$$\tan\frac{A}{2} = \frac{k}{AF}, \qquad \tan\frac{B}{2} = \frac{k}{BF}.$$

Substituting for AF and BF their values from (1), we have

$$\left.\begin{aligned} &\tan\frac{A}{2} = \frac{k}{s-a}, \\ &\tan\frac{B}{2} = \frac{k}{s-b}, \\ \text{similarly}\quad &\tan\frac{C}{2} = \frac{k}{s-c}, \end{aligned}\right\} \text{where } k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{2}$$

Again

$$\overline{AO}^2 = \overline{OF}^2 + \overline{AF}^2 = k^2 + (s-a)^2$$

$$= \frac{(s-a)(s-b)(s-c) + s(s-a)^2}{s} = \frac{bc(s-a)}{s}. \tag{3}$$

$$\sin\frac{A}{2} = \frac{OF}{AO} = \frac{k}{AO}, \qquad \cos\frac{A}{2} = \frac{AF}{AO} = \frac{s-a}{AO}.$$

Substituting for AO its value from (3) and reducing the result gives,—

$$\left.\begin{aligned} &\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, &\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \\ \text{Similarly}\quad &\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, &\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \\ &\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}, &\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}. \end{aligned}\right\} \tag{4}$$

In each of the radicals the positive root is to be taken, since each of the angles $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$, is necessarily less than 90°.

The formulas (2) and (4) are adapted for the logarithmetic computation of the angles of a triangle when the sides are given. For the quantities s, $s-a$, $s-b$, $s-c$, can easily be found, and with these known, the right member of each formula involves only multiplications, divisions and the extraction of square roots.

In general, the angles may be found from either the sine, cosine or tangent formulas, but since an angle near 90° cannot be accurately found from its sine, nor a very small angle from its cosine, the sine formulas are to be avoided when the angle is greater than 45° and the cosine formulas when the angle is less than 45°. When all the angles are required, the tangent formulas (2) should be used, since they require but four logarithms to be taken from the tables, that is, the logarithms of s, $s-a$, $s-b$ and $s-c$, while the sine and cosine formulas (4) require in addition the logarithms of a, b and c.

Exercise 33

1. Express in words each of the rules for the area of a triangle in Art. 67.

2. From each of the formulas (2) and (3) in Art. 67, write down two others by a cyclic advance of letters.

3. By comparing the formulas (2) and (4) in Art. 67 show that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Why could not the angles be found by this formula as well as by (2) or (4), Art. 68?

4. By comparing the expressions for sin A in Problem 3 with the expressions for $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ in (4), Art. 68, show that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}.$$

5. By forming the ratio of $\sin A : \sin B$, using the values given in Problem 3, show that

$$\sin A : \sin B = a : b.$$

This constitutes another proof of the law of sines.

6. From (3), Art. 62, we have $\sin A = \frac{a}{D}$, where D is the diameter of the circumscribed circle. By substituting this value for $\sin A$ in (2), Art. 67 show that

$$T = \frac{abc}{2D}.$$

7. From (5), Art. 67, we have for the radius of the inscribed circle

$$k = \frac{T}{s}.$$

By using the relation (Fig. 70)

$$ABC = ABO + ACO - BCO$$

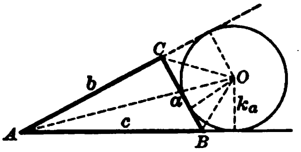


Fig. 70.

show that

$$k_a = \frac{T}{s-a}, \quad k_b = \frac{T}{s-b}, \quad k_c = \frac{T}{s-c},$$

where k_a, k_b, k_c are the radii of the escribed circles, touching the sides a, b, c respectively, externally.

8. Prove that

$$\frac{1}{k} = \frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c}.$$

9. Prove that $\qquad k_a + k_b + k_c - k = 2D.$

10. Let $ABCD$ (Fig. 71) be an inscribed quadrilateral, a, b, c, d its sides and Q its area. Show that

$$Q = \tfrac{1}{2}(ad + bc)\sin A. \qquad (a)$$

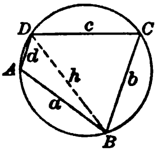


Fig. 71.

Also by comparing the two expressions for the diagonal

$$h^2 = a^2 + d^2 - 2\,ad\cos A = b^2 + c^2 - 2\,bc\cos C$$

show that

$$\cos A = \frac{a^2 - b^2 - c^2 + d}{2(ad + bc)}, \qquad (b)$$

and by substituting this value in $\sin^2 A = 1 - \cos^2 A$, show that

$$\sin A = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ad + bc}, \qquad (c)$$

where $\qquad s = \tfrac{1}{2}(a + b + c + d).$

By substituting (c) in (a), show that

$$Q = \sqrt{(s-a)(s-b)(s-c)(s-d)}. \qquad (d)$$

11. It was shown in Art. 65, II, that if two sides b, c, of a triangle ABC, and the angle B opposite one of these sides, are given, the third side a may be found from the relation

$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin B}.$$

Interpret the result geometrically.

Ans. The two terms on the right are respectively the distances from B and C to the foot of the perpendiculars drawn from A to BC or BC produced.

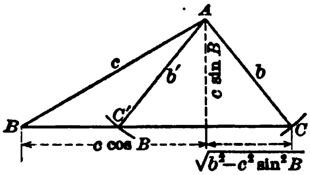


Fig. 72.

CHAPTER VIII

SOLUTION OF OBLIQUE TRIANGLES

69. Solution of Oblique Triangles. In the present chapter we shall illustrate the computation of each case of oblique triangles by a numerical example. Since five-place tables are used in the computation, the results are given to only five significant figures and the angles to the nearest second. In every case a check has been applied to the results obtained. Such a check is to be looked upon as an essential step in the solution, since no computation, no matter by whom, can be relied upon if it has not been checked. Instead of the analytic checks here given, graphic checks are often resorted to in practice, but such checks, while more easily applied, are of course open to errors of construction and are therefore unsatisfactory, except as checks against gross errors.

The solutions given below are arranged in a form which may serve as a model to beginners. It is customary for computers to make out a complete schedule of work (as is illustrated in the first case below) before referring to the tables, so that when the tables are once opened no writing remains to be done except that of filling in the numbers taken from the tables.

70. Case I. Given two angles and one side, as A, B and c.

(1) To find C, apply the relation $A + B + C = 180°$.
(2) To find a and b, apply the law of sines.
(3) To check, use the double formula.

EXAMPLE 1.

Given

$A = 71° \, 13' \, 30''$,
$B = 40° \, 34' \, 15''$,
$c = 236.54$.

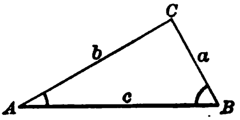


Fig. 73.

Required

$a = 241.18$,
$b = 165.68$,
$C = 68° \, 12' \, 15''$.

Schedule of Work.

(1) To find C. $\quad C = 180° - (A + B) = 68° \, 12' \, 15''$.
(2) To find a and b. By the law of sines, —

$$\frac{a}{c}=\frac{\sin A}{\sin C}\text{ or } a=\frac{c\sin A}{\sin C}. \qquad \frac{b}{c}=\frac{\sin B}{\sin C}\text{ or } b=\frac{c\sin B}{\sin C}.$$

$$\begin{aligned}
\log c &= \\
\log\sin A &= \\
\text{colog}\,\sin C &= \underline{\hspace{2cm}} \\
\log a &= \\
a &=
\end{aligned}
\qquad\qquad
\begin{aligned}
\log c &= \\
\log\sin B &= \\
\text{colog}\,\sin C &= \underline{\hspace{2cm}} \\
\log b &= \\
b &=
\end{aligned}$$

(3) Check.* By the double formula (6), left,

$$\frac{c}{a-b}=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\quad\text{or}\quad a-b=\frac{c\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)},$$

$$\tfrac{1}{2}(A-B)=15^\circ\,19'\,37'',\qquad \tfrac{1}{2}(A+B)=55^\circ\,53'\,52''.$$

$$\begin{aligned}
\log c &= \\
\log\sin\tfrac{1}{2}(A-B) &= \\
\text{colog}\,\sin\tfrac{1}{2}(A+B) &= \underline{\hspace{2cm}} \\
\log(a-b) &= \qquad\qquad\qquad a-b=
\end{aligned}$$

If the computation is correct this value of $a-b$ must agree with the value of $a-b$ obtained from (2) above.

Having completed a schedule as above, we now turn to the tables and complete the solution by filling in the missing numbers as follows:

Solution.

$$\begin{aligned}
\log c &= 2.37388 \\
\log\sin A &= 9.97626-10 \\
\text{colog}\,\sin C &= \underline{0.03221\hspace{1cm}} \\
\log a &= 2.38235 \\
a &= 241.18.
\end{aligned}
\qquad\qquad
\begin{aligned}
\log c &= 2.37388 \\
\log\sin B &= 9.81318-10 \\
\text{colog}\,\sin C &= \underline{0.03221\hspace{1cm}} \\
\log b &= 2.21927 \\
b &= 165.68.
\end{aligned}$$

Check.

$$\begin{aligned}
\log c &= 2.37388 \\
\log\sin\tfrac{1}{2}(A-B) &= 9.42214-10 \\
\text{colog}\,\sin\tfrac{1}{2}(A+B) &= \underline{0.08195\hspace{1cm}} \\
\log(a-b) &= 1.87797, \qquad\qquad a-b=75.50.
\end{aligned}$$

Beginners will do well to follow the above form. Expert computers save the repetition of recurring numbers such as log c in the above example by employing a more compact arrangement. For instance, the above solution and check can be put in the following

---

* Some authors check by the law of sines, $a : b = \sin A : \sin B$, but this check is unreliable, for it fails to detect an error in either c or log sin C.

Compact Arrangement

	Solution.	Check.
$c = 236.54$	2.37388	2.37388
$A = 71^\circ\,13'\,30''$	9.97626	
$B = 40^\circ\,34'\,15''$	9.81318	
$C = 68^\circ\,12'\,15''$	0.03221	
$\frac{1}{2}(A - B) = 15^\circ\,19'\,37''$		9.42214
$\frac{1}{2}(A + B) = 55^\circ\,53'\,52''$		0.08195
$a = 241.18$	2.38235	
$b = 165.68$	2.21927	
$a - b = 75.50$		1.87797

EXAMPLE 2. Find the area of the triangle given in Example 1.

Solution. Since one side and two angles are given, we use formula (3), Art. 67,

$$T = \frac{c^2 \sin A \sin B}{2 \sin C}$$

$$\begin{aligned}
\log c &= 2.37388\\
\log c^2 = 2 \log c &= 4.74776\\
\log \sin A &= 9.97626 - 10\\
\log \sin B &= 9.81318 - 10\\
\operatorname{colog} \sin C &= 0.03221\\
\operatorname{colog} 2 &= \underline{9.69897 - 10}\\
\log T &= 4.26838, \qquad T = 18552.
\end{aligned}$$

EXERCISE 34

The student must check his results when no answer is given.

1. Given $A = 46^\circ\,36'$, $B = 54^\circ\,18'$, $c = 479$.
   Find $a = 345$, $b = 396$, $C = 79^\circ\,06'$.
2. Given $A = 79^\circ\,59'$, $B = 44^\circ\,41'$, $a = 79.5$.
   Find $b = 56.8$, $c = 66.4$, $C = 55^\circ\,20'$.
3. Given $A = 54^\circ\,34'$, $B = 43^\circ\,56'$, $c = 67.9$. Find a, b, C.
4. Given $A = 69^\circ\,30.2'$, $B = 66^\circ\,39.4'$, $c = 438.3$.
   Find $a = 592.7$, $b = 581.0$, $C = 43^\circ\,50.4'$.
5. Given $A = 29^\circ\,41.2'$, $B = 37^\circ\,50.4'$, $a = 32.84$.
   Find $b = 40.68$, $c = 61.27$, $C = 112^\circ\,28.4'$.
6. Given $B = 78^\circ\,45.6'$, $C = 63^\circ\,32.9'$, $a = 8.875$. Find b, c, A.
7. Given $A = 64^\circ\,56'\,18''$, $B = 47^\circ\,29'\,11''$, $c = 913.45$.
   Find $a = 895.14$, $b = 728.40$, $C = 67^\circ\,34'\,31''$.

8. Given $B = 48^\circ\ 24'\ 15''$, $C = 31^\circ\ 13'\ 00''$, $c = 926.74$.
   Find $b = 1337.2$, $a = 1758.9$, $A = 100^\circ\ 22'\ 45''$.
9. Find the radius of the circumscribed circle in 8.
   Ans. $R = 894.06$.
10. Find the area of the triangle in 7. *Ans.* Area = 301,360.
11. Find the area and the radius of the circumscribed circle in 2. Check by using the relation in Problem 6, Exercise 33.

71. Case II. Given two sides and the angle opposite one of them, as *a*, *b*, *A*.

(1) To find B, apply the law of sines.
(2) To find C, apply the relation $A + B + C = 180^\circ$.
(3) To find c, apply the law of sines.
(4) To check, apply the double formula or the law of tangents.

When an angle is determined from its sine, as the angle B above, it admits of two values which are supplements of each other. Whether one or the other or both of these values are to be used depends on the conditions imposed by the problem. By constructing the triangle graphically, it is seen that various cases may arise depending on the relations between the given parts a, b and A.

Construct angle BAC equal to the given angle A, making CA equal to b, and from C as a center, with a radius equal to a, draw a circle.

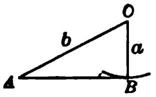
If a is less than the perpendicular distance from C to AB, the circle will not cut the line AB. In this case it is impossible to construct a triangle having the given parts, that is, the given data are inconsistent.

If a is equal to the perpendicular distance from C to AB, the circle will be tangent to the line AB at B (Fig. 74), and the resulting triangle will have a right angle at B. Now the perpendicular distance from C to AB is $b \sin A$, hence in this case $a = b \sin A$.

If a is greater than the perpendicular distance from C to AB but less than b, the circle will cut AB in two points B and B' (Fig. 75), and there are two solutions, namely, the triangle ABC and the triangle $AB'C$.

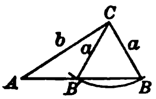
If a is equal or greater than b, the circle will cut the line AB in a single point B (Fig. 76), the second point of intersection falling on or to the left of A. In this case there will be but one solution.

So far we have assumed the given angle A to be acute. If A is obtuse, a must be greater than b (since A is greater than B, and in any triangle the greater angle is opposite the greater side). In this case there can be but one solution. (Fig. 77.)



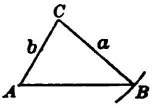
$a = b \sin A$.

Fig. 74.



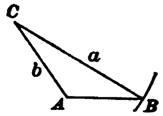
$b \sin A < a < b$.

Fig. 75.



$a \geqq b$.

Fig. 76.



$a > b$.

Fig. 77.

If we disregard the case in which the triangle is right-angled (Fig. 74) as not properly constituting a case of oblique triangles, we have the following simple test for the number of possible solutions,—

$a \geqq b$, one solution,

$a < b$, two solutions.

EXAMPLE 1.

Given

$a = 345.46$,

$b = 531.75$,

$A = 26° 47' 32''$.

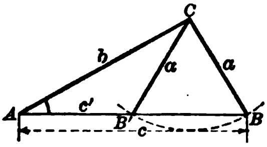


Fig. 78.

Required

$B = 43° 56' 00''$,

$B' = 136° 04' 00''$,

$C = 109° 16' 28''$,

$C' = 17° 08' 28''$,

$c = 723.45$.

$c' = 225.88$.

Solution. $a < b$, hence there are two solutions.

(1) To find B and B'. By the law of sines,—

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{or} \quad \sin B = \frac{b \sin A}{a}.$$

$$\begin{aligned} \log b &= 2.72571 \\ \log \sin A &= 9.65394 - 10 \\ \operatorname{colog} a &= \underline{7.46160 - 10} \\ \log \sin B &= 9.84125 - 10 \\ B &= 43° 56' 00''. \qquad B' = 136° 04' 00''. \end{aligned}$$

(2) To find C and C'.

$C = 180° - (A + B) = 109° 16' 28''$. $C' = 180° - (A + B') = 17° 08' 28''$.

(3) To find c and c'. By the law of sines $\frac{c}{a} = \frac{\sin C}{\sin A}$, or

$$c = \frac{a \sin C}{\sin A}. \qquad\qquad c' = \frac{a \sin C'}{\sin A}.$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log a = & 2.53840 & \log a = & 2.53840 \\
\log \sin C = & 9.97495 - 10 & \log \sin C' = & 9.46942 - 10 \\
\text{colog} \sin A = & \underline{0.34606\qquad} & \text{colog} \sin A = & \underline{0.34606\qquad} \\
\log c = & 2.85941 & \log c' = & 2.35388 \\
c = & 723.45. & c' = & 225.88.
\end{array}$$

(4) Check. By the double formula,—

$$\frac{c}{b-a} = \frac{\sin \frac{1}{2}(B+A)}{\sin \frac{1}{2}(B-A)}$$

or

$$\begin{aligned}
c \sin \tfrac{1}{2}(B-A) &= (b-a) \sin \tfrac{1}{2}(B+A), \\
\tfrac{1}{2}(B+A) &= 35^\circ\, 21'\, 46'', \\
\tfrac{1}{2}(B-A) &= 8^\circ\, 34'\, 14'', \\
c' \sin \tfrac{1}{2}(B'-A) &= (b-a) \sin \tfrac{1}{2}(B'+A), \\
\tfrac{1}{2}(B'+A) &= 81^\circ\, 25'\, 46'', \\
\tfrac{1}{2}(B'-A) &= 54^\circ\, 38'\, 14'', \\
b-a &= 186.29.
\end{aligned}$$

$$\begin{array}{rl@{\qquad}rl}
\log c = & 2.85941 & \log (b-a) = & 2.27019 \\
\log \sin \frac{1}{2}(B-A) = & \underline{9.17327 - 10} & \log \sin \frac{1}{2}(B+A) = & \underline{9.76249 - 10} \\
 & 2.03268 & & 2.03268 \\
\log c' = & 2.35388 & \log (b-a) = & 2.27019 \\
\log \sin \frac{1}{2}(B'-A) = & \underline{9.91143} - 10 & \log \sin \frac{1}{2}(B'+A) = & \underline{9.99513} - 10 \\
 & 2.26531 & & 2.26532
\end{array}$$

Compact Arrangement

	Solution.		Check.
$b = 531.75$	log 2.72571		
$A = 26^\circ\, 47'\, 32''$	log 9.65394	colog 0.34606	
$a = 345.46$	colog $\underline{7.46160}$	log 2.53850	
$B = 43^\circ\, 56'\, 00''$	log 9.84125		
$C = 109^\circ\, 16'\, 28''$		log $\underline{9.97496}$	
$c = 723.45$		log 2.85941	log 2.85941
$\frac{1}{2}(B-A) = 8^\circ\, 34'\, 46''$			log 9.17327
$\frac{1}{2}(B+A) = 35^\circ\, 21'\, 46''$			colog 0.23751
$b-a = 186.29$			colog $\underline{7.72981}$
			0.00000

$B' = 136^\circ\,04'\,00''$		
$C' = 17^\circ\,08'\,28''$	$\log \underline{9.46942}$	
$c' = 225.88$	$\log 2.35388$	$\log 2.35388$
$\frac{1}{2}(B' - A) = 54^\circ 38' 14''$		$\log 9.91143$
$\frac{1}{2}(B' + A) = 81^\circ 25' 46''$		$\operatorname{colog} 0.00487$
$b - a = 186.29$		$\operatorname{colog} \underline{7.72981}$
		9.99999

EXERCISE 35

Solve the following triangles:

1. $a = 840,\quad b = 485,\quad A = 21^\circ\,31'.$
   Ans. $B = 12^\circ\,14',\quad C = 146^\circ\,15',\quad c = 1272.$
2. $a = 41.4,\quad b = 52.8,\quad A = 40^\circ\,19'.$
   Ans. $B = 55^\circ 36'\quad C = 84^\circ\,05',\quad c = 63.6.$
   $B' = 124^\circ\,24',\quad C' = 15^\circ\,17',\quad c' = 16.9.$
3. $a = 3.25,\quad b = 2.57,\quad A = 32^\circ\,54'.$
4. $a = 242,\quad b = 767,\quad B = 36^\circ\,53'.$
   Ans. $A = 10^\circ\,55',\quad C = 132^\circ\,12',\quad c = 947.$
5. $a = 91.97,\quad b = 93.99,\quad B = 120^\circ\,35'.$
   Ans. $A = 57^\circ\,23.7',\quad C = 2^\circ\,01.3',\quad c = 3.85.$
6. $b = 978.7,\quad c = 871.6,\quad C = 38^\circ\,14.2'.$
   Ans. $B = 44^\circ\,01.5',\quad A = 97^\circ\,44.3',\quad a = 1395.$
   $B' = 135^\circ\,58.5',\quad A' = 5^\circ\,47.3',\quad a' = 142.$
7. $b = 678.5,\quad c = 423.1,\quad C = 53^\circ\,23.4'.$
8. $a = 48.134,\quad b = 35.826,\quad A = 36^\circ\,24'\,00''.$
   Ans. $B = 26^\circ\,12'\,38'',\quad C = 117^\circ\,23'\,22'',\quad c = 72.022.$
9. $b = 216.45,\quad c = 177.01,\quad C = 35^\circ\,36'\,20''.$
   Ans. $B = 45^\circ\,23'\,28'',\quad A = 99^\circ\,00'\,12'',\quad a = 300.29.$
   $B' = 134^\circ\,36'\,32'',\quad A' = 9^\circ\,47'\,08'',\quad a' = 51.67.$
10. $b = 14.332,\quad c = 13.617,\quad C = 45^\circ\,23'\,54''.$
11. $a = 342.6,\quad b = 745.9,\quad A = 43^\circ\,35.6'.$
    Ans. Impossible.

72. Case III. Given two sides and the included angle, as *a*, *b*, *C*.

To find A and B, we first find $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$.

(1) To find $\frac{1}{2}(A + B)$, apply the relation $A + B + C = 180^\circ$.

(2) To find $\frac{1}{2}(A - B)$, apply the law of tangents, Art. 66.

(3) $A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$, $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$.

(4) To find c, apply the law of sines.

(5) To check, apply one of the double formulas.

EXAMPLE I.

Given
$a = 12.346$,
$b = 5.7213$,
$C = 65° 30' 10''$.

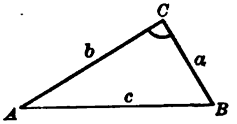


Fig. 79.

Required
$A = 86° 55' 57''$,
$B = 27° 33' 53''$,
$c = 11.250$.

Solution.

(1) To find $\frac{1}{2}(A + B)$. $\frac{1}{2}(A + B) = \frac{1}{2}(180° - C) = 57° 14' 55''$.

(2) To find $\frac{1}{2}(A - B)$. By the law of tangents

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \quad \text{or} \quad \tan \frac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \frac{1}{2}(A + B).$$

$a - b = 6.6247$, $a + b = 18.0673$.

$$\begin{aligned} \log(a - b) &= 0.82117 \\ \text{colog}(a + b) &= 8.74310 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= 0.19162 \\ \hline \log \tan \tfrac{1}{2}(A - B) &= 9.75589 - 10 \\ \tfrac{1}{2}(A - B) &= 29° 41' 02''. \end{aligned}$$

(3) To find A and B.

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = 86° 55' 57'',$$
$$B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = 27° 33' 53''.$$

(4) To find c. From the law of sines we have either

$$c = a\frac{\sin C}{\sin A}, \quad \text{or} \quad c = b\frac{\sin C}{\sin B},$$

but since A is an angle near 90° it is preferable to find c from the second expression.

$$\begin{aligned} \log b &= 0.75749 \\ \log \sin C &= 9.95903 - 10 \\ \text{colog} \sin B &= 0.33464 \\ \hline \log c &= 1.05116 \\ c &= 11.250. \end{aligned}$$

(5) Check. By the double formula (Art. 66, (6)),

$$\frac{c}{a-b} = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

$$\begin{array}{rlrl} \log c = & 1.05116 & \log(a-b) = & 0.82117 \\ \log\sin\frac{1}{2}(A-B) = & \underline{9.69480-10} & \log\sin\frac{1}{2}(A+B) = & \underline{9.92481-10} \\ & 0.74596 & & 0.74598 \end{array}$$

Compact Arrangement

	Solution.	Check.	
$a = 12.346$			
$b = 5.7213$		log 0.75749	
$a - b = 6.6247$	log 0.82117		log 0.82117
$a + b = 18.0673$	colog 8.74310		
$C = 65° 30' 10''$		log sin 9.95903	
$\frac{1}{2}(A + B) = 57° 14' 55''$	log tan $\underline{0.19162}$		log sin 9.92481
$\frac{1}{2}(A - B) = 29° 41' 02''$	log tan 9.75589		colog sin $\underline{0.30520}$
$A = 86° 55' 57''$			
$B = 27° 33' 53''$		colog sin $\underline{0.33464}$	
$c = 11.250$		1.05116	1.05118

EXERCISE 36

1. Given $a = 486$, $b = 347$, $C = 51° 36'$.
   Find $A = 83° 15'$, $B = 45° 09'$, $c = 383.5$.
2. Given $a = 364$, $b = 640$, $C = 53° 14'$.
   Find $A = 34° 38'$, $B = 92° 08'$, $c = 513$.
3. Given $a = 875$, $b = 567$, $C = 34° 52'$. Find A, B, c.
4. Given $a = 233.4$, $b = 557.2$, $C = 18° 23.0'$.
   Find $A = 12° 22.0'$, $B = 149° 15.0'$, $c = 343.7$.
5. Given $b = 145.9$, $c = 39.90$, $A = 92° 11.3'$.
   Find $B = 72° 40.7'$, $C = 15° 08.0'$, $a = 152.7$.
6. Given $c = 453.9$, $a = 478.1$, $B = 35° 37.9'$. Find C, A, b.
7. Given $a = 51.269$, $b = 14.687$, $C = 62° 09' 24''$.
   Find $A = 101° 32' 32''$, $B = 16° 18' 04''$, $c = 46.269$.
8. Given $b = 467.92$, $c = 612.34$, $A = 45° 29' 16''$.
   Find $B = 49° 34' 05''$, $c = 84° 56' 39''$, $a = 438.36$.

9. Given $c = 345.67$, $a = 654.32$, $B = 67^\circ\ 45'\ 45''$. Find C, A, b.

0. Given $a = 447.45$, $b = 216.45$, $C = 116^\circ\ 30'\ 20''$. Find the area.
Ans. $T = 43336$.

11. Show that when the included angle is a right angle, the law of tangents gives

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b}.$$

73. Case IV. Given three sides, *a*, *b*, *c*.

Each of the angles A, B and C is found by applying one of the formulas (2) or (4), Art. 68, for the tangent, sine or cosine of half the respective angle, but for the reasons stated in Art. 68 the tangent formulas are generally to be preferred.

To check, apply the relation $A + B + C = 180^\circ$.

EXAMPLE 1.

Given
$a = 12.653$,
$b = 17.213$,
$c = 23.106$.

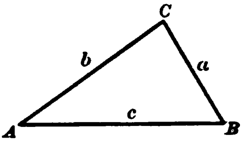


Fig. 80.

Required
$A = 32^\circ\ 36'\ 22''$,
$B = 47^\circ\ 08'\ 42''$,
$C = 100^\circ\ 14'\ 56''$.

Solution. By formula (2), Art. 68,

$$\tan\frac{A}{2} = \frac{k}{s-a}, \quad \tan\frac{B}{2} = \frac{k}{s-b}, \quad \tan\frac{C}{2} = \frac{k}{s-c},$$

where

$$s = \frac{a+b+c}{2}, \quad k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\begin{array}{rlrl} s &= 26.486 & \operatorname{colog} s &= 8.57698 - 10 \\ s - a &= 13.833 & \log(s-a) &= 1.14092 \\ s - b &= 9.273 & \log(s-b) &= 0.96722 \\ s - c &= 3.380 & \log(s-c) &= \underline{0.52892} \\ & & \log k^2 &= 1.21404 \\ & & \log k &= 0.60702 \end{array}$$

$$\begin{array}{rlrlrl} \log k &= 0.60702 & \log k &= 0.60702 & \log k &= 0.60702 \\ \log(s-a) &= \underline{1.14092} & \log(s-b) &= \underline{0.96722} & \log(s-c) &= \underline{0.52892} \\ \log\tan\frac{1}{2}A &= 9.46610 & \log\tan\frac{1}{2}B &= 9.63980 & \log\tan\frac{1}{2}C &= 0.07810 \\ \frac{1}{2}A &= 16^\circ\ 18'\ 11'' & \frac{1}{2}B &= 23^\circ\ 34'\ 21'' & \frac{1}{2}C &= 50^\circ\ 07'\ 28'' \\ A &= 32^\circ\ 36'\ 22'', & B &= 47^\circ\ 08'\ 42'', & C &= 100^\circ\ 14'\ 56''. \end{array}$$

Check. $A + B + C = 32^\circ\ 36'\ 22'' + 47^\circ\ 08'\ 42'' + 100^\circ\ 14'\ 56'' = 180^\circ$.

Compact Arrangement

	Solution.	Check.
$a = 12.653$		
$b = 17.213$		
$c = 23.106$		
$s = 26.486$	colog 8.57698	
$s - a = 13.833$	log 1.14092	
$s - b = 9.273$	log 0.96722	
$s - c = 3.380$	log 0.52892	
k^2	log 1.21404	
k	log 0.60702	
$\frac{1}{2}A = 16^\circ\, 18'\, 11''$	log tan 9.46610	$A = 32^\circ\, 36'\, 22''$
$\frac{1}{2}B = 23^\circ\, 34'\, 21''$	log tan 9.63980	$B = 47^\circ\, 08'\, 42''$
$\frac{1}{2}C = 50^\circ\, 07'\, 28''$	log tan 0.07810	$C = 100^\circ\, 14'\, 56''$
		$180^\circ\, 00'\, 00''$

EXAMPLE 2. Find the area of the triangle and the radii of the inscribed and escribed circles for the triangle in Example 1.

Solution. $a = 12.653, \quad b = 17.213, \quad c = 23.106.$

The area of a triangle in terms of the sides is given by (4), Art. 67.

$$T = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

The formulas for the radii of the inscribed and escribed circles are given in Problem 7, Exercise 33.

$$k = \frac{T}{s}, \quad k_a = \frac{T}{s-a}, \quad k_b = \frac{T}{s-b}, \quad k_c = \frac{T}{s-c}.$$

Using the results of Example 1, we have

$\log s = 1.42302$	
$\log (s-a) = 1.14092$	
$\log (s-b) = 0.96722$	
$\log (s-c) = 0.52892$	
$\log T^2 = 4.06008$	
$\log T = 2.03004$	$T = 107.16,$
$\log k = 0.60702$	$k = 4.046,$
$\log k_a = 0.88912$	$k_a = 7.747,$
$\log k_b = 1.06282$	$k_b = 11.556,$
$\log k_c = 1.50112$	$k_c = 31.704.$

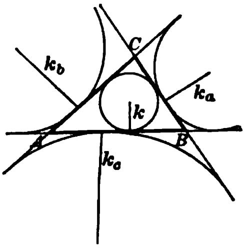


Fig. 81.

Check. A convenient check is obtained by using the relation in Problem 8, Exercise 33,

$$\frac{1}{k} = \frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c}.$$

$$\log\frac{1}{k} = \text{colog}\, k = 9.39298 - 10 \qquad \frac{1}{k} = 0.24716$$

$$\log\frac{1}{k_a} = \text{colog}\, k_a = 9.11088 - 10 \qquad \frac{1}{k_a} = 0.12909$$

$$\log\frac{1}{k_b} = \text{colog}\, k_b = 8.93718 - 10 \qquad \frac{1}{k_b} = 0.08653$$

$$\log\frac{1}{k_c} = \text{colog}\, k_c = 8.49888 - 10 \qquad \frac{1}{k_c} = 0.03154$$

$$\frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c} = 0.24716 = \frac{1}{k}.$$

EXERCISE 37

1. Given $a = 286$, $b = 321$, $c = 463$.
Find $A = 37° 34'$, $B = 43° 11'$, $C = 99° 15'$.

2. Given $a = 3.21$, $b = 3.61$, $c = 4.02$.
Find $A = 49° 24'$, $B = 58° 38'$, $C = 71° 58'$.

3. Given $a = 74.6$, $b = 81.9$, $c = 90.0$. Find the angles.

4. Given $a = 354.4$, $b = 277.9$, $c = 401.3$.
Find $A = 59° 39.5'$, $B = 42° 35.3'$, $C = 77° 45.2'$.

5. Given $a = 1.961$, $b = 2.641$, $c = 1.354$.
Find $A = 46° 03.7'$, $B = 104° 07.6'$, $C = 29° 48.8'$.

6. Given $a = 87.06$, $b = 9.16$, $c = 79.02$. Find A, B, C.

7. Given $a = 3359.4$, $b = 4216.3$, $c = 4098.7$.
Find $A = 47° 38' 00''$, $B = 68° 01' 06''$, $C = 64° 20' 54''$.

8. Given $a = 33.112$, $b = 44.224$, $c = 55.336$.
Find $A = 36° 45' 14''$, $B = 53° 03' 08''$, $C = 90° 11' 38''$.

9. Given $a = 14.493$, $b = 55.436$, $c = 66.913$. Find the angles.

10. Given $a = 46.78$, $b = 35.90$, $c = 77.00$. Find the area.
Ans. $T = 573.91$.

11. Find the area in Problem 2. Check.

12. Find the radii of the inscribed circle, of the escribed circles and the circumcircle of the triangle in problem 10.

Check by using the relation $k_a + k_b + k_c - k = 2\ D = 4\ R$, Problem 9, Exercise 33.

13. $a = 4$, $b = 5$, $c = 6$. Find the angles by applying the law of cosines directly, that is,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \cos B = \text{etc.}$$

14. The sides of an inscribed quadrilateral taken in order are

$$a = 56,\ b = 33,\ c = 16,\ d = 63.$$

Required the area Q of the quadrilateral and the angles A, B, C, D, the notation being that employed in Fig. 71, Problem 10, Exercise 33.

Ans. $Q = 1428$, $A = 44° 46'$, $B = 90°$, $C = 135° 14'$, $D = 90°$.

74. Practical Applications. The solution of oblique triangles finds numerous applications in the various arts and sciences. Chief among these are surveying, engineering, physics, astronomy and navigation. The simpler applications which involve the solution of a single triangle need no explanation, since they may be immediately referred to some one of the four cases treated in the preceding sections. Many of the practical applications, however, involve several triangles which must be successively solved in whole or in part before the required distance or angle can be ascertained. Sometimes the intermediate triangles to be solved are not apparent from the figure, but must be sought by some auxiliary geometrical construction. Again, it may happen that no single triangle exists containing the requisite number of parts; in such cases the solution is effected by solving the equations which arise by applying the formulas of Chapter VII so as to involve the unknown parts. We shall illustrate each of these cases by an example.

(*a*) *System of triangles.* EXAMPLE 1. Given one side c of a quadilateral $ABC'C$ (Fig. 82) and two angles A and B adjacent to this side, also the angles α, β which the diagonals d_1, d_2, drawn from A and B respectively, make with the given side, to determine the side x opposite the given side c. This problem is sometimes known as *Hansen's problem.*

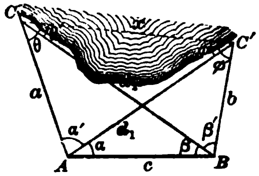


Fig. 82.

Analysis. Let $AC = a$, $BC' = b$, angle $CAC' = \alpha'$, angle $CBC' = \beta'$, angle $ACC' = \theta$, angle $BCC' = \theta'$, angle $AC'C = \phi$, angle $BC'C = \phi'$.

(1) In the triangle ABC, one side c and two adjacent angles A, β are known, hence d_2 may be found.

(2) In the triangle ABC', one side c and two adjacent angles α, B are known, hence b may be found.

(3) Then, in the triangle CBC', two sides d_2, b and the included angle $\beta' = B - \beta$ * are known, hence x may be found.

(4) Check. Compute x again, using the triangle CAC', having previously found a and d_1 from the triangles ABC and ABC' respectively.

Illustration. In order to determine the length CC' (Fig. 82) of a trestle to be built across the end of a lagoon, a distance AB, 500 ft. long, was measured off, and the following angles were measured with a transit:

$$CAB = 105^\circ\, 30' = A, \quad C'BA = 95^\circ\, 50' = B,$$
$$C'AB = 35^\circ\, 17' = \alpha, \quad CBA = 47^\circ\, 32' = \beta.$$

Required the distance $CC' = x$.

Solution. (1) Triangle ABC. By the law of sines

$$d_2 = c\frac{\sin A}{\sin ACB}, \quad \text{and} \quad a = c\frac{\sin \beta}{\sin ACB}.$$

$$ACB = 180^\circ - (A + \beta) = 26^\circ\, 58'.$$

$$\begin{array}{rlrl}
\log c &= 2.69897 & &= 2.69897 \\
\log \sin A &= 9.98391\dagger & \log \sin \beta &= 9.86786 \\
\operatorname{colog} \sin ACB &= \underline{0.34345} & &= \underline{0.34345} \\
\log d_2 &= 3.02633 & \log a &= 2.91028 \\
d_2 &= 1062.5. & a &= 813.36.
\end{array}$$

---

* It is assumed that the points A, B, C, C' are in the same plane, otherwise the angles CAC' and CBC' must be given in addition to the data of the problem.

† 9.98391 − 10. In this and the following problems the − 10's are omitted when this may be done without danger of confusion. This is a common practice among computers.

(2) Triangle ABC'. By the law of sines

$$b = c\,\frac{\sin\alpha}{\sin AC'B}, \quad \text{and} \quad d_1 = c\,\frac{\sin B}{\sin AC'B}.$$

$$AC'B = 180^\circ - (B + \alpha) = 48^\circ\ 53'.$$

$$\begin{array}{rl@{\qquad\qquad}rl}
\log c = & 2.69897 & = & 2.69897 \\
\log \sin\alpha = & 9.76164 & \log \sin B = & 9.99775 \\
\operatorname{colog} \sin AC'B = & \underline{0.12299} & = & \underline{0.12299} \\
\log b = & 2.58360 & \log d_1 = & 2.81971 \\
b = & 383.35. & d_1 = & 660.25.
\end{array}$$

(3) Triangle CBC'. Triangle CAC'.

By the law of tangents

$$\tan\frac{\phi' - \theta'}{2} = \frac{d_2 - b}{d_2 + b}\tan\frac{\phi' + \theta'}{2}, \qquad \tan\frac{\phi - \theta}{2} = \frac{a - d_1}{a + d_1}\tan\frac{\phi + \theta}{2}.$$

$\frac{1}{2}(\phi'+\theta') = \frac{1}{2}(180^\circ - \beta) = 48^\circ\ 18'$. $\frac{1}{2}(\phi+\theta) = \frac{1}{2}(180^\circ - \alpha') = 54^\circ 53.5'$.

$d_2 - b = 679.15$, $d_2 + b = 1445.85$, $a - d_1 = 153.11$, $a + d_1 = 1473.61$,

$$\begin{array}{rl@{\qquad}rl}
\log(d_2 - b) = & 2.83196 & \log(a - d_1) = & 2.18501 \\
\operatorname{colog}(d_2 + b) = & 6.83988 & \operatorname{colog}(a + d_1) = & 6.83162 \\
\log\tan\frac{1}{2}(\phi' + \theta') = & \underline{0.34836} & \log\tan\frac{1}{2}(\phi + \theta) = & \underline{0.15302} \\
\log\tan\frac{1}{2}(\phi' - \theta') = & 0.02020 & \log\tan\frac{1}{2}(\phi - \theta) = & 9.16965 \\
\frac{1}{2}(\phi' - \theta') = & 46^\circ\ 19'\ 55''. & \frac{1}{2}(\phi - \theta) = & 8^\circ\ 24'\ 25''.
\end{array}$$

By the double formula involving the sines,

$$x = (d_2 - b)\,\frac{\sin\frac{1}{2}(\phi' + \theta')}{\sin\frac{1}{2}(\phi' - \theta')}, \qquad x = (a - d_1)\,\frac{\sin\frac{1}{2}(\phi + \theta)}{\sin\frac{1}{2}(\phi - \theta)}.$$

$$\begin{array}{rl@{\qquad}rl}
\log(d_2 - b) = & 2.83196 & \log(a - d_1) = & 2.18501 \\
\log\sin\frac{1}{2}(\phi' + \theta') = & 9.96022 & \log\sin\frac{1}{2}(\phi + \theta) = & 9.91278 \\
\operatorname{colog}\sin\frac{1}{2}(\phi' - \theta') = & \underline{0.14065} & \operatorname{colog}\sin\frac{1}{2}(\phi - \theta) = & \underline{0.83505} \\
\log x = & 2.93283 & \log x = & 2.93284 \\
x = & 856.7, & x = & 856.7 \text{ (check).}
\end{array}$$

(*b*) *Auxiliary geometrical constructions.* EXAMPLE 2. In Example 1 it was shown how the distance CC' may be found without leaving the line AB. Another important problem is to determine one's position from the angles which the sides of a known triangle subtend from that position. This is known among surveyors as the three-point problem, sometimes as *Pothenot's problem.** It may be stated thus: Given three points A, B, C whose mutual distances are known, to find the distance of a fourth point P from either of the points A, B, C, having given the angles, which the sides AB, BC, CA subtend at P.

Analysis. Let A, B, C be the three points whose mutual distances are, —

$$AB = c, BC = a, CA = b,$$

and let P represent the fourth point at which AC and BC subtend the angles α and β respectively. It is required to find the distances d_1, d_2, d_3 of P from A, B and C respectively.

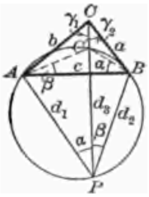


Fig. 83.

Circumscribe a circle about the triangle ABP, and let C' be the point at which this circle cuts PC or PC produced. Draw AC' and BC'. Then angle BAC' = angle $BPC = \beta$, being inscribed angles subtended by the same arc, and likewise angle ABC' = angle $APC = \alpha$.

(1) In the triangle ABC', one side c and the two adjacent angles α, β are known, hence AC' can be found.

(2) In the triangle ABC, the sides are known, hence A, the angle opposite the side a, can be found.

(3) In the triangle ACC', two sides b and AC' and the included angle $CAC' = A - \beta$ are known, hence angle $ACC' = \gamma_1$ can be found.

(4) In the triangle APC, one side b and two angles α, γ_1 are known, hence the sides d_1, d_3 can be found.

(5) Similarly, d_2 and d_3 can be found from the triangle BPC, having previously computed the triangle BCC'.

(6) Check. Compare the value of d_3 found in (4) with the value of d_3 in (5).

Illustration. A, B, C (Fig. 83) are three hostile forts whose mutual distances are known to be $AB = 4$ miles, $BC = 2$ miles, $CA = 3$ miles. From a battery planted at P, AC subtends an angle $34° 30'$ and BC an angle of $23° 45'$. Find the distance of the battery from each of the forts.

---

* Pothenot's problem and Hansen's problem are named after the men who were supposed to have first formulated and solved these problems. It is now known that both of these problems were previously solved by Snellius, the first in 1617, the other in 1627. Hence, if any one's name is to be associated with these problems hereafter it ought to be that of Snellius.

Solution. With the notation indicated in the figure, we have given

$$a = 2, \quad b = 3, \quad c = 4, \quad \alpha = 34^\circ\ 30', \quad \beta = 23^\circ\ 45';$$

to find d_1, d_2, d_3.

(1) Triangle ABC'. By the law of sines

$$AC' = c\frac{\sin\alpha}{\sin AC'B} = c\frac{\sin\alpha}{\sin(\alpha+\beta)}, \qquad BC' = c\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

$$\alpha + \beta = 34^\circ\ 30' + 23^\circ\ 45' = 58^\circ\ 15'.$$

$$\begin{array}{rlrl}
\log c &= 0.60206 & &= 0.60206\\
\log\sin\alpha &= 9.75313 & \log\sin\beta &= 9.60503\\
\operatorname{colog}\sin(\alpha+\beta) &= \underline{0.07040} & &= \underline{0.07040}\\
\log AC' &= 0.42559 & \log BC' &= 0.27749\\
AC' &= 2.6644. & BC' &= 1.8945.
\end{array}$$

(2) Triangle ABC. By the law of cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} \qquad\qquad \cos B = \frac{c^2 + a^2 - b^2}{2\,ca}$$

$$= \frac{3^2 + 4^2 - 2^2}{2\cdot 3\cdot 4} = 0.8750 \qquad\qquad = \frac{4^2 + 2^2 - 3^2}{2\cdot 4\cdot 2} = 0.6875$$

$$A = 28^\circ\ 57'. \qquad\qquad B = 46^\circ\ 34'.$$

(3) Triangle ACC'. Triangle BCC'.

By the law of tangents

$$\tan\frac{AC'C - \gamma_1}{2} \qquad\qquad \tan\frac{BC'C - \gamma_2}{2}$$

$$= \frac{b - AC'}{b + AC'}\tan\frac{ACC' + \gamma_1}{2}, \qquad = \frac{a - BC'}{a + BC'}\tan\frac{BC'C + \gamma_2}{2}.$$

$$CAC' = A - \beta = 5^\circ\ 12', \qquad CBC' = B - \alpha = 12^\circ\ 04'.$$

$$\frac{AC'C + \gamma_2}{2} = \frac{180^\circ - 5^\circ\ 12'}{2} = 87^\circ\ 24', \quad \frac{BC'C + \gamma_2}{2} = \frac{180^\circ - 12^\circ\ 04'}{2} = 83^\circ\ 58',$$

$$\begin{array}{rlrl}
b - AC' &= 0.3356, & a - BC' &= 0.1055,\\
b + AC' &= 5.6644. & a + BC' &= 3.8945.\\
\log(b - AC') &= 9.52582 & \log(a - BC') &= 9.02325\\
\operatorname{colog}(b + AC') &= 9.24685 & \operatorname{colog}(a + BC') &= 9.40954\\
\log\tan\dfrac{AC'C + \gamma_1}{2} &= \underline{1.34285} & \log\tan\dfrac{BC'C + \gamma_2}{2} &= \underline{0.97596}
\end{array}$$

$$\log\tan\frac{AC'C-\gamma_1}{2}=0.11552. \qquad \log\tan\frac{BC'C-\gamma_2}{2}=9.40875$$

$$\frac{AC'C-\gamma_1}{2}=55^\circ\,32' \qquad \frac{BC'C-\gamma_2}{2}=14^\circ\,23'$$

$$\gamma_1=\frac{AC'C+\gamma_1}{2}-\frac{AC'C-\gamma_1}{2} \qquad \gamma_2=\frac{BC'C+\gamma_2}{2}-\frac{BC'C-\gamma_2}{2}$$

$$=34^\circ\,52'. \qquad =69^\circ\,35'.$$

(4) Triangle APC. Triangle BPC.

By the law of sines

$$d_1=b\frac{\sin\gamma_1}{\sin\alpha}, \qquad d_2=a\frac{\sin\gamma_2}{\sin\beta},$$

$$d_3=b\,\frac{\sin CAP}{\sin\alpha}=b\,\frac{\sin(\alpha+\gamma_1)}{\sin\alpha}=a\,\frac{\sin(\beta+\gamma_2)}{\sin\beta}.$$

$$\alpha+\gamma_1=34^\circ\,30'+34^\circ\,52'=69^\circ\,22',$$
$$\beta+\gamma_2=23^\circ\,45'+69^\circ\,35'=93^\circ\,20'.$$

$$\begin{aligned}\log b&=0.47712 &\qquad \log a&=0.30103\\ \log\sin\gamma_1&=9.75714 & \log\sin\gamma_2&=9.97182\\ \operatorname{colog}\sin\alpha&=\underline{0.24687} & \operatorname{colog}\sin\beta&=\underline{0.39497}\\ \log d_1&=0.48113 & \log d_2&=0.66782\\ d_1&=3.0277. & d_2&=4.6539.\end{aligned}$$

$$\begin{aligned}\log b&=0.47712 &\qquad \log a&=0.30103\\ \log\sin(\alpha+\gamma_1)&=9.97121 & \log\sin(\beta+\gamma_2)&=9.99925\\ \operatorname{colog}\sin\alpha&=0.24687 & \operatorname{colog}\sin\beta&=0.39497\\ \log d_3&=\underline{0.69520} & \log d_3&=\underline{0.69526}\\ d_3&=4.957. & d_3&=4.957\text{ (check).}\end{aligned}$$

(*c*) *Solution by solving a system of simultaneous equations.*

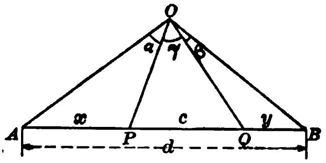


Fig. 84.

Example 3. From a point O (Fig. 84), the segments AP, PQ, QB of a straight line subtend the angles α, γ, β respectively. The distances $AB=d$, and $PQ=c$ are known; required the distances $AP=x$ and $QB=y$.

Solution. In this case no single triangle contains a sufficient number of known parts to afford a solution, but on applying the law of sines to the various triangles, we obtain the following equations,—

$$\frac{x}{\sin\alpha} = \frac{OP}{\sin A}, \qquad (1) \qquad\qquad \frac{x+c}{\sin(\alpha+\gamma)} = \frac{OQ}{\sin A}, \qquad (2)$$

$$\frac{y}{\sin\beta} = \frac{OQ}{\sin B}, \qquad (3) \qquad\qquad \frac{y+c}{\sin(\beta+\gamma)} = \frac{OP}{\sin B}. \qquad (4)$$

Dividing (1) by (2), and (3) by (4), we obtain

$$\frac{x}{x+c}\cdot\frac{\sin(\alpha+\gamma)}{\sin\alpha} = \frac{OP}{OQ}, \quad (5) \qquad \frac{y}{y+c}\cdot\frac{\sin(\beta+\gamma)}{\sin\beta} = \frac{OQ}{OP}. \quad (6)$$

Multiplying (5) by (6) gives, after a slight reduction,

$$\frac{xy}{(x+c)(y+c)} = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\gamma)\sin(\beta+\gamma)}. \qquad (7)$$

From the conditions of the problem we also have

$$x + y = d - c. \qquad (8)$$

The simultaneous equations (7) and (8) contain but two unknowns, x and y, which may be found from them by the familiar methods of algebra.

Second solution. Call the common altitude of the triangles h (Fig. 84). The area of each triangle may be expressed in two ways, first as one-half the product of the base by the altitude, second as one-half the product of two sides into the sine of the angle included by these sides (Art. 67, (2)).

Accordingly we have

$$2\cdot\text{area } APO = xh = AO\cdot PO\cdot\sin\alpha \qquad (9)$$

$$2\cdot\text{area } QPO = yh = QO\cdot BO\cdot\sin\beta \qquad (10)$$

$$2\cdot\text{area } PQO = ch = PO\cdot QO\cdot\sin\gamma \qquad (11)$$

$$2\cdot\text{area } ABO = dh = AO\cdot BO\cdot\sin(\alpha+\gamma+\beta) \qquad (12)$$

Dividing the product of (9) and (10) by the product of (11) and (12), and canceling the factors which appear in both the numerator and the denominator, gives

$$\frac{xy}{cd} = \frac{\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta+\gamma)}, \quad \text{or} \quad xy = cd\frac{\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta+\gamma)}. \qquad (13)$$

Equation (13), together with equation (8), is sufficient for the determination of x and y.

Illustration. An island PQ (Fig. 84) one mile wide lies in a direct line between two cities A and B on opposite shores of a river. From a point O up the river the channels AP and QB subtend angles of $45^\circ 35'$ and $25^\circ 10'$ respectively, while the island subtends an angle of $8^\circ 28'$. The cities are known to be 11 miles apart. Find the widths of the channels.

Solution. With the notation in Fig. 84, we have

$$d = 11,\ c = 1,\ \alpha = 45^\circ 35',\ \beta = 25^\circ 10',\ \gamma = 8^\circ 28'.$$

$$\alpha + \gamma = 54^\circ 03', \qquad \beta + \gamma = 33^\circ 38'.$$

$$\begin{aligned} \log \sin \alpha &= 9.85386 \\ \log \sin \beta &= 9.62865 \\ \text{colog} \sin(\alpha + \gamma) &= 0.09177 \\ \text{colog} \sin(\beta + \gamma) &= \underline{0.25659} \end{aligned}$$

$$\log \frac{\sin \alpha \sin \beta}{\sin(\alpha + \gamma)\sin(\beta + \gamma)} = 9.83087 = \log \frac{xy}{(x + c)(y + c)}$$

$$\frac{xy}{(x + c)(y + c)} = 0.67743 = k, \quad x + y = d - c = 10.$$

Eliminating y between these two equations gives

$$x^2 - 10x + \frac{11k}{1 - k} = 0.$$

Putting for k its value, we have the equation

$$x^2 - 10x + 23.10 = 0,$$

from which $\quad x = 3.62, \text{ or } 6.38,$

and hence $\quad y = 6.38, \text{ or } 3.62.$

Check. We compute xy by the second method and compare this product with the value of xy from the results just obtained.

$$xy = 3.62 \times 6.38 = 23.10.$$

Applying (13),

$$\alpha + \beta + \gamma = 45^\circ 35' + 25^\circ 10' + 8^\circ 28' = 79^\circ 13'.$$

$$\begin{aligned} \log c &= 0.00000 \\ \log d &= 1.04139 \\ \log \sin \alpha &= 9.85386 \\ \log \sin \beta &= 9.62865 \\ \text{colog} \sin \gamma &= 0.83199 \\ \text{colog} \sin(\alpha + \beta + \gamma) &= \underline{0.00774} \\ \log xy &= 1.36363, \qquad xy = 23.10. \end{aligned}$$

In the following exercises the student is expected to check his results whenever no answer is given. Reread Article 44 for the number of significant figures to be retained in the answers. Only as many problems need be worked out in detail as are necessary to make the student reasonably familiar with the methods of computation with logarithms; after that an additional number of problems may be selected to be analyzed only. The student is not expected to work or analyze problems which involve principles from physics or astronomy with which he is not familiar.

75. Miscellaneous Heights and Distances.

EXERCISE 38

1. A man walking along a straight road, observes a church at an angle of 45° with the direction of the road. After walking another mile, the same church makes an angle of 30° with the opposite direction of the road. How far is the church from the road?

Ans. 0.37 miles.

2. The angle of elevation of an aeroplane from a point due west of it is 50°, and from a point due east of it is 43°. The distance between the two points is 1000 ft. Find the approximate height of the plane. *Ans.* 523.2 ft.

3. Two mountains are 9 and 13 miles respectively from a town, and the angle subtended by them is 71° 40′. Find the distance between the mountains.

4. From a point 3 miles from one end of an island and 7 miles from the other end, the island subtends an angle of 33° 45′. Find the length of the island. *Ans.* 4.8 miles.

5. From a town two roads making an angle of 54° cross the same river, the first at a distance of 8 miles, the other at a distance of 12 miles. Find the distance between the points where the roads meet the river. *Ans.* 9.75 miles.

6. The distances between three ports A, B, C, are as follows: $AB = 71.2$ miles, $BC = 28.9$ miles, $CA = 60.1$ miles. C is due north of A; in what direction from A is B? *Ans.* N. 23° 32′ E.

7. Two stations A and B on opposite sides of a mountain are both visible from a third station C. The distances AC, CB and the angle ACB were measured and were found to be 11.5 miles, 13.4 miles and 59° 34′ respectively. Find the distance from A to B.

8. The two slopes of an embankment measure 48.5 ft. and 84.0 ft. respectively. The inclination of the first slope is 21° 31′. Find the inclination of the second slope and the width of the embankment at its base. *Ans.* 12° 14′, 127 ft.

9. What is the largest circular track that can be laid out within a triangular field whose sides are 229 yards, 109 yards and 312 yards respectively? *Ans.* Radius of track = 28.8 yards.

10. A person walking along a straight road observes the angle of elevation of the summit of the hill to be 25°. After walking 2 miles on an incline of 10° towards the summit, the angle of elevation is observed to be 37°. Find the air-line distance of the summit from the second position of the observer.

11. A person standing on the bank of a river observes the angle subtended by a tower on the opposite bank of the river to be 65° 00′, and when he recedes 50 ft. from the river, he finds the angle to be 32° 00′. Find the height of the tower and the width of the river.

Suggestion. Call the height of the tower x, and the width of the river y. Set up two equations involving x and y and solve.
Ans. $x = 44.1$ ft. $y = 20.6$ ft.

12. From a point A two straight roads extend at an angle of 28° 30′. At B and C they meet the same crossroad at distances $AB = 4500$ yds., $AC = 6450$ yds. Two cyclists leave A at the same time along different roads; when they reach the crossroad they turn towards each other and meet just halfway between B and C 25 minutes after they left A. Find the speed of each cyclist.
Ans. 11.04 miles and 8.38 miles per hr. respectively.

13. In example 12, where would the cyclists have met had their speeds been the same? *Ans.* 671 yds. from C.

14. A launch whose speed is 6 miles per hour sails in the direction N.W. A second launch having a speed of 8 miles per hour leaves at the same time from a point due south of the first launch and overtakes it in 5 hours. What was the course of the second launch, and how far apart were the launches when they started?

15. Two streets OA and OB (Fig. 85) meet at an angle of 65° 35′. The corner lot $AOBC$ extends 142 feet along OA and 97 feet along OB. Find the dimensions AC and BC of the lot, AC and BC being

at right angles to OA and OB respectively. Find also the area of the lot.

Suggestion. Solve (a) triangle AOB, (b) triangle ABC. To find the area we may apply formula (d), Problem 10, Exercise 33 (Why?).

Ans. $AC = 42.1$ ft., $BC = 111.9$ ft., area $= 8416.2$ sq.ft.

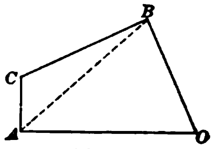


Fig. 85.

16. A, B, C are three mountain tops whose distances from one another are known to be $AB = 10.65$ miles $= c$, $BC = 17.15$ miles $= a$, $CA = 9.32$ miles $= b$. From a point O at which C appears in a straight line with A, the horizontal angle COB measures $15° \, 35' = \gamma$.

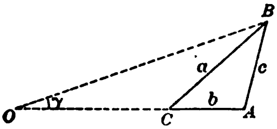


Fig. 86.

Find the distance $BO = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{b \sin \gamma} = 34.94.$

17. From a point at the edge of a moat the angle of elevation of the top of a wall on the opposite side of the moat is $32° \, 40' = \alpha$. Receding from the edge in a horizontal direction a distance of 50 ft. $= a$, the angle of elevation becomes $21° \, 35' = \beta$. Find the width x of the moat, and the height y of the wall on the opposite side.

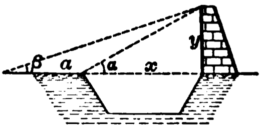


Fig. 87.

$$\textit{Ans.}\quad x = \frac{a \tan \beta}{\tan \alpha - \tan \beta} = 80.5 \text{ ft.}, \quad y = \frac{a \tan \alpha \tan \beta}{\tan \alpha - \tan \beta} = 51.6 \text{ ft.}$$

18. From a point A at sea level, the angles of elevation α, α' of two hills P, P' in the same straight line are $\alpha = 25° \, 30'$ and $\alpha' = 34° \, 20'$ respectively. Walking towards the summits at an inclination $\gamma = 10°$, the angle of elevation was again observed at the moment that P' became hidden from view by P, and was found to be $\beta = 42° \, 15'$. P is known to be $h = 1595$ ft. above sea level. Find the altitude of h' of P'.

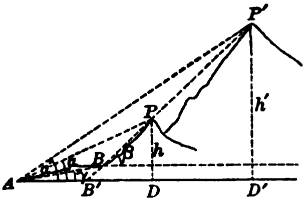


Fig. 88.

Suggestion. $\dfrac{h'}{h} = \dfrac{P'B'}{PB'} = \dfrac{P'B'}{AB'} \cdot \dfrac{AB'}{PB'}$. The law of sines will give each of the ratios $\frac{P'B'}{AB'}$ and $\frac{AB'}{PB'}$ in terms of known angles, hence

$$h' = h\frac{\sin\alpha'}{\sin\alpha}\cdot\frac{\sin(\beta-\alpha)}{\sin(\beta-\alpha')} = 4372 \text{ ft.}$$

19. The angular elevation of a standpipe S, from a place O due east of it, is $\alpha = 35° 27'$, and from a place O' due south of O and at a distance $a = 200$ ft. from it, the angular elevation is $\alpha' = 26° 16'$. Find the height h of the standpipe, assuming the foot F of the standpipe and the points O and O' to be in the same horizontal plane.

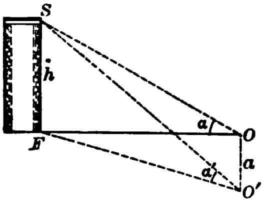


Fig. 89.

Suggestion.

From the triangle SFO, $h = \overline{OF}\cdot\tan\alpha$,

From the triangle SFO', $h = \overline{O'F}\cdot\tan\alpha'$,

From the triangle OFO', $\overline{O'F}^2 - \overline{OF}^2 = a^2$.

Eliminate OF and OF' and solve for h.

$$\textit{Ans.}\quad h = \frac{a\tan\alpha\tan\alpha'}{\sqrt{\tan^2\alpha - \tan^2\alpha'}} = 136.9 \text{ ft.}$$

20. From two points O and O' 200 ft. apart, a spire S subtends equal angles $\alpha = 21° 13'$. The bearings of S from O and O' are W. and N. 25° W. respectively. Find the height h of the spire and its distance from O, assuming that the points O and O' are in the same horizontal plane with the foot of the spire.

76. Applications from Physics.

EXERCISE 39

1. Two forces, one of 470 lbs., the other of 520 lbs., act on the same point at an angle of 37° 27'. Find the resultant force.

Ans. $R = 938$ lbs.

2. If two equal forces meet at an angle θ, what angle will the resultant force make with either force and what will be the magnitude of the resultant? *Ans.* $\frac{1}{2}\theta$, $\sqrt{2(1+\cos\theta)}$ times either force.

3. A force of 100 lbs. is resolved into two components which make angles of 25° 00' and 35° 00' respectively with the direction of the original force. Find the magnitude of each component.

Ans. 48.8, 66.2.

4. A force of 500 dynes is resolved into two components, one of which is 200 dynes and makes an angle of 37° 50′ with the original force. Find the other component.

5. Two forces of 15 lbs. and 25 lbs. respectively combine so as to form a resultant force of 30 lbs. Find the direction of the resultant with each of its components. *Ans.* 56° 15.0′, 29° 55.6′.

6. A ferry crosses a stream 360 yds. wide so as to make an angle of 125° with the direction of the current. The actual motion of the ferry is at an angle of 95° with the direction of the current. Find the actual distance the ferry made in crossing the river, and the distance it would have made had there been no current. *Ans.* 361.4 yds., 439.5 yds.

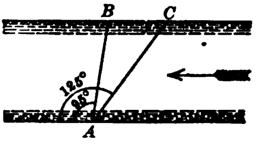


Fig. 90

7. Find the current of the river, Problem 6, and the speed of the ferry independent of the current if it required 5 minutes to cross the river. *Ans.* 1.24, 3 miles per hr.

8. If in Problem 6 the speed of the ferry in still water is 5 miles per hour, find the time it would take to cross the river and the velocity of the current.

9. A launch leaves the shore of a river running from west to east, and after two hours reaches an island a distance of 20 miles and in a direction due N.E. from the starting point. Assuming that the launch sailed in a straight line and with a speed which would have taken it 16 miles per hour in still water, find the velocity of the current.

Ans. 10.8 miles or 3.33 miles per hour, according as the course of launch was west or east of north.

10. A ray of light passes through a glass plate 12 mm. thick. The angle of incidence is 29° and the index of refraction is $\frac{3}{2}$. How far is it from the point where the ray leaves the plate to the point where it would have left it if it had suffered no refraction? *Ans.* 2.6 mm.

11. From a point P at the bank of a river the angle of elevation of a flagstaff AB on the opposite side of the river as measured with a transit is $\alpha = 47° 32'$, and the angle of depression of the image of the flagstaff as seen in the water is $\delta = 54° 36'$. The axis of the instrument

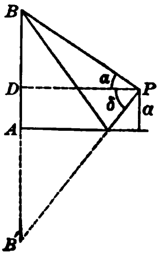


Fig. 91.

is $a = 4$ ft. 6 in. from the level of the water. Find the height of the flagstaff.

Suggestion. From the law of reflection of light, $AB' = AB$, hence

from the triangle BDP, $x - a = DP \cdot \tan \alpha$,

and from the triangle $B'DP$, $x + a = DP \cdot \tan \delta$.

Eliminate DP and solve for x. *Ans.* 36.1 ft.

12. Two billiard balls A, B are located at distances $a = 2.53$ and $b = 4.13$ respectively from the same cushion. Their distance apart is $c = 5.84$. At what angle must the first ball strike the cushion in order that on rebounding it may strike the second ball?

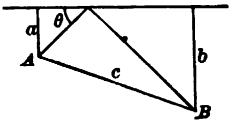


Fig. 92.

$$\textit{Ans.}\quad \tan\theta = \frac{a+b}{\sqrt{(a-b+c)(-a+b+c)}}.$$

13. A, B, C are three points whose distances are $AB = 25.0$, $BC = 20.0$, $CA = 10.0$. What must be the inclination of a mirror at C to AB in order that a point of light at A may be reflected to B?

Ans. $13° 34'$.

14. In Problem 13, what will be the distance between A and its reflection on AB or AB produced, if the mirror at C is inclined $25° 45'$ to the line AB?

15. A billiard table is $p = 6.00$ ft. long. Two balls are placed at P and P' whose distances from AD and AB are $a = 2$ ft. 0 in., $b = 1$ ft. 8 in., and $a' = 4$ ft. 0 in., $b' = 3$ ft. 4 in, respectively. At what angle must the ball at P strike the cushion AB so that after it rebounds from the cushion BC it may hit P'? Find also the distance of the points R and S from B at which the ball strikes the cushion BC.

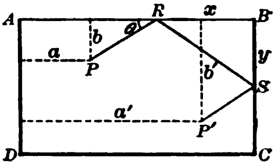


Fig. 93.

$$\textit{Ans.}\quad \tan\theta = \frac{b+b'}{2p-(a+a')},\ x = (p-a) - b\cot\theta = 2.00\text{ ft.},$$

$$\theta = 39° 48' \quad y = b' - (p-a')\tan\theta = 1\text{ ft. } 8\text{ in.}$$

16. A billiard table is p ft. long and q ft. wide. At what angle must a ball strike the long cushion in order that after striking each cushion in turn it may pass through its first position?

Ans. $\theta = \tan^{-1}\frac{q}{p}$.

17. Two parallel forces $p = 15$, $q = 25$, act on levers of lengths $a = 10$, $b = 3$, respectively, on the common fulcrum o. The force P makes an angle $\theta = 30°$ with the lever a. At what angle ϕ must the levers be inclined in order to produce equilibrium? *Ans.* $\phi = 60°$.

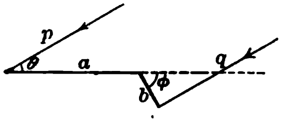


Fig. 94.

77. Applications from Surveying and Engineering.

Exercise 40

1. In running a line AB an obstacle was encountered beyond B. The distance BP 200 ft. was measured, making the angle $ABP = 154° = \theta$. At P an angle $BPC = 65° = \phi$ was turned off. Required the distance PC and the angle PCD in order that CD may be the prolongation of AB.

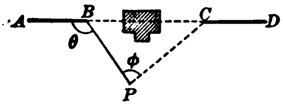


Fig. 95.

$$Ans.\ PC = \frac{BP \cdot \sin\theta}{\sin(\theta - \phi)} = 87.7, \text{ angle } PCD = 91°.$$

2. In surveying a boundary $ABCD$ an obstacle was met between B and C. Instead of measuring BC directly, $BP = 255$ ft. was measured, making the angle $ABP = 95°\ 30'$. Then $PC = 234.5$ ft. was measured, making the angle $BPC = 104°\ 34'$. Required the distance BC and the angle ABC.

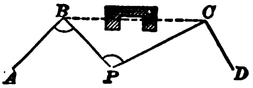


Fig. 96.

Ans. $BC = 387.4$ ft., angle $ABC = 131°\ 21.7'$.

3. In running a traverse $ABCD$ an impenetrable thicket is encountered between B and C. From B a road extends to P, where it is met by a crossroad which leaves the thicket at Q. The following measures were taken:

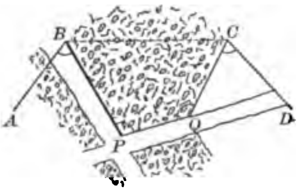


Fig. 97.

Angle $ABP = 35°\ 45'$,
$BP = 674$ ft.,
angle $BPQ = 110°\ 37'$,

$PQ = 890$ ft., angle $PQC = 115°\ 38'$, $QC = 345.4$ ft., angle $QCD = 33°\ 23'$. Required the distance BC and the angles ABC and BCD.

Ans. $BC = 1316$ ft., angle $ABC = 91°\ 05'$, angle $BCD = 111°\ 48'$.

4. B is a point, invisible and inaccessible from A, whose distance from A was to be found. From A a line was run to two points P and Q from each of which B was visible, and the following measures were taken: $AP = 236.7$ ft., angle $APB = 142^\circ\ 37.3'$, $PQ = 215.9$ ft., angle $AQB = 76^\circ\ 13.8'$. Required AB.

Ans. $AB = 441.0$ ft.

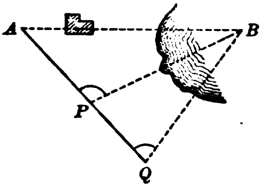


Fig. 98.

5. A surveyor wished to find the distance of an inaccessible point O from each of two points A and B, but had no instrument with which to measure angles. He measured $AA' = 150$ ft. in a straight line with OA and $BB' = 250$ ft. in a straight line with OB. He then measured $AB = 279.5$ ft., $BA' = 315.8$ ft., $A'B' = 498.7$ ft. Required AO and BO.

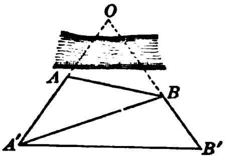


Fig. 99.

6. Two inaccessible points A and B were visible from C, but no other point could be found from which both A and B were visible. A point P was selected from which both A and C were visible. $CP = 425.3$ ft., angle $APC = 37^\circ\ 15.4'$, angle $ACP = 42^\circ\ 35.3'$. A second point Q was selected from which both B and C were visible. $CQ = 405.4$ ft., angle $BQC = 53^\circ\ 14.8'$, angle $BCQ = 58^\circ\ 04.7'$, angle $ACB = 65^\circ\ 10.5'$. Required AB.

Ans. $AB = 336.8$ ft.

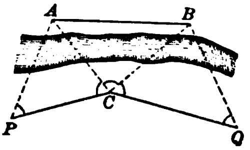


Fig. 100.

7. It was required to find the distance between two inaccessible points A and B. No point could be found from which both A and B were visible. Two points P and Q were selected from which A could be seen, and two other points R and S from which B could be seen. The following measures were taken:

$PQ = 200$ ft., angle $PQA = 61^\circ\ 30'$, angle $QPA = 70^\circ\ 25'$,
$RS = 250$ ft., angle $RSB = 29^\circ\ 16'$, angle $SRB = 72^\circ\ 10'$,
$PR = 300$ ft., angle $PRS = 121^\circ\ 36'$, angle $RPA = 92^\circ\ 54'$.

Find AB.

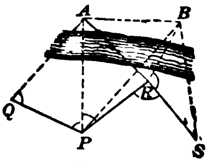


Fig. 101.

Suggestion. (*a*) In triangle QPA, find PA; (*b*) In triangle APR, find AR; (*c*) In triangle RSB, find RB; (*d*) In triangle ARB, find AB. To check, compute AB again using the triangle APB.

Ans. $AB = 480$ ft.

8. A circular tower subtends a horizontal angle of $2\alpha = 20° 13'$ at a distance of $a = 198$ ft. from the edge of the tower, measured on a diameter produced. Find the diameter of the tower.

$$\textit{Ans.}\ \text{Diameter} = \frac{2a \sin \alpha}{1 - \sin \alpha} = 84.3 \text{ ft.}$$

9. Three stations A, B, C are to be connected by a juncture O, equally distant from each of them. The distance of A from B is 25.3 miles; of B from C, 17.8 miles; and of C from A, 19.4 miles. Find the distance of O from A. *Ans.* 12.7 miles.

10. Fig. 102 represents a cross-section map. The horizontal scale is 1 in. = 100 ft., the vertical scale is 1 in. = 10 ft. $AB = 5$ in., $BC = 6.35$ in., $CD = 4.13$ in., $BB' = 4.98$ in., $CC' = 1.76$ in., $DD' = 3.87$ in. A and D' are to be connected by a street of uniform grade. Required the inclination of AD' to AD, the cut $B'B''$ that must be made at B' and the fill $C'C''$ at C'.

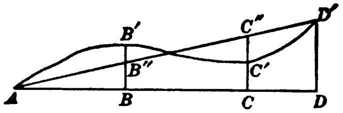


Fig. 102.

11. On a map drawn to a scale 1 : 5000, three points A, B, C are at distances $AB = 5.03$ in., $BC = 4.23$ in., $CA = 3.54$ in. The points A', B', C' of which A, B, C are the projections, lie 357.25, 713.27, and 623.53 ft. respectively above an arbitrarily chosen horizontal plane, called the datum plane. Required the actual lengths of the lines $A'B'$, $B'C'$, $C'A'$, the angles formed by these lines, and their inclinations to the datum plane.

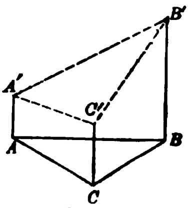


Fig. 103.

Ans. $A'B' = 2540.1$ ft., $8° 03' 26''$, $C'A'B' = 55° 18' 56''$.
$C'B' = 2116.9$ ft., $2° 25' 46''$, $A'B'C' = 44° 02' 58''$.
$A'C' = 1789.9$ ft., $8° 33' 20''$, $B'C'A' = 80° 38' 14''$.

12. From a hilltop H two landmarks A and B in the valley, which are known to be a mile apart, subtend a horizontal angle $\theta = 105° 34.4'$ and their angles of depression are $\alpha = 27° 27.5'$ and

$\beta = 37° 54.9'$ respectively. Find the height h of the hill above the valley.

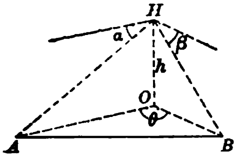


Fig. 104.

Suggestion. Denote OB and OA by a and b respectively. Then

$$a = h \cot \beta, \qquad b = h \cot \alpha,$$

and by the law of tangents

$$\tan \tfrac{1}{2}(A - B) = \frac{h \cot \beta - h \cot \alpha}{h \cot \beta + h \cot \alpha} \cdot \tan \tfrac{1}{2}(A + B)$$

$$= \frac{\tan \alpha - \tan \beta}{\tan \alpha + \tan \beta} \cdot \tan (90 - \tfrac{1}{2}\theta),$$

from which A = angle OAB and B = angle OBA may be found. Then

$$a = AB \cdot \frac{\sin A}{\sin \theta}, \text{ and finally } h = a \tan \beta.$$

Ans. $h = 2043$ ft.

13. From a mountain top 3257 ft. high, two hotels are observed whose altitudes are 627 and 937 ft. respectively. The angles of depression of the hotels are 45° 34′ and 58° 27′ respectively. Find the distance (horizontal) between the hotels.

14. A tower ED is situated on the side of a hill whose summit C is 550 ft. above BA. At a point B from which the top of the tower and the top of the hill appear in a straight line, the angles of elevation of the foot and top of the tower are 15° 30′ and 29° 27′ respectively. The distance of B from the foot of the hill at A is 600 ft. Find the height of the tower and its distance from the foot of the hill. *Ans.* $ED = 212$ ft., $AD = 248$ ft.

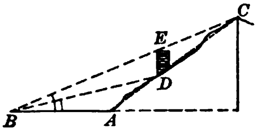


Fig. 105.

15. To determine the height H of a building above the steps, the horizontal distance d between two stations A' and A'' was measured with a tape, the heights h' and h'' of these stations were determined by level readings on the top of the steps, and the angles of elevation θ' and θ'' of the top of the building as seen from A' and A'' were measured by means of a transit. Show that

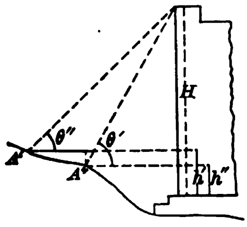


Fig. 106.

$$H = \frac{h'' \tan\theta' - h' \tan\theta'' - d \tan\theta' \tan\theta''}{\tan\theta' - \tan\theta''}.$$

16. To determine the area of a triangular field, the sides were measured with a 100 ft. steel tape, which was afterwards found to be 0.3 ft. too long. The sides as measured were 276.33 ft., 391.81 ft. and 318.00 ft. Determine the area as measured and also the corrected area.

Ans. Measured area = 43,527 sq. ft.,

$$\text{Corrected area} = \frac{43{,}527 \times \overline{100.3}^2}{100^2} = 43{,}789 \text{ sq. ft.}$$

What would have been the corrected area if the tape had been found 0.3 ft. too short?

17. In running a 7° curve (radius 819.02 ft.) an obstacle was encountered beyond the point A. In order to locate a point P beyond the obstacle, the tangent at A was produced to B, a distance of 500 ft. At B an angle ABC was turned off equal to 37°. Required the distance BP.

Suggestion. From the right triangle OAB, find OB and the angle OBA, then use the triangle OPB to find BP. *Ans.* $BP = 153.28$ ft.

The equations give a second solution. Interpret it.

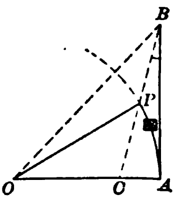


Fig. 107.

18. The inaccessible distance $PQ = x$ may be found by measuring $PA = a$, $QB = b$, in a straight line with PQ, and the angles α, β, γ which a, b, and x subtend at some accessible point O. Show that x is given by the equation

$$\frac{(a + x)(b + x)}{\sin(\alpha + \gamma)\sin(\beta + \gamma)} = \frac{ab}{\sin\alpha \sin\beta}.$$

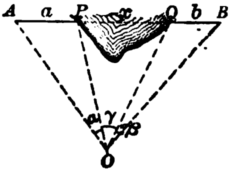


Fig. 108.

Suggestion. Find the ratio $PO : QO$, first from the triangles OAP and OAQ and second from the triangles OBQ and OBP, and compare the results.

19. From a boat P over a submerged rock, a sextant is used to measure the angles subtended by three known points A, B and C on shore. The readings give $APB = 57° \, 13.2'$, $BPC = 65° \, 21.3'$. From

a previous survey it is known that angle $ABC = 116° \; 53.6'$, $AB = 1846$ ft., and $BC = 1493$ ft. Find the distances AP, BP and CP. (This is another illustration of the three-point problem, the three points this time being given by two distances and an angle.)

Ans. $AP = 2137$ ft., $BP = 1582$ ft.,
$CP = 1061$ ft.

20. In the quadrilateral Ames-Blair-Cole-Davis, of a triangulation system, the angles were measured and adjusted with the following results:

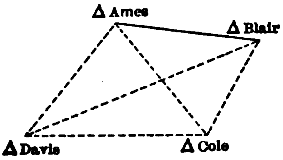


Fig. 109.

At Ames

To Blair	$00° \; 00' \; 00''$,
To Cole	$48° \; 53' \; 31''$,
To Davis	$109° \; 12' \; 37''$.

At Blair

To Cole	$00° \; 00' \; 00''$,
To Davis	$31° \; 15' \; 59''$,
To Ames	$60° \; 41' \; 57''$.

At Cole

To Davis	$00° \; 00' \; 00''$,
To Ames	$49° \; 13' \; 02''$,
To Blair	$119° \; 37' \; 34''$.

At Davis

To Ames	$00° \; 00' \; 00''$,
To Blair	$41° \; 21' \; 25''$,
To Cole	$70° \; 27' \; 52''$.

The distance Ames-Blair measured 12,362 ft. Find all the other lines in the figure.

Ans. Blair-Cole = 9,887 ft., Ames-Cole = 11,443 ft.,
Davis-Cole = 10,549 ft., Ames-Davis = 9,194 ft.,
Davis-Blair = 17,667 ft.

21. After adjustment the angles of the adjacent triangulation figure were found to be as follows:

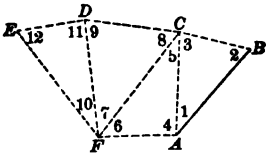


Fig. 110.

(1) $37° \; 44' \; 16''$,
(2) $63° \; 21' \; 53''$,
(3) $78° \; 53' \; 51''$,
(4) $93° \; 41' \; 23''$,
(5) $25° \; 18' \; 17''$,
(6) $61° \; 00' \; 20''$,
(7) $57° \; 31' \; 03''$,
(8) $68° \; 10' \; 11''$,
(9) $54° \; 18' \; 46''$,
(10) $35° \; 08' \; 11''$,
(11) $101° \; 53' \; 06''$,
(12) $42° \; 58' \; 43''$.

The side AB measured 4403.2 ft. Find EF.

$$EF = \frac{AB \cdot \sin(2) \sin(4) \sin(8) \sin(11)}{\sin(3) \sin(6) \sin(9) \sin(12)} = 7507.6 \text{ ft.}$$

22. It is required to find the sides of the triangle EFG, from the following adjusted results furnished by a survey:

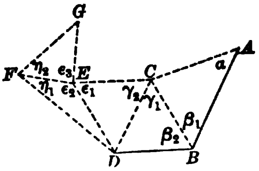


Fig. 111.

$\alpha = 40^\circ\,31'\,15''$, $\epsilon_1 = 54^\circ\,08'\,07''$,
$\beta_1 = 65^\circ\,15'\,16''$, $\epsilon_2 = 151^\circ\,20'\,11''$,
$\beta_2 = 54^\circ\,50'\,50''$, $\epsilon_3 = 65^\circ\,54'\,59''$,
$\gamma_1 = 59^\circ\,10'\,02''$, $\eta_1 = 25^\circ\,25'\,25''$,
$\gamma_2 = 74^\circ\,13'\,27''$, $\eta_2 = 80^\circ\,34'\,10''$.

The base line $AB = 76{,}542$ ft.

$$EF = \frac{AB \cdot \sin\alpha \sin\beta_2 \sin\gamma_2 \sin(\epsilon_2 + \eta_1)}{\sin(\alpha + \beta_1)\sin(\beta_2 + \gamma_1)\sin\epsilon_1 \sin\eta_1} = 7231.5 \text{ ft.}$$

$$EG = \frac{EF \cdot \sin\eta_2}{\sin(\eta_2 + \epsilon_3)} = 12{,}920 \text{ ft.}, \quad FG = \frac{EF \cdot \sin\epsilon_3}{\sin(\eta_2 + \epsilon_3)} = 11{,}957 \text{ ft.}$$

23. The sides b and d and the area of a quadrilateral field are to be computed, the following data having been determined by measurement. $a = 403.9$ ft., $c = 200.9$ ft., $\alpha = 115^\circ\,12.3'$, $\beta_1 = 40^\circ\,25.3'$, $\beta_2 = 42^\circ\,37.5'$, $\gamma = 121^\circ\,12.1'$.

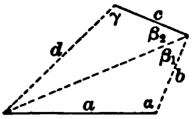


Fig. 112.

Check the result for d.

$$\text{Area} = \tfrac{1}{2}\,a\,\frac{\sin\alpha}{\sin\beta_1}\,[a\sin(\alpha + \beta_1) + c\sin\beta_2] = 85{,}308 \text{ sq. ft.}$$

24. A surveyor measured the following distances and their respective bearings:

$AB = 413$ ft., bearing N. 72° 15′ E.
$BC = 395$ ft., bearing N. 37° 27′ W.
$CD = 525$ ft., bearing S. 68° 13′ W.

What distance and bearing will carry him to the starting point?

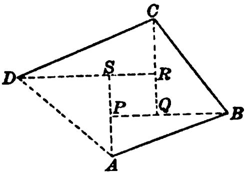


Fig. 113.

Suggestion. The distance DA and its bearing are most easily computed from the right triangle DAS.

$AS = AP + QC - RC = AB \cos 72^\circ\,15' + BC \cos 37^\circ\,27' - DC \cos 68^\circ\,13'$,
$SD = PB - QB - DR = AB \sin 72^\circ\,15' - BC \sin 37^\circ\,27' - CD \sin 68^\circ\,13'$.

Ans. $DA = 688.8$ ft. Bearing S. 69° 01.6′ E.

AP, QC and CR are called the ***latitudes***, PB, BQ and RD the ***departures*** of the courses AB, BC and CD respectively. The latitudes are considered positive or negative according as the course bears north or south, the departures positive or negative according as the course bears east or west. In the example given, AS is positive, SD negative, hence the bearing of AD is N.W. and the bearing of DA is S.E.

25. After adjusting the field measurements in a traverse survey around an area, the following bearings and distances were obtained:

Stations.	Bearing.	Distance.	Latitude.	Departure.
1–2	N. 44° 38.5′ E.	287.3	+ 204.4	+ 201.9
2–3	N. 52° 42.5′ E.	451.8	+ 273.7	+ 359.4
3–4	S. 2° 00.0′ W.	921.3	− 920.7	− 32.15
4–5	N. 88° 34.0′ W.	212.0	+ 5.30	− 211.9
5–6	S. 88° 50.5′ W.	317.3	− 6.41	− 317.2
6–1	N. 00° 00.0′	443.7	+ 443.7	000.0

Compute the latitudes and departures given in the columns above. Draw a figure and compute the area.

78. Applications from Navigation.

Exercise 41

1. A cape as seen from board ship bears N. by E. The ship sails N.W. 35 miles and then the cape bears exactly east. How far is the cape from the second point of observation? *Ans.* 29.7 miles.

2. To determine the distance between two ships at sea, an observer noted the interval of time between the flash and report of a gun fired on board each ship, and measured the angle which the two ships subtended. The intervals were 5 and 7 seconds respectively, and the angle was $37° 45'$. Assuming the velocity of sound as 1090 ft. per second, find the distance between the ships.

Ans. 4707 ft.

Discuss the accuracy of your result.

3. From a ship two rocks are seen in a straight line bearing S. $22° 30'$ E. After the ship sailed 8 miles S.E. the first rock bore N. $65° 13'$ W. and the second N. $76° 46'$ W. Find the distance between the rocks.

$$\textit{Ans.}\quad \frac{8 \cdot \sin 22° 30' \sin 11° 33'}{\sin 42° 43' \sin 54° 16'} = 1.11 \text{ miles.}$$

4. A privateer, 8 miles S.W. of a harbor, sees a ship leave the harbor. The course of the ship is S.E by S. and its speed 10 miles. In

what direction and at what speed must the privateer sail in order to overtake the ship in 2 hrs.?

5. A ship sails N.N.E. at a rate of 12 mi. per hr. A second ship making 15 miles per hour starts at the same time from a point 50 miles east of the first ship. In what direction must the second ship sail in order to meet the first ship, and how long will it take before they meet? *Ans.* N. 55° 33′ W. 2 hr. 57 min.

6. A ship sailing east sees two lighthouses 15 miles apart in a line E.N.E. After sailing an hour, one lighthouse bears N. by W. and the other N.E. by N. Find the speed of the ship.

7. From the top of a mast 70 ft. above the level of the sea, the light of a distant lighthouse is seen just above the horizon, and after the ship has sailed directly toward the lighthouse for 45 min. the lighthouse becomes visible from the deck of the ship, which is 12 ft. above the level of the sea. Find the rate at which the ship is moving, considering that the earth is a sphere 8000 miles in diameter.

Ans. 8 miles per hour.

8. A ship leaves Cape Flattery (48° 23′ N. Lat., 124° 44′ W. Long.) and sails W. 100 miles, N. by E. 150 miles, S.E. 113 miles. Find the position reached. *Ans.* 49° 10′ N. Lat., 124° 30′ W. Long.

Suggestion. 1 mi. = 1′ of arc.

79. Problems from Astronomy and Meteorology. In the following problems the planets are assumed to move in the ecliptic and their orbits are taken to be circles with the sun at the center.

EXERCISE 42

1. The mean distance of Venus from the sun is 36,000,000 miles and that of the earth from the sun 92,900,000 miles. How far is Venus from the earth when its angular distance from the sun is 13° 35′?

Ans. Either 61,700,000 or 119,000,000 miles.

2. The mean distance of Mars from the sun is 141,500,000 mi., What is the angular distance between Mars and the sun when the distance between Mars and the earth is 122,400,000 miles?

3. What is the distance between Jupiter and the earth, when its angular distance from the sun is 123° 12′, the heliocentric longitude of the earth being 67° 59′ and that of Jupiter 21° 23′?

Ans. 381,000,000 miles.

4. The mean distance of Saturn from the sun is 886,000,000 miles that of Uranus is 1,781,900,000 miles. What is the difference in the heliocentric longitude of these planets if their angular distance is 47° 45′?

5. From the top of a mountain a miles high the corrected angular depression of a line tangent to the earth's surface is found to be θ. Show that the diameter of the earth is

$$d = \frac{2a\cos\theta}{1-\cos\theta}.$$

6. Two observers A and B observe the same meteor. As seen by A, the altitude $\alpha = 61°\,35'$, the azimuth $\zeta_1 = 119°\,56'$. As seen by B, who is stationed 18.5 miles north of A, the azimuth of the meteor is $\zeta_2 = 85°\,32'$. Compute the height h of the meteor and its altitude α' as seen by B.

$$h = \frac{a\sin\zeta_2\tan\alpha}{\sin(\zeta_1+\zeta_2)} = 62.9 \text{ miles}, \quad \alpha' = \tan^{-1}\frac{\sin\zeta_2\tan\alpha}{\sin\zeta_1} = 25°\,11'.$$

7. An observer measures the angle of elevation of a sharply defined cloud which is in the same vertical plane with the sun. The angle measures 48.9°. The altitude of the sun at the same time measures 56.6°. The shadow of the cloud was 875 ft. from the observer. Find the height of the cloud. *Ans.* 4110 ft.

8. From two points O and O' on the same meridian, the zenith distances $ZOM = 35°\,25'\,20''$, $Z'O'M = 40°\,10'\,50''$, of the moon are measured. The difference in latitude between the points of observation is $ZCZ' = 74°\,26'\,18''$. Find the distance of the moon from the earth, assuming the radius of the earth as 3958.8 miles.

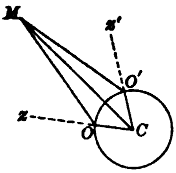


Fig. 114.

Ans. 238,840 miles.

9. On a clear day, twilight ceases when the sun has reached a position 18° below the horizon ($HAS = 18°$). Find the height AE of the atmosphere which is sufficiently dense to reflect the sun's rays. Take $OC = 4000$ miles. The result must be diminished by 20 % to allow for refraction.

Ans. 40 miles.

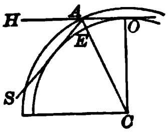


Fig. 115.

80. Geometrical Applications.

Exercise 43

1. Two sides and the included angle of a parallelogram are 156, 237 and 37° 50′ respectively. Find the length of the diagonals and the area of the parallelogram.

Ans. Diagonals, 148.7, 372.7; area, 22,680.

2. Find the area of a regular pentagon whose side is 12 ft.

Ans. 274.74 sq. ft.

3. The sides of a triangle are 17, 20, 27 respectively. Find the length of the median to the longest side.

4. In Problem 3, find the length of the bisector of the greatest angle.

5. Given a side c and two adjacent angles A and B of a triangle. Find expressions for the perpendiculars p_A, p_B, p_C drawn from the vertices A, B, C to the opposite sides.

$$\textit{Ans.}\quad p_A = c \sin B, \quad p_B = c \sin A, \quad p_C = \frac{c \sin A \sin B}{\sin (A + B)}.$$

6. Find the area of the segment of a circle whose radius is 15.6 and the base of the segment 21.3.

7. Find the length of the diagonals of a regular pentagon whose side is 10.

Ans. 16.18.

8. Find the lengths of the diagonals of a regular heptagon whose side is 12.

Ans. 26.96, 21.62.

9. Given the perimeter 25 and two angles of a triangle 55° and 65° respectively, to find the sides to three places of decimals.

Ans. 7.902, 8.743, 8.355.

10. Given one side a of a triangle, the sum t of the other two sides and the angle A opposite to the first side, to find the angles of the triangle.

$$\textit{Ans.}\quad \cos \tfrac{1}{2}(B - C) = \frac{t}{a} \sin \tfrac{1}{2} A, \qquad \tfrac{1}{2}(B + C) = 90^\circ - \tfrac{1}{2} A,$$

from which B and C may be found.

11. Two sides of a triangle are 13.5 and 17.6 respectively and the included angle is 35° 16′. Find the side of an equilateral triangle having the same area.

Ans. 6.293.

12. In Problem 11, find the side of an equilateral triangle having the same perimeter. *Ans.* 13.77.

13. The area of a regular polygon inscribed in a circle is to the area of a regular polygon of the same number of sides circumscribed about the circle as 3 is to 4. Find the number of sides of the polygon.

14. Two parallel chords on the same side of the center subtend angles of 100° and 50° respectively. Find the distance between the chords when the radius is 15.34. *Ans.* 4.042.

15. A regular pentagon and a regular decagon have equal perimeters. Find the ratios of their areas. Find the area of each, if the sum of their areas is 536 sq. in.

Ans. Ratio, 1.0487; area of pentagon, 274.36;
area of decagon, 261.64.

16. The coördinates of the vertices of a triangle are (0, 0) (471.32, 0), (896.41, 230.00) respectively. Find the coördinates of the intersection point of the bisectors of the angles.

Ans. (456.7, 57.66).

17. The side of a regular inscribed heptagon is approximately equal to one-half the side of an inscribed equilateral triangle. Find the error of the angle at the center, subtended by one of the sides as determined by this rule. Find the error in the length of the side of the heptagon if the radius of the circumscribed circle is 100.

18. The following rule gives approximately the side of a regular inscribed nine-sided polygon. On the diameter AB as a base (Fig. 116) construct a triangle ABC, having for the other two sides BC and AC, the sides of a regular inscribed hexagon and square respectively. From A as a center and a radius equal to AC, draw an arc cutting the diameter in D. CD will be approximately equal to the side of a regular nine-sided polygon. Determined the error in the central angle of the polygon as determined by this rule, and the error in the length of the side.

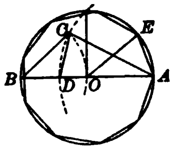


Fig. 116.

Suggestion. Apply the law of cosines to determine the angle A, then $CD = 2\,AC \cos \frac{1}{2} A$, and the angle at the center $AOE = 2 \cos^{-1}\frac{CD}{2r}$, where r is the radius of the circle.

Ans. Error in central angle $= 0^\circ 9.4'$,
Error in length of side $= 0.00254 \times r$.

19. The following rule gives approximately the side of a regular inscribed eleven-sided polygon. Let A, B, C, D be four consecutive vertices of a regular inscribed hexagon. Bisect the diagonal BD in E. Draw AE. From C as a center and radius equal to a side of the hexagon, draw an arc cutting AE in F. EF will be approximately equal to a side of the regular inscribed eleven-side. Compute the error in the central angle and in the side of the polygon thus determined.

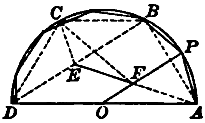


Fig. 117.

Suggestion. ABE is a right triangle, the two legs of which are readily found in terms of the radius r. Hence angle AEB can be found. Then in the triangle CEF, angle $CEF = 90^\circ +$ angle AEB, and the two sides CE and CF can be found, hence EF can be computed.

CHAPTER IX

THE GENERAL ANGLE AND ITS MEASURES

81. General Definition of an Angle. In elementary geometry an angle is defined as the difference in direction, or the amount of opening, between two lines which meet or tend to meet in a point. Obviously, two lines differ most in direction, the opening between them is greatest, when they extend in opposite directions. According to this definition, no angle can be greater than a straight angle, and in fact every angle considered in plane geometry is less than two right angles.

For the purposes of trigonometry and higher mathematics in general it is convenient to think of angles as formed by revolving a line in a plane about a fixed point. The line in its first or initial position is called the *initial line*, in its final position it is called the *terminal line*, the fixed point is called the *vertex*. *The amount of rotation which brings the line from its initial position to its terminal position is called the angle between the two lines.*

With this definition, it is plain that an angle may have any magnitude whatever, for their is no limit to the amount of rotation which a line may undergo. The angle described by the big hand of a clock increases so long as the clock continues to run,— when it has run 15 minutes the angle described is a right angle; when it has run 30 minutes the angle described is a straight angle; after an hour the big hand has returned to its initial position, yet the angle which it has described is not zero but four right angles. As the hand continues to move, the angle between the hand and its initial position still increases. After two complete revolutions this angle is equal to eight right angles; after three complete revolutions, to twelve right angles, and so on.

Before we can assign a magnitude to an angle in this wider sense, we must know more about it than merely the position of its sides. By the angle AOB (Fig. 118) may be meant an angle of 45°, as indicated by the short arrow, or an angle of 405°, as indicated by the long arrow, or

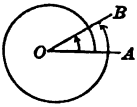


Fig. 118.

an angle of 45° increased by any number times 360°, depending on how many complete revolutions OB has described in reaching its final position. The angles 45°, 405°, and all the other angles which AOB might represent, are said to be *coterminal.* In elementary geometry, and generally when nothing to the contrary is said or implied, in referring to an angle, the numerically smallest of all the coterminal angles is understood. This value is known as the *principal value* of the angle.

82. Positive and Negative Angles. Sometimes it is convenient to distinguish between the two directions in which rotation of a line about a point in a plane may take place. This is done by prefixing a plus sign to the resulting angle if it has been described by rotation in a direction contrary to that in which the hands of a clock move (counterclockwise), Fig. 119, and a minus sign if the rotation has been in the opposite direction (clockwise), Fig. 120. When the sign is considered, the initial line is always read first; thus, each of the angles in figures 119 and 120 is read "angle AOB."

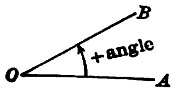


Fig. 119.

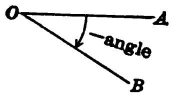


Fig. 120.

83. Complement and Supplement. If the algebraic sum of two angles is equal to a right angle, each angle is said to be the *complement* of the other; if their algebraic sum equals two right angles each is said to be the *supplement* of the other. That is to say,—

If $A + B = 90^\circ$, A and B are complementary angles,

If $A + B = 180^\circ$, A and B are supplementary angles.

A or B may have any magnitude, positive or negative. Thus, since

$95^\circ + (-5^\circ) = 90^\circ$, 95° and -5° are complementary angles,

and since

$-110^\circ + 290^\circ = 180^\circ$, -110° and 290° are supplementary angles.

84. Angles in the Four Quadrants. Let the vertex of the angle be taken as the origin of a system of rectangular coördinate axes, and the initial side of the angle for the positive direction of the x-axis. It follows that every angle less than 90°, as angle AOP_1, Fig. 121, has its terminal side in the first quadrant, and for this reason it is said to be an angle of the first quadrant. Every angle which is greater than a right angle but less than a straight angle, as angle AOP_2, is said to lie in, or to be an angle of, the second quadrant. Similarly, the angle AOP_3 is an angle of the third quadrant, angle AOP_4 an angle of the fourth quadrant, but an angle greater than four right angles and less than five right angles falls again in the first quadrant, and so on. Negative angles also lie in the quadrants determined by their terminal sides. Thus, while angle AOB (Fig. 119) lies in the first quadrant, the angle AOB (Fig. 120) lies in the fourth quadrant.

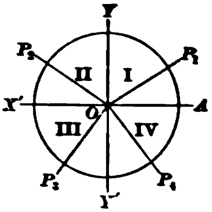


Fig. 121.

85. Sexagesimal Measure of Angles. It is familiar to all that different units of measures may be employed in the measurement of a given magnitude. Thus, any distance may be expressed in inches, feet, yards, rods and miles of the English system, or in millimeters, centimeters, meters and kilometers of the metric system. Values may be expressed in dollars and cents, in pounds and shillings, in francs and centimes, etc. Just so, angular magnitudes may be expressed in terms of different units of measure. In order to express a measure given in one set of units in terms of the units of another system, it is only necessary to know the relation between the units of the two systems.

The system of angular measure which we have used thus far is called the *sexagesimal system.*[*] The total angular magnitude about a point is divided into four equal parts and each part is called a right angle. The ninetieth part of a right angle is called a degree (°), the sixtieth part of a degree is called a minute (′), and the sixtieth part of a minute is called a second (″). The sexagesimal system of angular measures is the one most frequently used in every-day life.

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[*] From the Latin word *sexagesimus* (*sexa*, six + *decimus*, one-tenth), meaning one-sixtieth.

The sexagesimal scale was once applied to many measures. The sixtieth part of a unit of time, and length and weight was called its primate or prime; each prime was divided into sixty seconds, each second into sixty thirds. The divisor 60 has now disappeared among Western nations, except in measures of angles and of time. It is probable that the division of the angular space about a point into 360 degrees originated with the Babylonians, whose year consisted of 360 days. The Latin word for degree is *gradus*, the Greek, *bathmos*, each of which means step; that is, a degree originally meant the daily step of the sun eastward among the stars. The Chinese knew many centuries ago that the year consisted more nearly of 365¼ days, so they divided the total angular space about a point into 365¼ degrees. The sexagesimal system is unscientific and is doomed to be replaced sooner or later by a better system.

86. Decimal Division of Degrees. Instead of subdividing into minutes and seconds, the degree is sometimes divided decimally, into tenths, hundredths and thousandths. The decimal division of the degree has been used more or less ever since the invention of decimal fractions in the sixteenth century. Tables based on the decimal division of the degree have been published at various times.*

87. Centesimal Measure of Angles. Another system of angular measure, known as the *centesimal*, is obtained by dividing the right angle into 100 equal parts, each of which is called a grade (g), each grade into 100 minutes (${}^{\backprime}$), and each minute into 100 seconds (${}^{\backprime\backprime}$). An angle of 25 grades 16 minutes and 78 seconds would be written

$$25^g\,16^{\backprime}\,78^{\backprime\backprime}, \text{ or more simply, } 25.1678^g.$$

The centesimal system of angular measures was introduced as a part of the metric system by the French reformers† at the time of the great revolution. It possesses many advantages over the sexagesimal system, but owing to the fact that nearly all reference books and tables, all records of observation, the graduation of all astronomical instruments and of most engineering instruments, as well as the scales of geographical and nautical maps and charts, are based on the sexagesimal system, the progress of the introduction has been slow. But in spite of the many obstacles to be overcome, the system is making steady gains in the countries where the metric system is used. The centesimal system is now used exclusively in the field surveys in France, Belgium, Hessia and Baden, and it is legally recognized in several other states. It is regularly taught in many European high schools and technical schools.

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* The decimal system is now taught alongside the sexagesimal system in Harvard University and a number of other Eastern institutions of learning.

† The invention of this system and the first attempt to introduce it dates back to 1783 and is due to a German by the name of Johann Karl Schultze.

88. The Circular or Natural System of Angular Measures. In many practical investigations and in nearly all theoretical work, it is convenient to employ what is known as the circular or natural system of angular measure.

It is shown in geometry that in concentric circles the arcs a and a' (Fig. 122) intercepted by any angle α at the center are proportional to the radii r and r' of the circles, that is,

$$\frac{a}{r} = \frac{a'}{r'}, \tag{1}$$

and again, that in the same circle two central angles α and β are to each other as their intercepted arcs a and b, that is,

$$\frac{\alpha}{\beta} = \frac{a}{b} = \frac{a}{r}\bigg/\frac{b}{r} \tag{2}$$

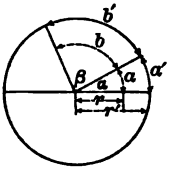

Fig. 122.

From (1) it follows that the ratio of the length of the arc to the radius of the circle is independent of the length of the radius, — in other words, that this ratio is constant so long as the angle is constant — and from (2) that this ratio varies as the angle, and may therefore be used as the measure of the angle. This ratio is known as the circular measure of the angle.

The circular measure of an angle is the ratio of the length of its intercepted arc, in a circle whose center is at the vertex of the angle, to the radius of the circle.

The unit of circular measure or *natural unit* is obtained by making a equal to r, that is, by taking the angle such that the intercepted arc equals the radius. This unit is called a radian.

A radian is an angle which, when placed with its vertex at the center of a circle, intercepts an arc equal in length to the radius of the circle.

Thus, if the arc AB (Fig. 123) is equal in length to the radius OA, the angle AOB measures one radian.

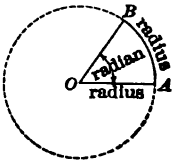

Fig. 123.

One peculiarity of the circular system is that it has no subsidiary or derived units; that is, no other units which are multiples or sub-multiples of a radian. All angles large or small are expressed in terms of this single unit, the radian.

89. Comparison of Sexagesimal and Circular Measure. To find the relation between the two kinds of units, degrees and radians, it is best to compare the two measures for the entire angular space about a point, that is, of four right angles. Expressed in circular measure, the measure of this angular space is equal to the circumference of a circle divided by the radius. Now the circumference of a circle is equal to the diameter, or twice the radius, multiplied by 3.14159+. Denoting this number by the Greek letter π,[*] we have for the circular measure of four right angles

$$\frac{\text{circumference}}{\text{radius}} = 2\,\pi.$$

Measured in degrees the same angular space is $360°$, hence we obtain the fundamental relation

$$2\,\pi \text{ radians} = 360 \text{ degrees},$$

or

$$\boldsymbol{\pi} \textbf{ radians} = \textbf{180 degrees}. \tag{1}$$

This relation enables us to reduce radians to degrees, and vice versa. Dividing both sides of (1) by the number π, we obtain

$$1 \text{ radian} = \frac{180}{\pi} \text{degrees},$$

hence, —

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[*] The number π is a most marvelous number. It is incommensurable, that is, it cannot be exactly expressed by a fraction whose terms are whole numbers. Nor can it be found by taking the square root, cube root or higher root of some commensurable number. Neither is it the root of any algebraic equation. For this reason it is called a transcendental number.

However, the value of π may be computed to any desired degree of accuracy. Archimedes showed that it is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. The former value is still commonly employed in rough approximations.

The Hindus, as early as the sixth century, computed the value of π from the perimeter of a regular inscribed polygon of 384 sides and found $\pi = 3.1416$; the value now generally used where $3\frac{1}{7} = 3.1428$ is not sufficiently accurate.

In recent times, the value of π has been computed to 707 places of decimals. The first 32 places were computed by Ludolph van Ceulen, a Dutchman, who devoted a good portion of his life to this task. For this reason π is frequently referred to as Ludolph's number. Its first 30 places are

$$\pi = 3.141{,}592{,}653{,}589{,}793{,}238{,}462{,}643{,}383{,}279.$$

To reduce radians to degrees, multiply the number of radians by $\frac{180}{\pi}$ (equals 57.3 nearly*).

Again, dividing both sides of (1) by 180 and writing the second member first, we obtain

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians};$$

hence,—

To reduce degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$ (equals 0.0175 nearly).

EXAMPLE 1. How many degrees in $\frac{3\pi}{2}$ radians?

Solution. Since π radians $= 180°$, $\frac{\pi}{2}$ radians $= 90°$,

and $\frac{3\pi}{2}$ radians $= 270°$.

EXAMPLE 2. How many radians in an angle of 60°?

Solution. Since $180° = \pi$ radians, $60° = \frac{\pi}{3}$ radians.

It is best to retain the letter π in the answers unless there is some reason to the contrary. For instance, putting for π its value 3.1416 and dividing by 3, the answer to the second example might have been written $60° = 1.0472$ radians, but $\pi/3$ radians is preferable. Likewise the first example might have been stated thus: How many degrees in 4.7144 radians? but the statement as first given is preferable.

When a number represents the measure of an angle, and no unit is expressed, the natural unit is understood. Thus, when we speak of the angle $\pi/2$, we mean not $\pi/2$ degrees but $\pi/2$ radians. The angle π means π radians or 180°, the angle 2 means not 2° but 2 radians, or 114.5916°. We may look upon $\frac{1}{2}\pi$, π, 2π, $2n\pi$, when referring to angles, as abbreviations of 90°, 180°, 360° and n times 360° respectively.

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* More accurately,

1 radian = 57.295,779,513 degrees,
1 degree = 0.017,453,290 radians,
1 minute = 0.000,290,888 radians,
1 second = 0.000,004,848 radians.

The value of the radian in degrees has been calculated to 43 places of decimals.

EXERCISE 44

(Use $3\frac{1}{7}$ for π and 57.3 for the value of 1 radian. n represents any positive integer.)

1. Express decimally the following angles:

$$45^\circ\,48'\,36,'' \; 185^\circ\,59'\,15'', \; 35^\circ\,30'\,30'', \; 375^\circ\,00'\,47''.$$

Ans. 45.81°, 185.9875°, 35.5083°, 375.01305°.

2. Express in degrees, minutes and seconds, 16.35°, 153.156°, 67.003°, $\frac{5}{7}$ of a right angle.

Ans. $16^\circ\,21'$, $153^\circ\,09'\,21.6''$, $67^\circ\,00'\,10.8''$, $64^\circ\,17'\,08.57''$.

3. Express in radians, 90°, 45°, $22\frac{1}{2}^\circ$, 60°, 15°, 30°, 135°, 270°. -315°, -75°, 360°. *Ans.* $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{8}$, $\frac{\pi}{3}$, $\frac{\pi}{12}$, etc.

4. Express in sexagesimal units the following angles:

$$\frac{\pi}{18},\ \frac{\pi}{10},\ \frac{\pi}{5},\ 2\tfrac{1}{2}\pi,\ \frac{2\pi}{3},\ 2\pi,\ 2n\pi,\ 2,\ \tfrac{1}{2},\ \frac{1}{\pi}.$$

Ans. 10°, 18°, . . . 114.6°, 28.65°, 18.23°.

5. Give a geometrical representation of each of the following angles: 340°, $\frac{5}{2}$ right angles, $-\frac{3}{4}\pi$, 725°, 2π, $2n\pi$, $(2n+1)\pi$.

6. State in which quadrant a line is after describing the following angles: 105°, -105°, $\frac{5\pi}{4}$, $-\frac{1}{2}\pi$, $\frac{2\pi}{3}$, 2, $+375^\circ$, $2n\pi - \frac{1}{4}\pi$.

Ans. 2d, 3d, . . . 1st, 4th.

7. Name three angles coterminal with an angle 30°.

8. Name two positive and two negative angles coterminal with -15°. *Ans.* 345°, 705°, -375°, -735°.

9. What is the smallest positive angle coterminal with 465°? With -465°. *Ans.* 105°, 255°.

10. Name the smallest negative angle coterminal with 735°? With -625°?

11. The principal value of an angle is $\frac{1}{4}\pi$; write in one formula all coterminal angles. *Ans.* $2n\pi + \frac{1}{4}\pi$.

12. If θ represents any angle less than 90°, all angles in the first quadrant can be expressed by $2n\pi + \theta$. Express similarly all angles in each of the other three quadrants.

Ans. II. $(2n+1)\pi - \theta$. III. $(2n+1)\pi + \theta$. IV. $2n\pi - \theta$.

13. What is the supplement of the angle $\frac{1}{2}\pi + \theta$? What is the complement? *Ans.* $\frac{1}{2}\pi - \theta$, $-\theta$.

14. What is the complement of $\frac{1}{4}\pi + \theta$? Of $\frac{\pi}{3} - \theta$?

Ans. $\frac{1}{4}\pi - \theta, \frac{\pi}{6} + \theta$.

15. Express both in radians and in degrees:
   (*a*) Each of the angles of an isosceles right triangle.
   (*b*) Each angle of a triangle whose angles are to each other as 1 : 2 : 3.

16. Find the number of radians in each of the angles of a regular pentagon, octagon, n-gon. *Ans.* $\frac{3\pi}{5}, \frac{3\pi}{4}, \frac{(n-2)\pi}{n}$.

17. Express decimally the following angles: $17^g\,18^{`}\,19^{``}$, $25^g\,65^{`}\,75^{``}$, $187^g\,05^{`}\,95^{``}$, $1^g\,00^{`}\,15^{``}$. *Ans.* 17.1819^g, 25.6575^g, etc.

18. Express $35^\circ\,30'\,30''$ and $35^g\,30^{`}30^{``}$ each as a fraction of a right angle. *Ans.* 0.394537, 0.35303.

19. Express $35^\circ\,30'\,30''$ in centesimal units. *Ans.* $39^g\,45^{`}\,37^{``}$.

20. Prove the following rule: To convert grades into degrees, diminish the number of grades by one-tenth of itself.

21. Make a corresponding rule for converting degrees into grades.

22. Find two regular polygons such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides of the first is to the number of sides of the second. *Ans.* Triangle and hexagon.

23. Find three pairs of regular polygons such that the number of degrees in an angle of one is equal to the number of grades in an angle of the other.

24. If g, d and r represent respectively the number of grades, degrees and radians in any angle, prove that

$$\pi(g - d) = 20r.$$

90. Relation Between Angle, Arc and Radius. If an angle θ is subtended by an arc 20 ft. long and the radius of the circumference of which the arc forms a part is 10 ft., the number of radians in θ is 20 ft. ÷ 10 ft. or 2, and generally, if an angle θ is subtended by an arc s units long and the radius measures r of the same units, the radian measure of the angle is

$$\theta = \frac{s}{r}. \qquad (1)$$

Solving this equation for s and r respectively, we get

$$s = r\theta, \qquad (2) \qquad\qquad r = \frac{s}{\theta}. \qquad (3)$$

Equation (1) enables us to find the angle when we know the length of the arc and the radius, (2) gives us the length of the arc in terms of the radius and the angle, and (3) gives the radius when the length of the arc is known as well as the angle which it subtends.

These three equations enable us to solve a great variety of interesting problems.

Example 1. The diameter of the earth subtends an angle of 17.5″ at the center of the sun. Assuming the diameter of the earth to be 7917.6 miles, what is the distance of the earth from the sun?

Solution. We may consider the diameter of the earth approximately equal to an arc which subtends an angle of 17.5″. Reducing 17.5″ to radians we find

$$\theta = \frac{17.5}{60 \times 60} \times \frac{\pi}{180};$$

hence, applying equation (1) we have for the required distance in miles

$$r = \frac{s}{\theta} = \frac{7917.6 \times 60 \times 60 \times 180}{17.5 \times \pi} = 93{,}322{,}000 \text{ miles.}$$

(By the use of logarithms, putting $\pi = 3.1416$.)

Example 2. A railroad train going due north strikes a curve of 3500 feet radius. The curve turns to the left and the train leaves the curve just 40 seconds after it entered the curve. In what direction does the train move after it leaves the curve, assuming that it is going at a uniform rate of 50 miles per hour.

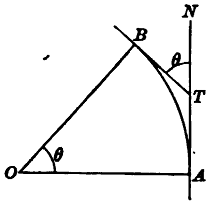

Fig. 124.

Solution. Let AN be the direction of the train on entering the curve and TB its direction when it leaves the curve at B.

It is required to find the angle NTB or θ.

Angle NTB = angle AOB (Why?), and by equation (1)

$$\theta = \text{angle } AOB \text{ (in radians)} = \frac{\text{arc } AB}{OA}.$$

OA is given and arc *AB* can be found, since we know the time which it takes the train to move over it at a given speed.

$$OA = \frac{3500}{5280} \text{ miles}, \qquad \text{arc } AB = \frac{50 \times 40}{60 \times 60} \text{ miles},$$

hence

$$\theta = \frac{50 \times 40}{60 \times 60} \div \frac{3500}{5280} = \frac{88}{105} \text{ radians},$$

$$= \frac{88}{105} \times \frac{180}{\pi} = \frac{88 \times 180 \times 7}{105 \times 22} = 48^\circ.$$

The train leaves the curve in the direction N. 48° W.

90a. Area of a Circular Sector. The area of a circular sector is equal to the area of a triangle with a base equal in length to the arc of the sector and an altitude equal to the radius, hence the area *A* is

$$A = \frac{rs}{2},$$

which by (2), Article 90, becomes

$$A = \tfrac{1}{2} r^2 \theta,$$

where θ is expressed in radians. (1)

EXAMPLE 1. Find the area of a circular sector bounded by an arc 10 feet in length, the radius of the circle being 50 feet.

Solution. By (1), Article 90, the radian measure of the angle $\theta = \frac{10}{50}$, and $r = 50$, hence by (1)

$$A = \frac{1}{2} \cdot 50^2 \cdot \frac{10}{50} = 250 \text{ sq. ft.}$$

EXERCISE 45

(To obtain the answers as given, use $\pi = 3\frac{1}{7}$).

1. Express in radians and in degrees the angle subtended by an arc 50 ft. long on a circle whose radius is 200 ft. *Ans.* $\frac{1}{4}$ radian = 14.3°.

2. What is the radius of a circle on which an arc 100 ft. long subtends an angle of 1°? *Ans.* $\frac{18,000}{\pi}$ ft.

3. How long is the arc subtended by a central angle of 75°, on a circumference whose radius is 100 ft.? *Ans.* 130.95 ft.

4. Find the length of an arc of 1° on the earth's surface ($r = 3960$ miles). *Ans.* $69\frac{1}{6}$ miles.

5. How long must a line be to subtend $1''$ at the distance of 1 mile? *Ans.* 0.3 inch (approximately).

6. A fly-wheel 10 ft. in diameter is revolving 200 revolutions per minute. Find the speed per second of a point on the rim.

Ans. $\frac{100\pi}{3}$ ft.

7. The diameter of the sun subtends an angle of $32'$ at the earth and the distance of the sun is about 93,000,000 miles. Required the diameter of the sun. *Ans.* 866,000 miles.

8. The earth moves around the sun once each year. Assuming its path to be a circle, find the velocity of the earth per second.

(Use the distance of the sun given in Problem 7.)

Ans. 18.5 miles.

9. The earth revolves on its axis once in 24 hours. Find its angular velocity per second, that is, the angle through which the earth turns per second, and hence the velocity in miles of a point on the equator. Use $r = 3960$ miles.

Ans. Angular velocity $15''$ per sec., linear velocity 0.288 mi. per sec.

10. The latitude of the city of Seattle is 47° 40′. Find the shortest distance from Seattle to the north pole. *Ans.* 2927 miles.

(Distances on the earth are of course measured along the arc of a great circle. Use $r = 3960$ mi.)

11. The diameter of a graduated circle is 12 inches and the graduations on its rim are $15'$. Find the approximate distance between two consecutive divisions on the rim. *Ans.* $\frac{1}{40}$ of an inch.

12. Two railroads meet at right angles at O. They are connected by a quadrant of a circle. The inner curve is 2000 ft. long. What is the distance of either point A or B from O?

Ans. $OA = \frac{4000}{\pi}$ ft.

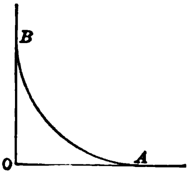

Fig. 125.

13. In Problem 12, if the rails are 4 ft. apart, how much longer will be the outer rail than the inner rail of the curve? *Ans.* 2π ft.

14. A circular sector, radius 10 inches, has an area of $26\frac{4}{21}$ sq. in. Find the angle of the sector. *Ans.* 30°.

15. A city lot has the shape of a circular sector, the curve bordering on the street. The straight sides of the lot are 100 ft. each and the angle between them is 60°. The lot was sold at $100 per foot frontage, what was the price per acre? *Ans.* $87,120.

91. Review.

1. (*a*) State and prove the law of sines. (*b*) State and prove the law of cosines. (*c*) Show that $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where R is the radius of the circumscribed circle. (*d*) Prove the projection theorem by means of the law of cosines.

2. (*a*) In any triangle show that $\sin(A + B) = \sin C$, $\cos(A + B) = -\cos C$, $\sin\frac{1}{2}(A + B) = \cos\frac{1}{2}C$, $\tan\frac{1}{2}(A + B) = \cot\frac{1}{2}C$. (*b*) Prove the double formula and the law of tangents. (*c*) Apply the law of tangents to the two legs a and b and the opposite acute angles of a right triangle and obtain $\tan\frac{1}{2}(A - B) = \frac{a - b}{a + b}$.

3. Express the area of a triangle: (*a*) In terms of two sides and the included angle. (*b*) In terms of the three sides. (*c*) Interpret geometrically the quantities $s - a$, $s - b$, $s - c$, and $k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

4. Show how to solve a triangle when the three sides are given, — (*a*) Without logarithms. (*b*) With logarithms. (*c*) Write down the formulas needed in (*a*) and (*b*). (*d*) Prove the half angle formulas in terms of the sides.

5. Discuss the solution of each of the four cases of oblique triangles giving in each case the formulas necessary for the logarithmic solution and a formula for checking the answers.

6. Review the method of solving each of the four cases of oblique triangles by division into right triangles.

7. Given the three sides of a triangle as follows:

$$a = 301.9, \qquad b = 673.1, \qquad c = 422.8.$$

Compute one or more of each of the following sets of related quantities:

(*a*) Angles,	$A = 18^\circ 12.4'$,	$B = 135^\circ 50.8'$,	$C = 25^\circ 56.9'$.
(*b*) Altitudes,	$h_a = 294.5$,	$h_b = 132.1$,	$h_c = 210.3$.
(*c*) Medians,	$m_a = 541.4$,	$m_b = 147.3$,	$m_c = 476.9$.
(*d*) Angle bisectors,	$b_A = 512.8$,	$b_B = 132.4$,	$b_C = 406.2$.
(*e*) Area and radii of inscribed and circumscribed circles,	$T = 44{,}458$,	$r = 63.61$,	$R = 483.1$.
(*f*) Radii of escribed circles,	$r_a = 112.0$,	$r_b = 1723$,	$r_c = 161.0$.

(*g*) Check results by the graphic method.

8. (*a*) Give a general definition of an angle of any magnitude. (*b*) Define a negative angle. (*c*) Give a general definition of the complement and supplement of an angle. (*d*) What is meant by the principal value of a set of coterminal angles. (*e*) Give three positive and three negative angles coterminal with 30°.

9. (*a*) Explain each system of angular measures and define the unit in each. (*b*) State some of the advantages of each system. (*c*) State the relation between radians and degrees. (*d*) Express the following relations in radian measure, $\sin(180^\circ - \theta) = \sin\theta$, $\tan(90^\circ - \theta) = \cot\theta$, $\cos 180^\circ = -1$.

10. Prove the following relations, where a denotes the side of any regular polygon, p the perimeter, n the number of sides, r the radius of the inscribed circle, R the radius of the circumscribed circle, and A the area of the polygon.

$$(a)\quad a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n}.$$

$$(b)\quad p = 2nR\sin\frac{\pi}{n} = 2nr\tan\frac{\pi}{n}.$$

$$(c)\quad A = nR^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{nar}{2} = nr^2\tan\frac{\pi}{n}.$$

CHAPTER X

FUNCTIONS OF ANY ANGLE

In this chapter a right angle will be denoted by R, rather than by 90° or $\pi/2$. The advantage of this notation is that it frees our results from any particular system of measure, that is, our results hold equally well whether the angles are expressed in degrees, grades or radians. Whenever R is expressed in a given unit it is understood that the other angles are expressed in the same unit. Thus, if we write $\sin(90^\circ - \theta)$ it is understood that θ is expressed in degrees, but if we write $\sin(\pi/2 - \theta)$, θ is to be expressed in radians, while the expression $\sin(R - \theta)$ does not specify the unit in which the angles are measured.

92. Definition of the Trigonometric Functions of Any Angle. The functions of any angle, positive or negative, are defined in exactly the same way as were the functions of an angle less than 180°. Let θ be any angle. Take the vertex O of the angle for an origin and the initial side of the angle for the positive direction of the x-axis. Let P represent any point (not the origin) on the terminal side of the angle. Let x and y denote the rectangular coördinates of P, and r its distance from the origin. Then whether θ is positive or negative, and whether P falls in the first, second, third or fourth quadrant, we have in every case,

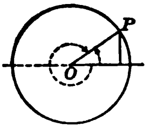

Fig. 126.

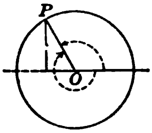

Fig. 127.

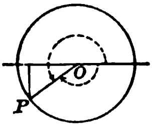

Fig. 128.

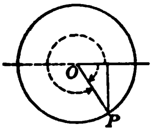

Fig. 129.

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc\theta = \frac{1}{\sin\theta},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

93. Signs of the Functions in Each of the Quadrants.

(*a*) Since the sine is the ratio of the ordinate of *P* to its distance from *O*, and the distance is always positive, the sign of the sine is the same as the sign of the ordinate, which is + in the first and second, − in the third and fourth quadrants.

(*b*) Since the cosine is the ratio of the abscissa of *P* to its distance from *O*, and the distance is always positive, the sign of the cosine is the same as the sign of the abscissa, which is + in the first and fourth, − in the second and third quadrants.

(*c*) Since the tangent is the ratio of the ordinate of *P* to the abscissa of *P*, the sign of the tangent is + when the ordinate and abscissa have like signs, that is, in the first and third quadrants, and − when they have unlike signs, that is, in the second and fourth quadrants.

(*d*) Any number and its reciprocal have like signs, hence the signs of the cosecant, secant and cotangent are the same as the signs of the sine, cosine and tangent respectively.

The student must make himself perfectly familiar with the signs of the functions in the various quadrants. The following figure will prove an aid to his memory.

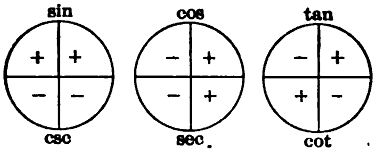

Fig. 130.

94. Periodicity of the Trigonometric Functions. If the terminal side of an angle is revolved in the plane of the angle through 4 *R*, or any number of times 4 *R*, it will return to the position from which it started, and this is true whether the revolution is in the positive (counterclockwise) or negative (clockwise) direction. It follows that the trigonometric functions of any angle remain unchanged when the angle is increased or diminished by 4 R or by any number of times 4 R. That is,

$$\sin(\theta \pm 4R) = \sin\theta,$$

and similarly for each of the other functions. In general,

$$\left.\begin{aligned} \sin(\theta \pm 4nR) &= \sin\theta, \\ \cos(\theta \pm 4nR) &= \cos\theta, \\ \tan(\theta \pm 4nR) &= \tan\theta, \text{etc.,} \end{aligned}\right\} \qquad (1)$$

where n is any positive or negative integer.

Thus,
$$\sin 375^\circ = \sin(375^\circ - 360^\circ) = \sin 15^\circ,$$
$$\sin(-15^\circ) = \sin(-15^\circ + 360^\circ) = \sin 345^\circ,$$
$$\sin\frac{9\pi}{4} = \sin\left(\frac{9\pi}{4} - 2\pi\right) = \sin\frac{\pi}{4},$$
$$\sin\frac{-7\pi}{6} = \sin\left(\frac{-7\pi}{6} + 2\pi\right) = \sin\frac{5\pi}{6},$$

and similarly for any other function.

It should be observed that by means of formula (1) the function of any negative angle can be replaced by the same function of some positive angle.

Since the trigonometric functions remain unchanged when the angle is increased or diminished by 4 R, they are called ***periodic functions*** with the ***period*** 4 R. It will be shown presently that the tangent and cotangent have the smaller period 2 R. There are other periodic functions besides the trigonometric functions.

95. Changes in the Values of the Functions while the Angle Changes from 0 to 4 R. Let a point P start from a position A and move in the positive direction along the circumference of a circle whose radius OA equals unity. Join P to the center O of the circle and let x and y represent the coördinates of P in its various positions with reference to O as origin and OA as the positive x-axis. Let us consider the changes in the values of the various functions of the angle $AOP = \theta$, as the point P moves along the circumference of the circle.

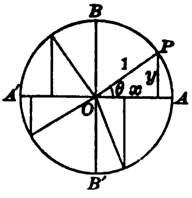

Fig. 131.

By definition $\sin\theta = \frac{y}{OP}$, $\cos\theta = \frac{x}{OP}$, $\tan\theta = \frac{y}{x}$. But OP was taken equal to 1, hence $\sin\theta$ is represented by the ordinate y, $\cos\theta$ by the abscissa x, and $\tan\theta$ by the ratio of the two.

First quadrant. While P moves from A to B, that is,
while θ changes from 0 to R,
y is positive and changes from 0 to 1,
hence $\sin\theta$ is positive and changes from 0 to 1;
x is positive and changes from 1 to 0,
hence $\cos\theta$ is positive and changes from 1 to 0;
y/x is positive and changes from 0 to ∞,
hence $\tan\theta$ is positive and changes from 0 to ∞.

Second quadrant. While P moves from B to A', that is
while θ changes from R to $2R$,
y is positive and changes from 1 to 0,
hence $\sin\theta$ is positive and changes from 1 to 0;
x is negative and changes from -0 to -1,
hence $\cos\theta$ is negative and changes from -0 to -1;
y/x is negative and changes from $-\infty$ to -0,
hence $\tan\theta$ is negative and changes from $-\infty$ to -0.

Third quadrant. While P moves from A' to B', that is,
while θ changes from $2R$ to $3R$,
y is negative and changes from -0 to -1,
hence $\sin\theta$ is negative and changes from -0 to -1;
x is negative and changes from -1 to -0,
hence $\cos\theta$ is negative and changes from -1 to -0;
y/x is positive and changes from 0 to ∞,
hence $\tan\theta$ is positive and changes from 0 to ∞.

Fourth quadrant. While P moves from B' to A, that is,
while θ changes from $3R$ to $4R$,
y is negative and changes from -1 to -0,
hence $\sin\theta$ is negative and changes from -1 to -0;
x is positive and changes from 0 to 1,
hence $\cos\theta$ is positive and changes from 0 to 1;
y/x is negative and changes from $-\infty$ to -0,
hence $\tan\theta$ is negative and changes from $-\infty$ to -0.

Cosecant, secant and cotangent. Since these functions are the reciprocals of the sine, cosine and tangent respectively, their variations can be immediately written down from the variations of the latter. Remember, —

(*a*) That the reciprocal of a number less than 1 is some number greater than 1, and vice versa.

(*b*) That the reciprocal of 0 is ∞, and vice versa.

(*c*) That reciprocals have like signs.

96. Changes in the Value of the Tangent of an Angle as the Angle Changes From 0 to 4 *R*. Some students find it difficult to follow the changes in the tangent from the ratio y/x when x and y both change. The following discussion is free from this difficulty.

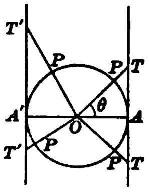

Fig. 132.

Let P (Fig. 132) move as in Fig. 131, but instead of the coördinates of the point P let us consider the coördinates of the point T or T' in which OP produced meets one of the tangents to the circle at A and A'.

First quadrant. While θ changes from 0 to R, AT changes from 0 to ∞, therefore $\tan\theta = \frac{AT}{OA} = \frac{AT}{1}$ changes from 0 to ∞.

Second quadrant. While θ changes from R to $2R$, $A'T'$ changes from $+\infty$ to $+0$, therefore $\tan\theta = \frac{A'T'}{OA'} = \frac{A'T'}{-1}$ changes from $-\infty$ to -0.

Third quadrant. While θ changes from $2R$ to $3R$, $A'T'$ changes from -0 to $-\infty$, therefore $\tan\theta = \frac{A'T'}{OA'} = \frac{A'T'}{-1}$ changes from $+0$ to $+\infty$.

Fourth quadrant. While θ changes from $3R$ to $4R$, AT changes from $-\infty$ to -0, therefore $\tan\theta = \frac{AT}{OA} = \frac{AT}{1}$ changes from $-\infty$ to -0.

97. Summary of Results. The results of the two preceding articles are brought together in the following table:

Quadrant.	I.	II.	III.	IV.
Angle	0 to R	R to $2R$	$2R$ to $3R$	$3R$ to $4R$
sin	$+0$ to $+1$	$+1$ to $+0$	-0 to -1	-1 to -0
cos	$+1$ to $+0$	-0 to -1	-1 to -0	$+0$ to $+1$
tan	$+0$ to $+\infty$	$-\infty$ to -0	$+0$ to $+\infty$	$-\infty$ to -0
csc	$+\infty$ to $+1$	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$
sec	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$	$+\infty$ to $+1$
cot	$+\infty$ to $+0$	-0 to $-\infty$	$+\infty$ to $+0$	-0 to $-\infty$

The student should observe, —

(*a*) Every sine and cosine has some value between $+1$ and -1.

(*b*) Every secant and cosecant has some value either greater than $+1$, or less than -1.

(*c*) A tangent or cotangent may have any value whatever.

(*d*) The functions change only at the points between the quadrants, that is, when the angle has one of the values R, $2R$, $3R$, $4R$, etc, and then only when the value of the function is either 0 or ∞.

(*e*) To a given value of a function correspond in general two different angles between 0 and $4R$. To a positive sine correspond two angles, one in the first the other in the second quadrant; to a negative sine correspond two angles, one in the third the other in the fourth quadrant. To a negative tangent correspond two angles, one in the second the other in the fourth quadrant, etc.

98. Fundamental Relations. All the fundamental relations between the functions of an acute angle (Article 12) hold true when the angle is unrestricted in magnitude. The argument is an exact repetition of that used in Article 56. It follows that all trigonometric identities which have been proven for the case when the angle does not exceed a right angle, hold universally, that is, whatever be the magnitude of the angle provided that radical expressions such as $\sqrt{1-\cos^2\theta}$, $\sqrt{1+\tan^2\theta}$, etc., be given the proper sign, $+$ or $-$, depending on the quadrant in which θ lies.

99. Representation of Trigonometric Functions by Lines. Until recent times the trigonometric functions were defined by lines connected with a circle as follows:

Let *AOP* be any angle, *AP* the arc which this angle intercepts on a circle drawn with *O* as a center and any length *OA* as a radius.

Draw the radius OA' perpendicular to OA, the initial side of the angle. Draw tangents to the circle at A and A', and produce OP to intersect these tangents in T and T' respectively. From P draw the perpendiculars PF and PF' to OA and OA' (produced if necessary) respectively.

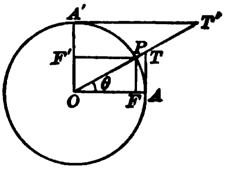

Fig. 133.

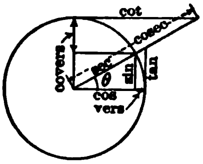

Fig. 134.

The former definitions were then as follows:

FP = sine of arc AP.
$F'P$ = sine of complementary arc $A'P$ = cosine of arc AP.
AT = tangent of arc AP.
$A'T'$ = tangent of complementary arc $A'P$ = cotangent of arc AP.
OT = secant of arc AP.
OT' = secant of complementary arc $A'P$ = cosecant of arc AP.
FA = versine of arc AP.
$F'A'$ = versine of complementary arc $A'P$ = coversine of arc AP.

The definitions just given apply to any arc, provided the conventions regarding the algebraic signs of the various lines be carefully observed. These conventions are, as already stated in Article 53, with the additional one that the distances OT, OT' are positive if they pass through the extremity of the arc in question, that is, if the point P lies between O and T or T'; and negative if they do not, that is, if the point O lies between P and T or T'. Thus for an arc AP in the second quadrant, Fig. 135,

FP, the sine, is positive;
OF, the cosine, is negative;
AT, the tangent, is negative;
$A'T'$, the cotangent, is negative;
OT, the secant, is negative;
OT', the cosecant, is positive.

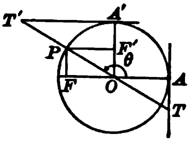

Fig. 135.

According to the old definitions, the length of each of the lines which defines the function depends on the length of the arc; and since the length of the arc depends on the radius, it was necessary to specify the length of the radius employed. If, however, all lengths are expressed in terms of the radius as unit, the old definitions agree with the modern definitions. Thus in Fig. 133,

$$\tan \text{arc } AP = AT \text{ (old definition).}$$

Now $\frac{\text{arc } AP}{OA} = \theta$, the measure of the angle which the arc subtends,

and $\frac{AT}{OA} =$ the measure of AT using OA as the unit of measure,

so that, if we substitute for the actual lengths of the arc AP and the line AT their measures in terms of the radius, the old definition becomes

$$\tan \theta = \frac{AT}{OA}, \text{ which is the modern definition.}$$

Similarly, the old and the new definitions of each of the other functions may be shown to agree.

The definitions of the functions as lines, while no longer used as definitions, are still useful in many ways. By their means the variation of the functions in the various quadrants is most readily traced, and the fundamental relations $\sin^2 \theta + \cos^2 \theta = 1$, $\tan^2 \theta + 1 = \sec^2 \theta$, $\cot^2 \theta + 1 = \csc^2 \theta$ become manifest at sight. Above all, they explain the origin of the names of the functions.*

EXERCISE 46

1. Make out a table giving each of the functions of 45°, 135°, 225°, 315°.

2. Make out a table giving each of the functions of 30°, 150°, 210°, 330°.

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* In the light of the historic definitions the origin of the terms *tangent* and *secant* is obvious. The origin of the terms *cosine, cotangent* and *cosecant* has already been explained. The origin of the term *sine* is probably as follows. The Latin word from which the word *sine* is derived is *sinus*, meaning "bay" or "bosom." The Arabic word was *dschiba*, meaning "half the chord of double an arc." Owing to the practice of the Arabs to omit the vowels in writing, *dschiba* was confused with *dschaib* meaning "bay" or "bosom," and it was this word *dschaib* which the Roman translators properly rendered *sinus*. The word *arc* comes from the Latin *arcus*, meaning "a bow." The *versed sine* was formerly called *sagitta*, an arrow, because it occupied the position of the arrow in a bow.

The modern conception of the functions as ratios dates from the second half of the seventeenth century. The old definitions modified by using unity for the radius of the circle were used by many writers less than twenty-five years ago.

3. Given $\sin\theta = \frac{1}{2}$, find the values of θ less than $4R$. What will be the values less than $4R$ which θ may have if $\cos\theta = \frac{1}{2}$?

Ans. 30°, 150°; 60°, 300°.

4. Find the values of the following functions: sin 390°, cos 765°, tan 405°, sin (– 45°), cos (– 30°), tan (– 135°).

Ans. $\frac{1}{2}, \frac{1}{2}\sqrt{2}, 1, -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{3}, 1$.

5. Trace the changes of the cosine through each of the four quadrants from the changes of the sine by means of the relation

$$\cos\theta = \sqrt{1 - \sin^2\theta}.$$

6. Trace the changes of the secant in the first quadrant from those of the tangent by means of the relation $\sec\theta = \sqrt{1 + \tan^2\theta}$.

7. Construct the lines representing the various functions of an arc in the third quadrant; of an arc in the fourth quadrant.

8. What sign must be attached to the radical in $\sin\theta = \sqrt{1 - \cos^2\theta}$, when θ is an angle in the second quadrant? In the third quadrant?

9. What sign must be attached to the radical in

$$\sec\theta = \sqrt{1 + \tan^2\theta},$$

when θ is an angle in the third quadrant? In the fourth quadrant?

100. Reduction of Trigonometric Functions to the First Quadrant. In Articles 57 and 58 it was shown how to express any function of an angle in the second quadrant in terms of functions of an angle less than R. It remains to be shown how functions of angles in the third and fourth quadrants may be expressed in terms of functions of angles less than R. When this has been done, the value of any function of any angle can be found from the tables which contain the functions of angles in the first quadrant, that is, from 0° to 90°.

101. Reductions from the Third Quadrant. Any angle θ_3 in the third quadrant lies between $2R$ and $3R$, hence every such angle may be expressed by either

$$\theta_3 = 2R + \theta, \quad \text{or} \quad \theta_3 = 3R - \phi,$$

where θ and ϕ is each less than R.

(*a*) $\theta_3 = 2R + \theta$. Let θ_3 = angle AOP_3 represent any angle in the third quadrant, and let θ = angle $A'OP_3$. Produce P_3O to P, making $OP = OP_3 = r$, then angle $AOP = \theta$. Let (x, y), (x_3, y_3) denote the coördinates of P and P_3 respectively. Draw PF and P_3F_3 perpendicular to AA', then the triangles OPF and OP_3F_3 are geometrically equal and $y_3 = -y$, $x_3 = -x$. Hence we have,—

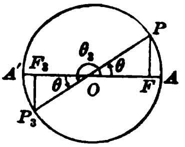

Fig. 136.

$$\left.\begin{aligned}
\sin\theta_3 &= \sin(2R+\theta) = \frac{y_3}{r} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,\\
\cos\theta_3 &= \cos(2R+\theta) = \frac{x_3}{r} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta,\\
\tan\theta_3 &= \tan(2R+\theta) = \frac{y_3}{x_3} = \frac{-y}{-x} = \frac{y}{x} = \quad\tan\theta.
\end{aligned}\right\}\quad (A)$$

From (A) we obtain

$$\left.\begin{aligned}
\csc\theta_3 &= \csc(2R+\theta) = \frac{1}{\sin(2R+\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\
\sec\theta_3 &= \sec(2R+\theta) = \frac{1}{\cos(2R+\theta)} = \frac{1}{-\cos\theta} = -\sec\theta,\\
\cot\theta_3 &= \cot(2R+\theta) = \frac{1}{\tan(2R+\theta)} = \frac{1}{\tan\theta} = \quad\cot\theta.
\end{aligned}\right\}\quad (A')$$

Observing that in both (A) and (A') the signs on the right are the signs of the functions in the third quadrant, we have the simple rule:

Any function of $(2R + \theta)$ *is equal to plus or minus the same function of* θ*, the sign being that of the function in the third quadrant.*

EXAMPLE. $\sin 204° = \sin(180° + 24°) = -\sin 24° = -0.4067,$
$\cos 204° = \cos(180° + 24°) = -\cos 24° = -0.9135,$
$\tan 204° = \tan(180° + 24°) = \tan 24° = 0.4453.$

(*b*) $\theta_3 = 3R - \phi$.

Put $R - \phi = \theta$, then $\sin\theta = \cos\phi$, $\cos\theta = \sin\phi$, $\tan\theta = \cot\phi$. Hence

$$\left.\begin{aligned}
\sin\theta_3 &= \sin(3R-\phi) = \sin(2R+\theta) = -\sin\theta = -\cos\phi,\\
\cos\theta_3 &= \cos(3R-\phi) = \cos(2R+\theta) = -\cos\theta = -\sin\phi,\\
\tan\theta_3 &= \tan(3R-\phi) = \tan(2R+\theta) = \quad\tan\theta = \quad\cot\phi,
\end{aligned}\right\}\quad (B)$$

and from (B)

$$\left.\begin{aligned}\csc\theta_3 &= \csc(3R-\phi) = \frac{1}{\sin(3R-\phi)} = \frac{1}{-\cos\phi} = -\sec\phi,\\ \sec\theta_3 &= \sec(3R-\phi) = \frac{1}{\cos(3R-\phi)} = \frac{1}{-\sin\phi} = -\csc\phi,\\ \cot\theta_3 &= \cot(3R-\phi) = \frac{1}{\tan(3R-\phi)} = \frac{1}{\cot\phi} = \tan\phi.\end{aligned}\right\}\quad(B')$$

In (B) and (B') the signs on the right are again the signs in the third quadrant of the functions on the left, hence the second rule:

Any function of $(3R-\phi)$ is equal to plus or minus the corresponding cofunction of ϕ, the sign being that of the functions in the third quadrant.

EXAMPLE.

$$\begin{aligned}\sin 204^\circ &= \sin(270^\circ - 66^\circ) = -\cos 66^\circ = -0.4067,\\ \cos 204^\circ &= \cos(270^\circ - 66^\circ) = -\sin 66^\circ = -0.9135,\\ \tan 204^\circ &= \tan(270^\circ - 66) = \cot 66 = 0.4453.\end{aligned}$$

102. Reductions from the Fourth Quadrant. Any angle θ_4 in the fourth quadrant lies between $3R$ and $4R$, hence every such angle can be expressed by either

$$\theta_4 = 4R - \theta, \qquad \text{or} \qquad \theta_4 = 3R + \phi,$$

where θ and ϕ are angles less than R.

(*a*) $\theta_4 = 4R - \theta$. Let θ_4 = angle AOP_4 be any angle in the fourth quadrant, and let us put angle $P_4OA = \theta$. Draw $OP = OP_4 = r$, making angle $AOP = \theta$. From P and P_4 draw perpendiculars to OA. Then the triangles OPF and OP_4F are geometrically equal, and if (x, y), (x_4, y_4) denote the coördinates of P and P_4 respectively, we have $y_4 = -y$, $x_4 = x$. Consequently

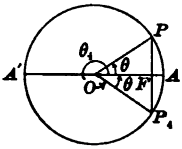

Fig. 137.

$$\left.\begin{aligned}\sin\theta_4 &= \sin(4R-\theta) = \frac{y_4}{r} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,\\ \cos\theta_4 &= \cos(4R-\theta) = \frac{x_4}{r} = \frac{x}{r} = \cos\theta,\\ \tan\theta_4 &= \tan(4R-\theta) = \frac{y_4}{x_4} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta.\end{aligned}\right\}\quad(A)$$

From (A) follows

$$\left.\begin{aligned}
\csc\theta_4 &= \csc(4R-\theta) = \frac{1}{\sin(4R-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\
\sec\theta_4 &= \sec(4R-\theta) = \frac{1}{\cos(4R-\theta)} = \frac{1}{\cos\theta} = \sec\theta,\\
\cot\theta_4 &= \cot(4R-\theta) = \frac{1}{\tan(4R-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta.
\end{aligned}\right\} (A')$$

In (A) and (A') the signs on the right are the signs of the functions in the fourth quadrant, hence

Any function of $(4R-\theta)$ is equal to plus or minus the same function of θ, the sign being that of the function in the fourth quadrant.

EXAMPLE. $\sin\frac{11\pi}{6} = \sin\left(2\pi-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2},$

$\cos\frac{11\pi}{6} = \cos\left(2\pi-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3},$

$\tan\frac{11\pi}{6} = \tan\left(2\pi-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{3}\sqrt{3}.$

(b) $\theta_4 = 3R+\phi.$

Put $\phi = R-\theta$, then $\sin\phi = \cos\theta$, $\cos\phi = \sin\theta$, $\cot\phi = \tan\theta$, hence

$$\left.\begin{aligned}
\sin\theta_4 &= \sin(3R+\phi) = \sin(4R-\theta) = -\sin\theta = -\cos\phi,\\
\cos\theta_4 &= \cos(3R+\phi) = \cos(4R-\theta) = \cos\theta = \sin\phi,\\
\tan\theta_4 &= \tan(3R+\phi) = \tan(4R-\theta) = -\tan\theta = -\cot\phi,
\end{aligned}\right\} (B)$$

and from (B)

$$\left.\begin{aligned}
\csc\theta_4 &= \csc(3R+\phi) = \frac{1}{\sin(3R+\phi)} = \frac{1}{-\cos\phi} = -\sec\phi,\\
\sec\theta_4 &= \sec(3R+\phi) = \frac{1}{\cos(3R+\phi)} = \frac{1}{\sin\phi} = \csc\phi,\\
\cot\theta_4 &= \cot(3R+\phi) = \frac{1}{\tan(3R+\phi)} = \frac{1}{-\cot\phi} = -\tan\phi.
\end{aligned}\right\} (B')$$

In (B) and (B') the signs on the right are the signs in the fourth quadrant of the functions on the left, hence

Any function of $(3R+\phi)$ is equal to plus or minus the corresponding cofunction of ϕ, the sign being that of the function in the fourth quadrant.

EXAMPLE. $\sin\frac{11\pi}{6} = \sin\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2},$

$$\cos\frac{11\pi}{6} = \cos\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{1}{2}\sqrt{3},$$

$$\tan\frac{11\pi}{6} = \tan\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{1}{3}\sqrt{3}.$$

103. Functions of Negative Angles.

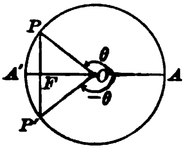

Fig. 138.

Let $-\theta$ = angle AOP' be any negative angle. Construct the angle $AOP = \theta$. Take $OP = OP' = r$, and let (x, y), (x', y') denote the coördinates of the points P and P' respectively, then $x' = x$, $y' = -y$, and we have

$$\left.\begin{aligned} \sin(-\theta) &= \frac{y'}{r} = \frac{-y}{r} = -\sin\theta, \\ \cos(-\theta) &= \frac{x'}{r} = \frac{x}{r} = \cos\theta, \\ \tan(-\theta) &= \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta, \end{aligned}\right\} \qquad (A)$$

and from (A)

$$\left.\begin{aligned} \csc(-\theta) &= \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta, \\ \sec(-\theta) &= \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta, \\ \cot(-\theta) &= \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta. \end{aligned}\right\} \qquad (A')$$

The signs on the right are the signs of the functions in the fourth quadrant, hence

Any function of $(-\theta)$ is equal to plus or minus the same function of θ, the sign being that of the function in the fourth quadrant.

EXAMPLE. $\sin(-13^\circ 25') = -\sin 13^\circ 25' = -0.2320,$

$$\cos(-13^\circ 25') = \cos 13^\circ 25' = 0.9727,$$

$$\tan(-13^\circ 25') = -\tan 13^\circ 25' = -0.2385.$$

104. Table of Principal Reduction Formulas and General Rules.

The principal results of the last three articles, together with the corresponding results for the first and second quadrants (Articles 10, 57, 58), are brought together in the following table for purposes of comparison and reference.

Quadrant I

$$\begin{array}{l|l}
 & \sin(R-\phi)=\cos\phi, \\
 & \cos(R-\phi)=\sin\phi, \\
 & \tan(R-\phi)=\cot\phi.
\end{array}$$

Quadrant II

$$\begin{array}{l|l}
\sin(2R-\theta)=\sin\theta, & \sin(R+\phi)=\cos\phi, \\
\cos(2R-\theta)=-\cos\theta, & \cos(R+\phi)=-\sin\phi, \\
\tan(2R-\theta)=-\tan\theta. & \tan(R+\phi)=-\cot\phi.
\end{array}$$

Quadrant III

$$\begin{array}{l|l}
\sin(2R+\theta)=-\sin\theta, & \sin(3R-\phi)=-\cos\phi, \\
\cos(2R+\theta)=-\cos\theta, & \cos(3R-\phi)=-\sin\phi, \\
\tan(2R+\theta)=\tan\theta. & \tan(3R-\phi)=\cot\phi.
\end{array}$$

Quadrant IV

$$\begin{array}{l|l}
\sin(4R-\theta)=\sin(-\theta)=-\sin\theta, & \sin(3R+\phi)=-\cos\phi, \\
\cos(4R-\theta)=\cos(-\theta)=\cos\theta, & \cos(3R+\phi)=\sin\phi, \\
\tan(4R-\theta)=\tan(-\theta)=-\tan\theta. & \tan(3R+\phi)=-\cot\phi.
\end{array}$$

We observe that each equation on the left involves a pair of same-named functions and the coefficients of R are even numbers, 2 or 4. On the right each equation involves a pair of conamed functions and the coefficients of R are the odd numbers 1 and 3. In either case the signs are the signs of the functions in the respective quadrants.

By increasing the angles on the left by multiples of $4R$ (which will not change the value of the functions), we obtain formulas for the functions of

$$6R-\theta,\ 6R+\theta,\ 8R-\theta,\ 8R+\theta,\ \ldots,\ \ldots,\ \ldots,\ 2nR\pm\theta,$$

and by increasing the angles on the right by multiples of $4R$, we obtain

$$5R+\phi,\ 5R-\phi,\ 7R+\phi,\ 7R-\phi,\ \ldots,\ \ldots,\ \ldots,\ (2n+1)R\pm\phi.$$

All the foregoing results are therefore included in the two formulas,

$$\textit{Any function } (2nR \pm \theta) = \pm \textit{ same function } \theta,$$

$$\textit{Any function } (\overline{2n+1}\,R \pm \phi) = \pm \textit{ cofunction } \phi,$$

the sign being the sign of the function on the left in the quadrant in which the angle falls when θ or ϕ are acute angles.*

EXERCISE 47

1. Express in terms of same-named functions of angles less than R,

$\sin 146^\circ$, $\cos 235^\circ$, $\tan 317^\circ$, $\sin\frac{5\pi}{4}$, $\cos\frac{7\pi}{8}$, $\tan\frac{15\pi}{4}$.

Ans. $\sin 34^\circ$, $-\cos 55^\circ$, $-\tan 43^\circ$, $-\sin\frac{\pi}{4}$, $-\cos\frac{\pi}{8}$, $-\tan\frac{\pi}{4}$.

2. Express in terms of cofunctions of angles less than R,

$\tan 95^\circ$, $\sin 272^\circ$, $\cos 115^\circ 10'$, $\sec\frac{2\pi}{3}$, $\sin\frac{28\pi}{3}$, $\cot 3\frac{1}{2}R$.

Ans. $-\cot 5^\circ$, $-\cos 2^\circ$, $-\sin 25^\circ 10'$, $-\csc\frac{\pi}{6}$, $-\cos\frac{\pi}{6}$, $-\tan\frac{R}{2}$.

3. Express in terms of functions of positive angles less than 45° $\sin 143^\circ 15'$, $\cos 143^\circ 15'$, $\tan 243^\circ 10' 15''$, $\sec 284^\circ 30'$, $\cot 127^\circ$.
Ans. $\sin 36^\circ 45'$, $-\cos 36^\circ 45'$, $\cot 26^\circ 49' 45''$, $\csc 14^\circ 30'$, $-\tan 37^\circ$.

4. Use natural functions table to find

$\sin 111^\circ 30'$, $\cos 253^\circ 12,'$ $\tan 134^\circ$, $\sin 317^\circ 15'$, $\cos 97^\circ 35'$.

Ans. 0.9304, -0.2890, -1.0355, -0.6788, -0.1320.

5. Find $\sin(-150^\circ)$, $\cos 3564^\circ$, $\tan(-5445^\circ)$, $\sin\left(-27\frac{\pi}{4}\right)$, $\cos(-100^\circ)$. *Ans.* $-\frac{1}{2}$, 0.8090, -1, $-\frac{1}{2}\sqrt{2}$, -0.1736.

6. If $\sin\theta = 0.5831$, what values less than $4R$ may θ have?

7. If $\tan\theta = -4.3897$, and $\sin\theta$ is known to be positive, find the value of θ. *Ans.* $\theta = 102^\circ 50'$.

8. If $\cos\theta = \sin 147^\circ$, show that one value of θ is 303°.

9. If $\sin\theta = \cos 5\theta$, show that one value of θ is 15°, and another 75°.

10. Given $\sin\phi = -0.4561$; find $\tan\phi$. *Ans.* $\tan\phi = 0.5125$.

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* These rules hold not only for the sine, cosine and tangent, but for the cosecant, secant and cotangent as well. The latter have been omitted from the summary on account of their lesser importance.

105. Generalization of the Preceding Reduction Formulas.

In the proof for the formulas for the functions of a negative angle (Article 103), θ was not restricted in magnitude. These formulas therefore hold true for any angle, but in the formulas for the functions of $2R + \theta$, $3R - \phi$ (Article 101), and of $4R - \theta$, $3R + \phi$ (Article 102), θ and ϕ were assumed to be angles between 0 and R. This restriction is not necessary and will now be removed. In other words, we will now show that the formulas of Articles 101 and 102 hold for any value of θ and ϕ. The complete list of reduction formulas includes the formulas for the functions of $2R - \theta$ and of $R \pm \phi$, we shall show that these formulas also hold for any value of the angles.

For the sake of brevity the proofs will be confined to the first one of the formulas in each set. The proofs of the other formulas are left as exercises for the student.

(*a*) *Functions of* $(2R + \theta)$. The angles $(2R + \theta)$ and θ differ by $2R$ no matter how large θ is and whether θ is positive or negative. Consequently the points P_3 and P (Fig. 136) must always lie on a straight line through the origin. The coördinates of P_3 and P will therefore be numerically equal but opposite in sign. Hence for any value of θ, positive or negative,

$$\sin(2R + \theta) = \frac{y_3}{r} = \frac{-y}{r} = -\sin\theta. \tag{1}$$

This establishes the first of the relations (A), Article 101, for every value of θ.

(*b*) *Functions of* $(2R - \theta)$. If in (1) we put for θ, $-\theta$, we obtain

$$\sin(2R - \theta) = -\sin(-\theta).$$

But by (A), Article 103, $\sin(-\theta) = -\sin\theta$ for every value of θ, therefore

$$\sin(2R - \theta) = \sin\theta. \tag{2}$$

This establishes the first of the relations in Article 57 for every value of θ.

(*c*) *Functions of* $(4R - \theta)$. The functions of an angle are not changed if the angle is increased or diminished by $4R$ (Article 94), hence

$$\sin(4R - \theta) = \sin(-\theta), \text{ for every value of } \theta.$$

But $\sin(-\theta) = -\sin\theta$, by Article 103,

therefore $\sin(4R - \theta) = -\sin\theta.$ (3)

This establishes the first of the relations (A), Article 102, for every value of the angle.

(d) *Functions of* $(R - \phi)$. Let ϕ be any angle and let ϕ' be the smallest positive angle coterminal with ϕ. Then ϕ' can be written in one of the forms

θ, $2R - \theta$, $2R + \theta$, $4R - \theta$, where θ is positive and less than R, according as ϕ' is an angle in the first, second, third or fourth quadrant.

If $\phi' = 2R - \theta$, then

$$\sin(R - \phi') = \sin(-R + \theta) = -\sin(R - \theta) = -\cos\theta = \cos\phi'.$$

If $\phi' = 2R + \theta$, then

$$\sin(R - \phi') = \sin(-R - \theta) = -\sin(R + \theta) = -\cos\theta = \cos\phi'.$$

If $\phi' = 4R - \theta$, then

$$\sin(R - \phi') = \sin(-3R + \theta) = -\sin(3R - \theta) = \cos\theta = \cos\phi'.$$

We see then that, whether ϕ' is in the first, second, third or fourth quadrant,

$$\sin(R - \phi') = \cos\phi',$$

and hence

$$\sin(R - \phi) = \cos\phi \tag{4}$$

is established for every value of ϕ.

(e) *Functions of* $(R + \phi)$, $(3R - \phi)$, $(3R + \phi)$. Let ϕ be any angle and put $\phi = R - \theta$. Then by applying the formulas whose generality has been already established, we find

$$\sin(R + \phi) = \sin(2R - \theta) = \sin\theta = \cos\phi \tag{5}$$

$$\sin(3R - \phi) = \sin(2R + \theta) = -\sin\theta = -\cos\phi, \tag{6}$$

$$\sin(3R + \phi) = \sin(4R - \theta) = -\sin\theta = -\cos\phi. \tag{7}$$

EXERCISE 48

Repeat the argument of the preceding article to show that for all values of θ and ϕ respectively,—

1. $\tan(2R \pm \theta) = \pm\tan\theta.$
2. $\tan(R \pm \phi) = \mp\cot\phi.$
3. $\tan(3R \pm \phi) = \pm\cot\phi.$

4. Show that for every value of θ,

$$\sin(\theta - R) = -\cos\theta,\ \sin(\theta - 2R) = -\sin\theta,\ \sin(\theta - 3R) = \cos\theta.$$
$$\cos(\theta - R) = \sin\theta,\ \cos(\theta - 2R) = -\cos\theta,\ \cos(\theta - 3R) = -\sin\theta.$$
$$\tan(\theta - R) = -\cot\theta,\ \tan(\theta - 2R) = \tan\theta,\ \tan(\theta - 3R) = -\cot\theta.$$

5. Prove geometrically that $\cos(R - \phi) = \sin\phi$,

(*a*) When ϕ is an angle in the second quadrant.
(*b*) When ϕ is an angle in the third quadrant.
(*c*) When ϕ is an angle in the fourth quadrant.

6. General proofs of the reduction formulas for the angles

$$(2R - \theta),\quad (4R - \theta),\quad (2R + \theta)$$

may be obtained from considerations of symmetry. Referred to the same origin and taking the initial line to coincide with the *x*-axis, then no matter how large θ, and whether positive or negative,

(*a*) The terminal sides of θ and $(2R - \theta)$ are symmetrically situated with respect to the *y*-axis.
(*b*) The terminal sides of θ and $(4R - \theta)$ are symmetrically situated with respect to the *x*-axis.
(*c*) The terminal sides of θ and $(2R + \theta)$ are symmetrically situated with respect to the origin.

It follows that the same functions of each pair of angles are numerically equal and that in

(*a*) The sines have like and the cosines opposite signs.
(*b*) The sines have opposite and the cosines like signs.
(*c*) The sines and cosines each have opposite signs.

In each case the sign of the tangent may be determined from the relation

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

CHAPTER XI

FUNCTIONS OF TWO OR MORE ANGLES

106. Addition Theorems for the Sine and Cosine.

First Proof. If a, b, c denote the sides of any triangle and A, B, C the angles opposite these sides, we have from the law of sines (Article 62, (3)),

$$a = D \sin A,$$
$$b = D \sin B,$$
$$c = D \sin C,$$

and by the projection theorem (Article 63, (4)),

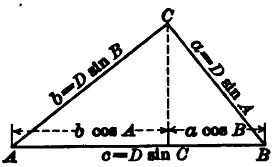

Fig. 139.

$$c = a \cos B + b \cos A.$$

Substituting in this equation the above values for a, b and c, and dividing out the constant factor D, we get

$$\sin C = \sin A \cos B + \cos A \sin B.$$

Now $C = 180° - (A + B)$, therefore $\sin C = \sin (A + B)$, whence

$$\sin (A + B) = \sin A \cos B + \cos A \sin B. \qquad (1)$$

Again

$$\begin{aligned} \cos^2 (A + B) &= 1 - \sin^2 (A + B) \\ &= 1 - (\sin A \cos B + \cos A \sin B)^2 \\ &= 1 - \sin^2 A \cos^2 B - 2 \sin A \sin B \cos A \cos B - \cos^2 A \sin^2 B \\ &= 1 - (1 - \cos^2 A) \cos^2 B - 2 (\ . \ . \ . \ ) - (1 - \sin^2 A) \sin^2 B \\ &= \cos^2 A \cos^2 B - 2 \sin A \sin B \cos A \cos B + \sin^2 A \sin^2 B \\ &= (\cos A \cos B - \sin A \sin B)^2. \end{aligned}$$

Taking the square root of both sides,

$$\cos (A + B) = \cos A \cos B - \sin A \sin B. \qquad (2)$$

Since the last expression was obtained by extracting a square root it may seem that the double sign, ±, should have been put before the right-hand member of (2), but on putting $B = 0$, the minus would yield the result $\cos A = -\cos A$, which shows that the minus sign cannot be used.

Formulas (1) and (2) embody the so-called *addition theorems* for the sine and cosine respectively. In words,—

The sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second plus the cosine of the first angle times the sine of the second.

The cosine of the sum of two angles is equal to the product of the cosines of the separate angles diminished by the product of their sines.

It is plain that by means of these theorems the sine and cosine of the sum of two angles may be found if the sines and cosines of each of the separate angles are known.

EXAMPLE 1. Given the functions of 45° and of 30°, to find the sine and cosine of 75°.

Solution.
$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6}+\sqrt{2}).\\ \cos 75^\circ &= \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6}-\sqrt{2}).\end{aligned}$$

107. Generalization of the Addition Theorems. In the foregoing demonstration, A and B are angles of a triangle, their sum is therefore necessarily less than $2R$. This restriction may be removed, in other words, the addition theorems hold for angles of any magnitude and whether positive or negative.

To prove this let A and B be two angles each less than R, so that their sum is less than $2R$, and let $A_1 = A \pm R$, then

$$\sin A_1 = \pm \cos A,\quad \cos A_1 = \mp \sin A,\quad \text{(Art. 104, and Ex. 48, 4)}$$

$$\begin{aligned}\sin(A_1 + B) &= \sin(A \pm R + B) = \pm\cos(A+B)\\ &= \pm(\cos A \cos B - \sin A \sin B)\\ &= \pm\cos A \cos B \mp \sin A \sin B\\ &= \sin A_1 \cos B + \cos A_1 \sin B. \qquad (1)\\ \cos(A_1 + B) &= \cos(A \pm R + B) = \mp\sin(A+B)\\ &= \mp(\sin A \cos B + \cos A \sin B)\\ &= \mp\sin A \cos B \mp \cos A \sin B\\ &= \cos A_1 \cos B - \sin A_1 \sin B\,^{*} \qquad (2)\end{aligned}$$

---

[*] If the student finds it difficult to follow the double signs, let him consider the two cases $A_1 = A + R$, $A_1 = A - R$, separately.

Equations (1) and (2) show that the addition theorems continue to hold if the angle A is increased or diminished by R, and the same reasoning applies to the angle B. By a repetition of the process just employed it is clear that the theorems will continue to hold true if A_1 is replaced by A_2, where $A_2 = A_1 \pm R = A_1 \pm 2R$, and generally that A may be replaced by

$$A_n = A \pm nR,$$

and B by

$$B_m = B \pm mR,$$

n and m being two arbitrary integers.

But this proves the theorems for all values of the angles, for any positive or negative angle may be put in the form $A \pm nR$, where n is some integer and A some angle less than R.

108. Addition Theorems. *Second Proof.* The addition theorems may be proved without making use of the law of sines and the projection theorem.

In Fig. 140, let XOM = angle A, and MON = angle B, then XON = angle $(A + B)$.

On ON take any point P, and from P draw PT and PQ perpendicular to OX and OM respectively.

From Q draw QR perpendicular to OX and QS parallel to OX.

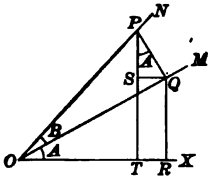

Fig. 140.

The triangles QOR and QPS are similar (Why?), hence angle QPS = angle A. Now

$$\sin(A+B) = \frac{TP}{OP} = \frac{TS+SP}{OP} = \frac{RQ}{OP} + \frac{SP}{OP}$$

$$= \frac{RQ}{OQ} \cdot \frac{OQ}{OP} + \frac{SP}{QP} \cdot \frac{QP}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{OT}{OP} = \frac{OR-TR}{OP} = \frac{OR}{OP} - \frac{SQ}{OP}$$

$$= \frac{OR}{OQ} \cdot \frac{OQ}{OP} - \frac{SQ}{QP} \cdot \frac{QP}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

In the figure we have taken $A + B$ less than a right angle, but the proof just given will hold for any angles, provided proper attention be given to the algebraic signs of the lines which enter the figure.

109. Subtraction Theorems for the Sine and Cosine. Since the addition theorems have been shown to hold for negative as well as for positive angles, we may replace B by $-B$. The equations (1) and (2), Article 106, then become

$$\sin (A - B) = \sin A \cos (-B) + \cos A \sin (-B)$$
$$\cos (A - B) = \cos A \cos (-B) - \sin A \sin (-B),$$

from which

$$\mathbf{\sin (A - B) = \sin A \cos B - \cos A \sin B,} \qquad (1)$$
$$\mathbf{\cos (A - B) = \cos A \cos B + \sin A \sin B.} \qquad (2)$$

These formulas enable us to compute the sine and cosine of the difference of two angles if the sines and cosines of the separate angles are known.

EXAMPLE. Given the functions of 45° and 30°; to find the sine and cosine of 15°.

Solution.

$$\sin 15^\circ = \sin (45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}).$$
$$\cos 15^\circ = \cos (45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).$$

EXERCISE 49

1. Find the sine and cosine of 15° from the relation, $15^\circ = 60^\circ - 45^\circ$.

2. Given $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{8}{17}$, $\cos y = \frac{15}{17}$; find $\sin (x + y)$ and $\cos (x + y)$.

3. Find sin 90° and cos 90° from the relation $90^\circ = 60^\circ + 30^\circ$.

4. Find sin 0° and cos 0° from the relation $0^\circ = 30^\circ - 30^\circ$.

5. Apply the addition and subtraction theorems to find the following:

$$\sin (90^\circ - x),\quad \cos (90^\circ + x),\quad \sin (180^\circ + x),\quad \cos (270^\circ - x),$$
$$\sin (360^\circ - y),\quad \cos (45^\circ - y),\quad \sin (30^\circ + y),\quad \cos (60^\circ - y).$$

6. Show that

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B,$$
$$\sin(A+B)-\sin(A-B) = 2\cos A\sin B,$$
$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B,$$
$$\cos(A+B)-\cos(A-B) = -2\sin A\sin B.$$

7. By putting $B = A$ in Problem 6 show that

$$\sin 2A = 2\sin A\cos A, \cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

8. Prove the subtraction theorems geometrically by means of Fig. 141. $XOQ =$ angle A, $MON =$ angle B, $XON =$ angle $(A - B)$. QR, QP, PT are perpendiculars to OX, OM, OX respectively. Angle $SQP =$ angle A (Why?).

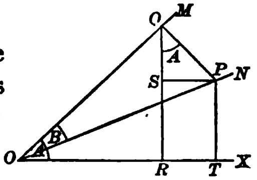

Fig. 141.

9. Show that the answers to Problem 17, Exercise 38, may be put in the forms

$$x = \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}, \qquad y = \frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

10. Show that the answer to Problem 19, Exercise 38, may be written

$$h = \frac{a\sin\alpha\sin\alpha'}{\sqrt{\sin(\alpha-\alpha')\sin(\alpha+\alpha')}}.$$

11. Show that

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A\cos B},$$

and hence that

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}.$$

12. Show that $\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta$,

and hence that $\sin\overline{n+1}^\circ = \sin n^\circ\cos 1^\circ + \cos n^\circ\sin 1^\circ$,

similarly $\cos\overline{n+1}^\circ = \cos n^\circ\cos 1^\circ - \sin n^\circ\sin 1^\circ$.

Hence, if the sine and cosine of 1° are known, those of 2°, 3°, etc., may be readily computed.

13. Show that $\cos(A + \frac{1}{4}\pi) + \sin(A - \frac{1}{4}\pi) = 0$,

and $\sin(A + \frac{1}{4}\pi) + \cos(A - \frac{1}{4}\pi) = \sqrt{2}(\sin A + \cos A)$.

14. Show that

$$\cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0,$$
$$\sin A \sin (B - C) + \sin B \sin (C - A) + \sin C \sin (A - B) = 0.$$

15. Two straight roads OA and OB (Fig. 142) cross at an angle α. From O, their point of intersection, a straight road is to be laid out to a point P, which is p miles from the first road and q miles from the second. Required the angle θ which OP will make with OA.

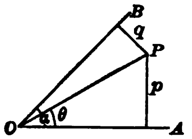

Fig. 142.

$$\frac{p}{q} = \frac{\sin \theta}{\sin (\alpha - \theta)}, \qquad \text{whence} \quad \tan \theta = \frac{p \sin \alpha}{p \cos \alpha + q}.$$

16. The area T of a triangle was computed from two sides, b and c, and the included angle A. Afterwards it was found that an error ϵ had been made in measuring the angle A. Show that the corrected area is given by the formula

$$T' = T (\cos \epsilon + \sin \epsilon \cot A).$$

17. Two parallel forces p and q act on levers of lengths a and b respectively, which are inclined at an angle α at the common fulcrum O. What angle θ must the forces make with the lever a in order that there may be equilibrium?

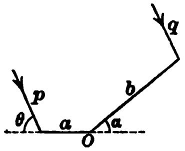

Fig. 143.

(Suggestion. Equating moments about O we have

$$ap \sin \theta = bq \sin (\alpha + \theta),$$

from which

$$\tan \theta = \frac{bq \sin \alpha}{ap - bq \cos \alpha}.)$$

18. Show that

$$\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A,$$
$$\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

19. x, y, z being any angles, show that

$$\sin (x + y + z) = \sin x \cos y \cos z + \sin y \cos z \cos x + \sin z \cos x \cos y - \sin x \sin y \sin z.$$

$$\cos (x + y + z) = \cos x \cos y \cos z - \cos x \sin y \sin z - \cos y \sin z \sin x - \cos z \sin x \sin y.$$

20. Show that

$$\sin(x+y-z)+\sin(x-y+z)+\sin(-x+y+z)=\sin(x+y+z)+4\sin x\sin y\sin z.$$

$$\cos(x+y-z)+\cos(x-y+z)+\cos(-x+y+z)=4\cos x\cos y\cos z-\cos(x+y+z).$$

21. By eliminating a, b, c from the equations (Art. 63),

$$c=a\cos B+b\cos A,$$
$$a=b\cos C+c\cos B,$$
$$b=c\cos A+a\cos C,$$

show that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

Solve this equation for $\cos C$, and obtain

$$\cos C=-\cos A\cos B\pm\sin A\sin B.$$

Remembering that $C=180^\circ-(A+B)$, and disregarding the lower sign which is inadmissible (Why?), we find

$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

This constitutes another *proof of the addition theorem* for the cosine.

110. Tangent of the Sum and Difference of Two Angles. If we divide the sine of the sum of two angles by the cosine we obtain the tangent, thus

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

On dividing both the numerator and denominator of the right-hand member by $\cos A\cos B$, we have

$$\tan(A+B)=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

that is

$$\boldsymbol{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}. \qquad (1)$$

To obtain the tangent of the difference of two angles, we need only put in (1) for B, $-B$, thus

$$\tan(A-B)=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)},$$

that is,

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \qquad (2)$$

Of course we might have deduced (2) just as we deduced (1) that is, by dividing $\sin (A - B)$ by $\cos (A - B)$.

111. Functions of Double an Angle. If $B = A$, the formulas for the sine, cosine and tangent of the sum of two angles become

$$\sin (A + A) = \sin A \cos A + \cos A \sin A,$$
$$\cos (A + A) = \cos A \cos A - \sin A \sin A,$$
$$\tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A},$$

that is,

$$\sin 2A = 2 \sin A \cos A, \qquad (1)$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1, \qquad (2)$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \qquad (3)$$

By means of these formulas the functions of twice an angle are easily computed, provided the functions of the single angle are known.

112. Functions of Half an Angle. It is frequently necessary to express the functions of half an angle in terms of the functions of the whole angle. This is most easily accomplished by means of (2), Article 111. Since these formulas hold for any value of the angle, we may replace A by $\frac{1}{2} A$, thus

$$\cos (2 \cdot \tfrac{1}{2} A) = 1 - 2 \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} A - 1,$$

or

$$\left.\begin{aligned} \cos A &= 1 - 2\sin^2 \tfrac{1}{2} A \\ &= 2 \cos^2 \tfrac{1}{2} A - 1. \end{aligned}\right\} \qquad (1)$$

If we solve the first of these equations for $\sin \frac{1}{2} A$, we obtain

$$\sin \tfrac{1}{2} A = \pm \sqrt{\frac{1 - \cos A}{2}}, \qquad (2)$$

and the second solved for $\cos \frac{1}{2} A$ gives

$$\cos \tfrac{1}{2} A = \pm \sqrt{\frac{1 + \cos A}{2}}. \qquad (3)$$

Dividing (2) by (3) gives

$$\tan \tfrac{1}{2} A = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}. \qquad (4)$$

EXERCISE 50

1. Given the tangents of 45° and 30°, compute the tangents of 75° and 15°.

Ans. $\tan 75° = \dfrac{3 + \sqrt{3}}{3 - \sqrt{3}}$, $\tan 15° = \dfrac{3 - \sqrt{3}}{3 + \sqrt{3}}$.

2. Given $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, find $\tan(A + B)$ and $\tan(A - B)$.

3. Show that

$$\tan(45° + A) = \frac{1 + \tan A}{1 - \tan A}, \qquad \tan(45° - A) = \frac{1 - \tan A}{1 + \tan A}.$$

4. Show that

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}, \qquad \cot(A - B) = \frac{\cot A \cot B + 1}{-\cot A + \cot B}.$$

5. Show that

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}.$$

6. Express $\sin 4A$, $\cos 4A$, $\tan 4A$ in terms of the functions of $2A$.

7. Given the functions of 30°; find the sine, cosine and tangent of 60°.

8. Given the functions of 45°; find the sine, cosine and tangent of $22\frac{1}{2}°$.

Ans. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $\tan 22\frac{1}{2}° = \sqrt{2} - 1$.

9. Given the functions of 30°; find the sine, cosine and tangent of 15°.

Ans. $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$, $\cos 15° = \frac{1}{2}\sqrt{2 + \sqrt{3}}$, $\tan 15° = 2 - \sqrt{3}$.

10. Express $\sin 3A$ in terms of $\sin A$.

Ans. $\sin 3A = 3 \sin A - 4 \sin^3 A$.

(Suggestion. $3A = 2A + A$.)

11. Express $\cos 3A$ in terms of $\cos A$.

Ans. $\cos 3A = 4 \cos^3 A - 3 \cos A$.

12. Express $\tan 3A$ in terms of $\tan A$.

$$Ans.\quad \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

13. Find the sine and cosine of 18°.

$$Ans.\quad \sin 18^\circ = \frac{\sqrt{5}-1}{4},\qquad \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

(Suggestion. Let $x = 18^\circ$, then $2x + 3x = 90^\circ$, $2x = 90^\circ - 3x$, $\sin 2x = \cos 3x$.

Now express $\sin 2x$ and $\cos 3x$ each in terms of functions of x, and solve for $\sin x$. Then $\cos x = \sqrt{1-\sin^2 x}$.)

14. Show that if t stands for $\tan A$,

$$\sin 2A = \frac{2t}{1+t^2},\qquad \cos 2A = \frac{1-t^2}{1+t^2},\qquad \tan 2A = \frac{2t}{1-t^2}.$$

15. Use Fig. 144 to prove that

$$\sin 2A = 2\sin A\cos A,$$
$$\cos 2A = \cos^2 A - \sin^2 A$$

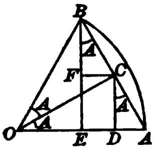

Fig. 144.

16. Use Fig. 145 to prove that

$$\sin \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1-\cos A)},$$
$$\cos \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1+\cos A)}.$$

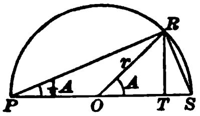

Fig. 145.

(Suggestion. $\sin \tfrac{1}{2}A = \dfrac{RS}{2r}$, $\overline{RS}^2 = \overline{OR}^2 + \overline{OS}^2 - 2\,OR\cdot OS\cos A$,

$\cos \tfrac{1}{2}A = \dfrac{RP}{2r}$, $\overline{RP}^2 = \overline{OR}^2 + \overline{OP}^2 + 2\,OR\cdot OP\cos A$.)

17. If $A + B + C = 180^\circ$, show that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

18. $\tan A = \dfrac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}$. Solve this equation for $\tan\frac{1}{2}A$, and identify your result with (4), Art. 112.

19. Show that the equation $a\tan x + b\cot x = c$, may be reduced to the form $(a-b)\cos 2x + c\sin 2x = a + b$.

20. A flagpole 50 ft. high stands on a tower 40 ft. high. At what distance from the foot of the tower will the flagpole and tower subtend equal angles? *Ans.* 120 ft.

21. A tower is situated at a distance of a ft. from the banks of a river b ft. wide. At what height on the tower will the river subtend an angle of 30°?

Ans. $\frac{1}{2} b \sqrt{3} \pm \frac{1}{2} \sqrt{3 b^2 - 4 a^2 - 4 ab}$. (Two solutions?)

22. The dial of a town clock has a diameter of $2r = 8$ ft. and its center is $h = 70$ ft. above the ground. At what distance from the foot of the tower will the dial be most plainly visible?

Ans. $x = \sqrt{(h + r)(h - r)} = 69.98$ ft.

(Suggestion. The angle subtended by the dial must be the largest possible.)

23. The height h of an object AB was computed from the distance d of a point O from the foot of the object and the angle θ which AB subtended at this point. It was found that an error ϵ had been made in measuring θ. Show that h must be corrected by an amount

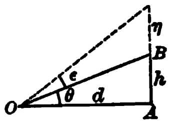

Fig. 146.

$$\eta = \frac{d \sin \epsilon}{\cos (\theta + \epsilon) \cos \theta}.$$

113. Sums and Differences Transformed into Products. From

$$\begin{aligned}\sin (A + B) &= \sin A \cos B + \cos A \sin B\\ \sin (A - B) &= \sin A \cos B - \cos A \sin B\\ \cos (A + B) &= \cos A \cos B - \sin A \sin B\\ \cos (A - B) &= \cos A \cos B + \sin A \sin B\end{aligned}$$

we obtain by addition and subtraction

$$\begin{aligned}\sin (A + B) + \sin (A - B) &= 2 \sin A \cos B\\ \sin (A + B) - \sin (A - B) &= 2 \cos A \sin B\\ \cos (A + B) + \cos (A - B) &= 2 \cos A \cos B\\ \cos (A + B) - \cos (A - B) &= - 2 \sin A \sin B.\end{aligned}$$

Let us now put

$$A + B = x \text{ and } A - B = y,$$

from which $A = \frac{1}{2}(x + y), \quad B = \frac{1}{2}(x - y),$

so that the preceding formulas become

$$\left.\begin{aligned} \sin x + \sin y &= 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) \\ \sin x - \sin y &= 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y) \\ \cos x + \cos y &= 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) \\ \cos x - \cos y &= -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y). \end{aligned}\right\} \quad (1)$$

These formulas are frequently used in the further study of mathematics. One of their uses is that they enable us to replace sums or differences by products and thus help us to adapt formulas to computation by logarithms.

EXAMPLE 1. Transform $\sin 2\phi + \sin 4\phi + \sin 6\phi$ into a product.

Solution. By the first formula (1)

$$\begin{aligned} \sin 2\phi + \sin 4\phi &= 2 \sin \tfrac{1}{2}(2\phi + 4\phi) \cos \tfrac{1}{2}(2\phi - 4\phi) \\ &= 2 \sin 3\phi \cos \phi \end{aligned}$$

and by (1), Art. 111,

$$\sin 6\phi = 2 \sin 3\phi \cos 3\phi.$$

Adding

$$\sin 2\phi + \sin 4\phi + \sin 6\phi = 2 \sin 3\phi (\cos 3\phi + \cos \phi).$$

We next apply the third of formula (1) to the expression in parentheses on the right

$$\cos 3\phi + \cos \phi = 2 \cos 2\phi \cos \phi,$$

so that finally

$$\sin 2\phi + \sin 4\phi + \sin 6\phi = 4 \sin 3\phi \cos 2\phi \cos \phi.$$

EXAMPLE 2. If $A + B + C = 180°$, show that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Solution.
$$\begin{aligned} \sin 2A + \sin 2B &= 2 \sin (A + B) \cos (A - B) \\ &= 2 \sin C \cos (A - B), \end{aligned}$$

and
$$\begin{aligned} \sin 2C &= 2 \sin C \cos C \\ &= -2 \sin C \cos (A + B); \end{aligned}$$

hence

$$\sin 2A + \sin 2B + \sin 2C = 2 \sin C\,[\cos (A - B) - \cos (A + B)];$$

but by the fourth of the formulas (1)

$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B,$$

therefore finally

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

EXERCISE 51

1. State in words the theorems embodied in the formulas (1), Article 113.

2. Show that

$$\begin{aligned} \sin (30^\circ + y) + \sin (30^\circ - y) &= \cos y, \\ \sin (30^\circ + y) - \sin (30^\circ - y) &= \sqrt{3} \sin y, \\ \cos (30^\circ + y) + \cos (30^\circ - y) &= \sqrt{3} \cos y, \\ \cos (30^\circ + y) - \cos (30^\circ - y) &= - \sin y. \end{aligned}$$

3. Show that

$$\begin{aligned} \sin 75^\circ + \sin 15^\circ &= \tfrac{1}{2}\sqrt{6}, \\ \sin 75^\circ - \sin 15^\circ &= \tfrac{1}{2}\sqrt{2}, \\ \cos 75^\circ + \cos 15^\circ &= \tfrac{1}{2}\sqrt{6}, \\ \cos 75^\circ - \cos 15^\circ &= -\tfrac{1}{2}\sqrt{2}. \end{aligned}$$

4. Express the following products as sums or differences of two functions,

$$\sin 10^\circ \cos 5^\circ,\ \cos 20^\circ \sin 10^\circ,\ \sin \tfrac{1}{2}\theta \sin \tfrac{1}{3}\theta,$$

$$\cos (\tfrac{1}{4}\pi - \theta) \cos (\tfrac{1}{4}\pi + \theta).$$

Ans. $\tfrac{1}{2}(\sin 15^\circ + \sin 5^\circ)$, ... $\tfrac{1}{2}(\cos \tfrac{1}{2}\pi + \cos 2\theta) = \tfrac{1}{2}\cos 2\theta$.

5. Show that

$$\sin 16^\circ + \sin 14^\circ = 2 \sin 15^\circ \cos 1^\circ,$$

$$\sin \frac{3x}{2} - \sin \frac{x}{2} = 2 \cos x \sin \frac{x}{2},$$

$$\sin (n + 1)x + \sin (n - 1)x = 2 \sin nx \cos x.$$

Prove the following identities:

6. $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

7. $$\frac{\cos A + \cos B}{\cos A - \cos B} = -\cot \tfrac{1}{2}(A + B) \cot \tfrac{1}{2}(A - B).$$

8. $$\frac{\sin A - \sin B}{\cos A - \cos B} = \tan \tfrac{1}{2}(A - B).$$

9. $$\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot \tfrac{1}{2}(A + B).$$

10. Show that

$$\sin(\alpha + x) \sin x = m \cos(\alpha + x) \cos x$$

may be transformed into

$$-\cos(\alpha + 2x) + \cos \alpha = m[\cos(\alpha + 2x) + \cos \alpha].$$

11. Show that

$$\tfrac{1}{2}(\cos 2x + \cos 2y) = \cos(x + y) \cos(x - y),$$
$$-\tfrac{1}{2}(\cos 2x - \cos 2y) = \sin(x + y) \sin(x - y).$$

12. Using the results of 11 show that

$$\cos(x+y) \sin(x-y) + \cos(y+z) \sin(y-z) + \cos(z+x) \sin(z-x) = 0.$$

13. Show that

$$\sin 100^\circ + \sin 40^\circ + \sin 60^\circ = 4 \cos 30^\circ \sin 50^\circ \cos 20^\circ,$$
$$\sin 2\theta + \sin 6\theta + \sin 8\theta = 4 \cos \theta \cos 3\theta \sin 4\theta.$$

14. Show that

$$\sin x + \sin(x - 2) = 2 \sin(x - 1) \cos 1,$$

whence $\sin x = 2 \sin(x - 1) \cos 1 - \sin(x - 2).$

Similarly $\sin x = 2 \cos(x - 1) \sin 1 + \sin(x - 2),$

$$\cos x = 2 \cos(x - 1) \cos 1 - \cos(x - 2),$$
$$\cos x = -2 \sin(x - 1) \sin 1 + \cos(x - 2).$$

These formulas enable us to compute the sine or cosine of x° if the sines and cosines of $(x - 1)^\circ$, $(x - 2)^\circ$, and 1° are known.

15. Assuming the functions of 1° and 2° as known, compute the sines and cosines of 3°, 4° and 5°.

(Suggestion. Apply the results of Problem 14.)

16. By the law of sines

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Taking this proportion by composition and division, we obtain

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

Apply formulas (1), Art. 113, to this result and deduce the law of tangents. (This constitutes an independent *proof of the law of tangents.* This proof is the one generally given in elementary textbooks on trigonometry.)

17. From the law of sines we obtain readily

$$\frac{c}{a-b} = \frac{\sin C}{\sin A - \sin B}, \qquad \frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B},$$

Hence deduce the *double formulas*

$$\frac{c}{a-b} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}, \qquad \frac{c}{a+b} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}.$$

18. In Fig. 83, let A' = angle PAC, B' = angle PBC, then by applying the law of sines to each of the triangles PAC and PBC, we find

$$d_3 = \frac{a \sin B'}{\sin \beta} = \frac{b \sin A'}{\sin \alpha}, \tag{1}$$

from which

$$\frac{\sin A'}{\sin B'} = \frac{a \sin \alpha}{b \sin \beta}.$$

By composition and division and the identity in Problem 6

$$\frac{\sin A' + \sin B'}{\sin A' - \sin B'} = \frac{a \sin \alpha + b \sin \beta}{a \sin \alpha - b \sin \beta} = \frac{\tan \frac{1}{2}(A'+B')}{\tan \frac{1}{2}(A'-B')}. \tag{2}$$

Now $A' + B'$ is known from the relation

$$A' + B' + C + \alpha + \beta = 360°,$$

and therefore (2) enables us to find $A' - B'$. Hence A' and B' may be found and with these known, the law of sines gives

$$d_1 = \frac{b \sin (A' + \alpha)}{\sin \alpha}, \qquad d_2 = \frac{a \sin (B' + \beta)}{\sin \beta},$$

This constitutes another solution of the *three-point problem.*

19. Prove the formulas (1), Article 113, geometrically by means of Fig. 147.

Suggestion. Take the radius of the arc equal to unity, that is, $OE = OD = 1$, then

$$\sin x + \sin y = 2 \cdot GF,$$
$$\sin x - \sin y = 2 \cdot LF,$$
$$\cos x + \cos y = 2 \cdot OG,$$
$$\cos x - \cos y = -2 \cdot LD.$$

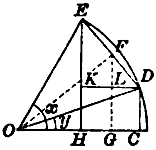

Fig. 147.

20. Show that

$$\frac{\tan(\alpha + x) + \tan x}{\tan(\alpha + x) - \tan x} = \frac{\sin(\alpha + 2x)}{\sin \alpha}.$$

CHAPTER XII

TRIGONOMETRIC EQUATIONS

114. Angles Corresponding to a Given Function. Every given angle has a single sine, cosine, tangent, etc. In Chapter X it was shown how these functions may be found from the tables by first expressing them in terms of an angle in the first quadrant. Suppose now that one of the functions of an angle is known and it is required to find the angle. If the angle is known to be less than R, it is at once found from the tables; if less than $2R$, there also is no uncertainty except in the case of the sine (and its reciprocal the cosecant), which may be either the angle given in the table or its supplement. But in case no restriction is imposed on the magnitude of the angle, the given function may belong to any one of an unlimited number of angles. We shall show how all the angles corresponding to a given function can in each case be expressed by a single formula from which their separate values may be written down when required.

115. Principal Value. Of all the angles which correspond to a given function, the one which has the least numerical value is called the principal value (see Article 81). If there are two least values with opposite signs, the positive angle is taken as the principal value.

Thus, if $\sin\theta = \frac{1}{2}$, $\theta = 30°$ or $150°$ or either of these values increased or diminished by any number of times $360°$. Of all these angles, $30°$ being the least numerically, is considered the principal value of the angles whose sine is $\frac{1}{2}$.

If $\sin\theta = -\frac{1}{2}\sqrt{3}$, $\theta = -60°$, $-120°$ or either of these values increased or diminished by any number of times $360°$. Here $-60°$, having the least numerical value, is the principal value of the angles whose sine is $-\frac{1}{2}\sqrt{3}$.

If $\cos\theta = \frac{1}{2}\sqrt{2}$, $\theta = 45°$ or $-45°$ or either of these increased or diminished by multiples of $360°$. In this case $45°$, not $-45°$, is the principal value of θ.

116. Formula for Angles Having a Given Sine.

Let it be required to determine θ from the equation

$$\sin \theta = k.$$

Let $\alpha =$ the principal value of θ. Then the other values of θ are

$$\begin{array}{cc}
\pi - \alpha, & -\pi - \alpha, \\
2\pi + \alpha, & -2\pi + \alpha, \\
3\pi - \alpha, & -3\pi - \alpha, \\
4\pi + \alpha, & -4\pi + \alpha, \\
5\pi - \alpha, & -5\pi - \alpha, \\
6\pi + \alpha & -6\pi + \alpha \\
\text{etc.} & \text{etc.}
\end{array}$$

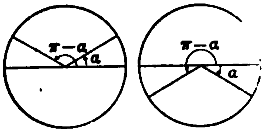

Fig. 148.

It will be observed that the coefficients of π are the integers

$$1, 2, 3, 4, 5, 6, \text{etc.}, \quad -1, -2, -3, -4, -5, -6, \text{etc.},$$

while the sign of α is positive or negative according as the coefficient of π is even or odd. Now $(-1)^n$ is always positive when n is an even integer, negative when n is an odd integer. Making use of this property, we can express the whole set of angles by the single formula

$$\theta = n\pi + (-1)^n \alpha,$$

where α is the principal value of the angles whose sine is k, and n any positive or negative integer.

When $n = 0$, we get the principal value of θ, for

$$0\pi + (-1)^0\alpha = \alpha.$$

117. Formula for Angles Having a Given Cosine.

If $\cos \theta = k$, and α the principal value of θ, the other values of θ are

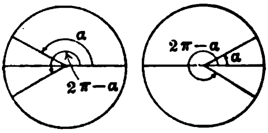

Fig. 149.

$$\begin{array}{cc}
2\pi - \alpha, & -2\pi - \alpha, \\
2\pi + \alpha, & -2\pi + \alpha, \\
4\pi - \alpha, & -4\pi - \alpha, \\
4\pi + \alpha, & -4\pi + \alpha, \\
6\pi - \alpha, & -6\pi - \alpha, \\
6\pi + \alpha, & -6\pi + \alpha, \\
\text{etc.} & \text{etc.}
\end{array}$$

All these values and no others are expressed by the single formula

$$\theta = 2n\pi \pm \alpha,$$

where $2n$ is necessarily an even number.

118. Formula for Angles Having a Given Tangent.

If $\tan\theta = k$, and α the principal value of θ, the other values of θ are

$$\begin{array}{ll} \pi + \alpha, & -\pi + \alpha, \\ 2\pi + \alpha, & -2\pi + \alpha, \\ 3\pi + \alpha, & -3\pi + \alpha, \\ 4\pi + \alpha, & -4\pi + \alpha, \\ 5\pi + \alpha, & -5\pi + \alpha, \\ \text{etc.} & \text{etc.} \end{array}$$

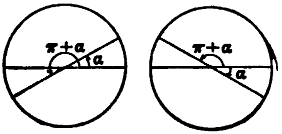

Fig. 150.

All these values and no others are expressed by the single formula.

$$\theta = n\pi + \alpha,$$

n being any positive or negative integer.

119. Summary of Results. Summing up the results of the last three articles we have

$$\left.\begin{aligned} \sin(n\pi + (-1)^n\alpha) &= \sin\alpha, \\ \cos(2n\pi \pm \alpha) &= \cos\alpha, \\ \tan(n\pi + \alpha) &= \tan\alpha, \end{aligned}\right\} \qquad (1)$$

where α is the principal value of the angle, and n any positive or negative integer.

These three formulas should be thoroughly understood and memorized by the student.

120. Trigonometric Equations Involving a Single Angle.

Equations involving one or more trigonometric functions of one or more unknown angles are called trigonometric equations. They may or may not involve other unknowns which are not angles.

To solve trigonometric equations involving a single unknown angle, we express each of the functions which occur in the equation in terms of some one of them, and solve the resulting equation algebraically, considering this function as the unknown. The corresponding angles may then be easily found from the formulas of Article 119.

EXAMPLE 1. Solve the equation $2 \sin x + \csc x = 3$.

Solution. Expressing the cosecant in terms of the sine, we have

$$2 \sin x + \frac{1}{\sin x} = 3.$$

Now consider $\sin x$ the unknown, call it s say, we then have

$$2s + \frac{1}{s} = 3.$$

Clearing of fractions and transposing,

$$2s^2 - 3s - 1 = 0.$$

Solving $$s = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2}}{4} = \frac{3 \pm 1}{4} = 1 \quad \text{or} \quad \tfrac{1}{2},$$

that is, $$\sin x = 1 \quad \text{or} \quad \tfrac{1}{2},$$

from which the principal values of x are $x = \frac{\pi}{2}$ or $\frac{\pi}{6}$,

and by Article 119 the general values of x are

$$x = n\pi + (-1)^n \frac{\pi}{2}, \quad \text{or} \quad n\pi + (-1)^n \frac{\pi}{6}.$$

The general values of x just written are equivalent to the two sets of values

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \text{etc.}, \frac{-3\pi}{2}, \frac{-7\pi}{2}, \frac{-11\pi}{2}, \text{etc.},$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{etc.}, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \frac{-19\pi}{6}, \text{etc.},$$

Check. If $\sin x = 1$, $\csc x = 1$, and $2 \sin x + \csc x = 2 . 1 + 1 = 3$.
If $\sin x = \frac{1}{2}$, $\csc x = 2$, and $2 \sin x + \csc x = 2 . \frac{1}{2} + 2 = 3$.

EXAMPLE 2. To find θ in the equation $\tan \theta + 3 \cot \theta = 4$.

Solution. Expressing the cotangent in terms of the tangent gives

$$t + \frac{3}{t} = 4,$$

where $t = \tan\theta$. Solving for t,

$$t^2 - 4t + 3 = 0,$$

$$t = \frac{4 \pm \sqrt{4^2 - 4 \cdot 3}}{2} = \frac{4 \pm 2}{2} = 3 \text{ or } 1,$$

that is, $\tan\theta = 3$ or 1,

from which the principal values of θ are

$$\theta = \tan^{-1} 3 = 71^\circ\, 33'\, 54'' \text{ or } \tan^{-1} 1 = 45^\circ.$$

By Article 119 the general values of θ are

$$\theta = n \cdot 180^\circ + 71^\circ\, 33'\, 54'' \text{ or } n \cdot 180^\circ + 45^\circ.$$

Check. If $\tan\theta = 3$, $\cot\theta = \frac{1}{3}$, $\tan\theta + 3\cot\theta = 3 + 3 \cdot \frac{1}{3} = 4.$
If $\tan\theta = 1$, $\cot\theta = 1$, $\tan\theta + 3\cot\theta = 1 + 3 \cdot 1 = 4.$

EXAMPLE 3. Find ϕ in $\tan\phi + 2\sqrt{3}\cos\phi = 0$.

Solution. $$\tan\phi = \frac{\sqrt{1 - \cos^2\phi}}{\cos\phi},$$

hence $$\sqrt{1 - \cos^2\phi} + 2\sqrt{3}\cos^2\phi = 0.$$

Transposing one term and squaring,

$$1 - \cos^2\phi = 12\cos^4\phi,$$

$$12\cos^4\phi + \cos^2\phi - 1 = 0,$$

$$\cos^2\phi = \frac{-1 \pm \sqrt{1 + 4 . 12}}{24} = \frac{-1 \pm 7}{24} = \frac{1}{4} \quad\text{or}\quad \frac{-1}{3},$$

$$\cos\phi = \tfrac{1}{2},\quad -\tfrac{1}{2},\quad \sqrt{-\tfrac{1}{3}},\quad -\sqrt{-\tfrac{1}{3}}.$$

The last two values of $\cos\phi$ are not admissible, since the cosine of a real angle cannot be imaginary, and the first two values each give

$$\phi = 2n\pi \pm \frac{\pi}{3}.$$

Check. $\cos\left(2n\pi \pm \frac{\pi}{3}\right) = \frac{1}{2}$, $\tan\left(2n\pi \pm \frac{\pi}{3}\right) = \pm\sqrt{3}$,

$\tan\phi + 2\sqrt{3}\cos\phi = \pm 3 + 2\sqrt{3 \cdot (\frac{1}{2})} = 0$ for the lower sign but not for the upper. Hence the general solution of the equation

$$\tan\phi + 2\sqrt{3}\cos\phi = 0 \quad\text{is}\quad \phi = 2n\pi - \frac{\pi}{3}.$$

This example clearly shows the necessity of verifying the results before accepting them as the solutions of a given equation.

EXAMPLE 4. Solve the equation $\sin x + \cos x = \sqrt{2}$.

First solution. $\sin x + \sqrt{1 - \sin^2 x} = \sqrt{2}$,

$$\sqrt{1 - \sin^2 x} = \sqrt{2} - \sin x.$$

Squaring $$1 - \sin^2 x = 2 - 2\sqrt{2}\sin x + \sin^2 x.$$

Solving $$2\sin^2 x - 2\sqrt{2}\sin x + 1 = 0,$$

$$\sin x = \frac{2\sqrt{2} \pm \sqrt{8 - 4 \cdot 2}}{4} = \frac{\sqrt{2}}{2},$$

$$x = n\pi + (-1)^n \frac{\pi}{4}.$$

Check. $\sin\left(n\pi + (-1)^n \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}$, $\cos\left(n\pi + (-1)^n \frac{\pi}{4}\right) = \pm\frac{1}{2}\sqrt{2}$,

according as n is even or odd. Therefore

$$\sin x + \cos x = \tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}\sqrt{2} = \sqrt{2} \quad \text{or} \quad 0$$

according as n is even or odd. We see that the lower sign does not satisfy the original equation, that is n cannot be odd; hence the general solution of the equation

$$\sin x + \cos x = \sqrt{2}$$

is $$x = 2n\pi + \tfrac{1}{4}\pi.$$

Second solution. We have given the solution which would most naturally suggest itself to the beginner. A more elegant solution of the foregoing equation is the following:

Since $\cos x = \sin(\frac{1}{2}\pi - x)$, the equation to be solved may be written

$$\sin x + \cos x = \sin x + \sin(\tfrac{1}{2}\pi - x) = \sqrt{2}.$$

The middle member is the sum of two sines, which by Article 113 may be transformed into a product, thus

$$\sin x + \sin(\tfrac{1}{2}\pi - x) = 2\sin\tfrac{1}{4}\pi\cos(x - \tfrac{1}{4}\pi) = \sqrt{2},$$

that is $$2 \cdot \tfrac{1}{2}\sqrt{2} \cdot \cos(x - \tfrac{1}{4}\pi) = \sqrt{2},$$

or $$\cos(x - \tfrac{1}{4}\pi) = 1,$$

from which $$x - \tfrac{1}{4}\pi = 2n \pm 0, \text{or } x = 2n + \tfrac{1}{4}\pi.$$

Example 5. Solve the equation $a \sin(x + \alpha) + b \cos(x + \beta) = 0$, a, b, α, β, being known constants.

Solution. $\sin(x + \alpha) = \sin x \cos \alpha + \cos x \sin \alpha,$
$\cos(x + \beta) = \cos x \cos \beta - \sin x \sin \beta.$

Substituting these values in the original equation,

$$a(\sin x \cos \alpha + \cos x \sin \alpha) + b(\cos x \cos \beta - \sin x \sin \beta) = 0.$$

Collecting the coefficients of $\sin x$ and $\cos x$ separately, we have

$$(a \cos \alpha - b \sin \beta) \sin x + (a \sin \alpha + b \cos \beta) \cos x = 0.$$

Dividing by $\cos x$, and solving for $\tan x$,

$$\tan x = -\frac{a \sin \alpha + b \cos \beta}{a \cos \alpha - b \sin \beta},$$

from which

$$x = n\pi - \tan^{-1}\left(\frac{a \sin \alpha + b \cos \beta}{a \cos \alpha - b \sin \beta}\right).$$

Check. To check complicated results like the one just obtained it is best to assume arbitrary values for the constants. Thus if

$$a = 2,\ b = 1,\ \alpha = 20^\circ,\ \beta = 15^\circ,\ \frac{a \sin \alpha + b \cos \beta}{a \cos \alpha - b \sin \beta} = 1.0181,$$

$$x = n\pi - \tan^{-1} 1.0181 = n\pi - 45^\circ 31',$$

$$x + \alpha = n\pi - 25^\circ 31', \quad x + \beta = n\pi - 30^\circ 31'.$$

Substituting these values in the original equation, we have

$$2(\pm 0.4308) + 1(\mp 0.8615) = 0.$$

Exercise 52

In the following find both the principal and the general value of the angle:

1. $\tan \theta = 2 \sin \theta.$

Ans. $\theta = n\pi$ or $2n\pi \pm \frac{\pi}{3}$; principal values 0 and $\frac{\pi}{3}$.

2. $3 \sin^2 x = \cos^2 x.$

Ans. $x = n\pi \pm \frac{\pi}{6}$; principal value $\frac{\pi}{6}$.

3. $3 \tan^2 x - 4 \sin^2 x = 1.$

Ans. $x = n\pi \pm \frac{\pi}{4}$; principal value $\frac{\pi}{4}$.

4. $2\sin^2\phi = 3\cos\phi$.

Ans. $\phi = 2n\pi \pm \frac{\pi}{3}$; principal value $\frac{\pi}{3}$.

5. $\tan y + \cot y = 2$. *Ans.* $y = \pi n + \frac{1}{4}\pi$; principal value $\frac{\pi}{4}$.

In the following problems find those values of the variable angle which are less than 360°:

6. $\tan^2 x + \csc^2 x = 3$. *Ans.* $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

7. $\sin\theta + \cos\theta = 1$. *Ans.* $\theta = 0, \frac{1}{2}\pi$.

8. $\csc x \cot x = 2\sqrt{3}$. *Ans.* $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

9. $\sin^2 t - 2\cos t + \frac{1}{4} = 0$. *Ans.* $t = \frac{\pi}{3}, \frac{5\pi}{3}$.

10. $3\sec^4\theta - 10\sec^2\theta + 8 = 0$.

Ans. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$.

11. $\tan x + \sec^2 x = 7$.

Ans. $x = 63^\circ 26' 04'', 108^\circ 26' 06'', 243^\circ 26' 04'', 288^\circ 26' 06''$.

12. $6\cos^2 x + 5\sin x = 7$.

Ans. $x = 19^\circ 28' 16'', 30^\circ, 150^\circ, 160^\circ 31' 44''$.

13. $\sin x + \csc x = \frac{5}{2}$. *Ans.* $x = 30^\circ, 150^\circ$.

14. $\sin x - \cos x = \frac{3}{2}$. *Ans.* No solution.

15. If $a\sin x + b\cos x = c$, show that $\sin x = \frac{ac \pm b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}$.

16. If $a\tan x + b\cot x = c$, show that $\tan x = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}$.

17. If $\sin(\alpha + x) = m\sin x$, show that $\sin x = \frac{\pm\sin\alpha}{\sqrt{m^2 - 2m\cos\alpha + 1}}$,

or more simply $\cot x = \frac{m - \cos\alpha}{\sin\alpha}$.

18. If $\tan(\alpha + x) = m\tan x$, show that

$$\tan x = \frac{(m-1) \pm \sqrt{(m-1)^2 - 4m\tan^2\alpha}}{2m\tan\alpha}.$$

19. If $\tan(\alpha + x)\tan x = m$, show that

$$\tan x = \frac{-(1+m)\tan\alpha \pm \sqrt{(1+m)^2\tan^2\alpha + 4m}}{2}.$$

By expressing each product as a sum or difference of functions show:

20. If $\sin(\alpha \pm x)\sin x = m$, then $\cos(\alpha \pm 2x) = \cos\alpha \mp 2m$.

21. If $\cos(\alpha \pm x)\cos x = m$, then $\cos(\alpha \pm 2x) = 2m - \cos\alpha$.

22. If $\sin(\alpha \pm x)\cos x = m$, then $\sin(\alpha \pm 2x) = 2m - \sin\alpha$.

23. If $\cos(\alpha \pm x)\sin x = m$, then $\sin(\alpha \pm 2x) = \sin\alpha \pm 2m$.

121. Trigonometric Equations Involving Multiple Angles.

When an equation involves multiple angles, as the equation

$$\cos 3\theta + \sin 2\theta = a, \quad \text{or} \quad \tan 2x = \cot 5x,$$

it can frequently be solved by two or more different methods, and the answer will appear in various forms according as one or the other of these methods has been employed in the solution. Generally the different forms of the answers can easily be identified, but in some cases considerable ingenuity is required to show that the different forms are really the same. Thus it is easy to see that $2n\pi \pm \frac{1}{2}\pi$ and $n\pi + \frac{1}{2}\pi$ (n being any integer, but not the same integer in both forms) express the same general value, but it is not so easy to see that

$$\theta = n\pi + \tfrac{1}{2}\pi \quad \text{or} \quad n\pi + (-1)^n \sin^{-1}\tfrac{1}{4}(-1 \pm \sqrt{5})$$

$$\text{and } \theta = 2n\pi - \tfrac{1}{2}\pi \quad \text{or} \quad \frac{2n\pi}{5} + \frac{\pi}{10}$$

are equivalent results.

Example 1. Solve the equation $\sin 2\theta = \cos\theta$.

First solution. Substituting for $\sin 2\theta$ its value in terms of the single angle θ (Article 111, (1)), we have

$$2\sin\theta\cos\theta = \cos\theta.$$

Transposing and factoring,

$$\cos\theta(2\sin\theta - 1) = 0,$$

from which

$$\cos\theta = 0, \quad \text{or} \quad \sin\theta = \tfrac{1}{2}.$$

Therefore

$$\theta = 2n\pi \pm \tfrac{1}{2}\pi = n\pi + \tfrac{1}{2}\pi^*, \text{ or } n\pi + (-1)^n\frac{\pi}{6}. \qquad (1)$$

---

[*] $\theta = 2n\pi + \frac{1}{2}\pi$ and $2n\pi - \frac{1}{2}\pi = (2n - 1)\pi + \frac{1}{2}\pi$. Now $2n$ represents every even and $2n - 1$ every odd number, hence we may write $\theta = n\pi + \frac{1}{2}\pi$, where n is any integer, even or odd.

Check.

$$\sin 2\theta = \sin(2n\pi + \pi) = 0, \text{ or } \sin\left(2n\pi + (-1)^n \frac{\pi}{3}\right) = \pm \tfrac{1}{2}\sqrt{3},$$

according as n is even or odd.

$$\cos\theta = \cos(n\pi + \tfrac{1}{2}\pi) = 0, \text{ or } \cos\left(n\pi + (-1)^n \frac{\pi}{6}\right) = \pm \tfrac{1}{2}\sqrt{3},$$

according as n is even or odd.

Substituting these values in the original equation, we have

$$0 = 0, \text{ or } \pm \tfrac{1}{2}\sqrt{3} = \pm \tfrac{1}{2}\sqrt{3}.$$

Second solution.

Since $\cos\theta = \sin(\tfrac{1}{2}\pi - \theta)$, we have

$$\sin 2\theta = \sin(\tfrac{1}{2}\pi - \theta),$$

from which

$$2\theta = n\pi + (-1)^n(\tfrac{1}{2}\pi - \theta),$$

or

$$\theta = \frac{n\pi + (-1)^n \tfrac{1}{2}\pi}{2 + (-1)^n}. \qquad (2)$$

Third solution. Transposing the second member of the equation, we have

$$\sin 2\theta - \cos\theta = \sin 2\theta - \sin(\tfrac{1}{2}\pi - \theta) = 0.$$

By Article 113, the difference of two sines may be tranformed into a product of a sine and cosine, thus

$$\sin 2\theta - \sin(\tfrac{1}{2}\pi - \theta) = 2\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)\sin(\tfrac{3}{2}\theta - \tfrac{1}{4}\pi) = 0,$$

from which

$$\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) = 0, \text{ or } \sin(\tfrac{3}{2}\theta - \tfrac{1}{4}\pi) = 0,$$

and

$$\tfrac{1}{2}\theta + \tfrac{1}{4}\pi = 2n\pi \pm \tfrac{1}{2}\pi = n\pi + \tfrac{1}{2}\pi, \text{ or } \tfrac{3}{2}\theta - \tfrac{1}{4}\pi = n\pi,$$

that is,

$$\theta = 2n\pi + \tfrac{1}{2}\pi, \text{ or } \tfrac{2}{3}n\pi + \tfrac{1}{6}\pi. \qquad (3)$$

Identification of results.

If in (2) n is odd, say $2m + 1$, we have

$$\theta = (2m + 1)\pi - \tfrac{1}{2}\pi = 2m\pi + \tfrac{1}{2}\pi,$$

and if n is even, say $2m$, we have

$$\theta = \frac{2m\pi + \tfrac{1}{2}\pi}{3} = \tfrac{2}{3}m\pi + \tfrac{1}{6}\pi.$$

This shows that the results (2) and (3) are the same.

Next consider the second value in (3). Every integer n is either some multiple of 3, say $3m$, or some multiple of 3 increased by unity, say $3m + 1$, or some multiple of 3 increased by 2, say $3m + 2$.

In the first case, $n = 3m$, we have from the second value of (3)

$$\frac{2n\pi}{3} + \frac{\pi}{6} = \frac{2.3m\pi}{3} + \frac{\pi}{6} = 2m\pi + \frac{\pi}{6}.$$

In the second case, $n = 3m + 1$,

$$\frac{2n\pi}{3} + \frac{\pi}{6} = \frac{2(3m+1)\pi}{3} + \frac{\pi}{6} = (2m+1)\pi - \frac{\pi}{6}.$$

In the third case, $n = 3m + 2$,

$$\frac{2n\pi}{3} + \frac{\pi}{6} = \frac{2(3m+2)\pi}{3} + \frac{\pi}{6} = (2m+1)\pi + \frac{\pi}{2}.$$

This last value, combined with the first value of θ in (3), gives $\theta = 2n\pi + \frac{1}{2}\pi$, or $(2m+1)\pi + \frac{1}{2}\pi$, that is, $\theta = n\pi + \frac{1}{2}\pi$ (n even or odd), and

$\theta = 2m\pi + \frac{1}{6}\pi$, or $(2m+1)\pi - \frac{1}{6}\pi$, may be written

$$\theta = n\pi + (-1)^n\frac{\pi}{6}.$$

This shows that the results (3) and (1) are the same.

Exercise 53

Find the general value of the angle in each of the following equations:

1. $\cos 2x + \cos x + 1 = 0$. *Ans.* $x = n\pi - \frac{1}{2}\pi$, or $2n\pi \pm \frac{2}{3}\pi$.
2. $\cos 5x = \sin 4x$. *Ans.* $x = 2n\pi - \frac{1}{2}\pi$, or $\frac{2n\pi}{9} + \frac{\pi}{18}$.
3. $\cos 5x = \cos 4x$. *Ans.* $x = 2n\pi$, or $\frac{2n\pi}{9}$.
4. $\sin 4x = \sin 5x$. *Ans.* $x = 2n\pi$, or $(2n+1)\frac{\pi}{9}$.
5. $\tan 5\theta = \cot 2\theta$. *Ans.* $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$.
6. $\sin 3x + \sin 2x + \sin x = 0$.
   Ans. $x = 2n\pi$, or $2n\pi \pm \frac{1}{2}\pi$, or $2n\pi \pm \frac{2\pi}{3}$.

7. $\cos x - \cos 3x = \sin 2x$.

$$\textit{Ans.}\ x = n\pi,\ \text{or}\ n\pi - \tfrac{1}{2}\pi,\ \text{or}\ n\pi + (-1)^n\frac{\pi}{6}.$$

8. $\cos 5y - \cos 3y + \sin y = 0$.

$$\textit{Ans.}\ y = n\pi,\ \text{or}\ \tfrac{1}{2}n\pi + (-1)^n\frac{\pi}{24}.$$

9. $\cos(60° - x) + \cos(60° + x) = \frac{1}{3}$.

$$\textit{Ans.}\ x = 2n\pi \pm \cos^{-1}\tfrac{1}{3} = 2n\,180° \pm 70°\,32'.$$

10. $\cos mx = \sin kx$, m and k being known.

$$\textit{Ans.}\ x = \frac{(n\pi + \frac{1}{2}\pi)}{(m \pm k)}.$$

11. $\cot\phi = \tan k\phi$, k being known.

$$\textit{Ans.}\ \phi = \frac{n\pi}{k+1} + \frac{\pi}{2(k+1)}.$$

12. $\tan 2x \tan x = 1$. *Ans.* $x = n\pi \pm \frac{\pi}{6}$.

Solve each of the following problems by each of three methods, and identify the results.

13. $\cos 2\theta = \sin\theta$. *Ans.* $\theta = 2n\pi - \frac{1}{2}\pi$, or $\frac{2}{3}n\pi + \frac{\pi}{6}$.

14. $\cos 3x = \sin 2x$. *Ans.* $x = 2n\pi - \frac{1}{2}\pi$, or $\frac{2}{5}n\pi + \frac{\pi}{10}$.

Suggestion. $\frac{1}{4}(-1 + \sqrt{5}) = \sin\frac{\pi}{10}$, $\frac{1}{4}(-1 - \sqrt{5}) = \sin\left(-\frac{3\pi}{10}\right)$.

15. $\sin 3x = \cos 2x$. *Ans.* $x = 2n\pi + \frac{1}{2}\pi$, or $\frac{2}{5}n\pi + \frac{\pi}{10}$.

122. Trigonometric Equations Involving Two or More Variables. When there are given two or more trigonometric equations, involving two or more variables, the solution, if it is possible at all, generally depends upon more or less ingenious combinations of the equations, for which no definite rules can be given. We shall illustrate the various methods and devices commonly employed by a few examples chosen from among those which most frequently occur in applied trigonometry.

EXAMPLE 1. To find r and θ from the equations

$$r \sin \theta = a, \qquad (1)$$

$$r \cos \theta = b, \qquad (2)$$

a and b being known constants.

First solution. Dividing the first equation by the second, we have

$$\tan \theta = \frac{a}{b}$$

from which θ may be found. r is then found from either (1) or (2),

$$r = \frac{a}{\sin \theta} = \frac{b}{\cos \theta}.$$

Since the angles θ as determined from the tangent differ by multiples of π, $\sin \theta$ and $\cos \theta$ will have two values each which are numerically equal but opposite in sign. Therefore r will have two values which are equal and opposite in sign. If from the outset it is known that r can be positive only, which is the case frequently, then $\sin \theta$ must have the same sign as a and $\cos \theta$ the same sign as b. θ must then be limited to the quadrants determined by these signs.

Second solution. Squaring each equation and adding the results gives

$$r^2 = a^2 + b^2,$$

from which r is obtained. Then θ is found from either of the equations

$$\sin \theta = \frac{a}{r}, \quad \cos \theta = \frac{b}{r}$$

Of the two methods the first is better adapted to the use of logarithms than the second.

EXAMPLE 2. Find r, θ and ϕ from the equations

$$r \cos \theta \cos \phi = a \qquad (1)$$

$$r \cos \theta \sin \phi = b \qquad (2)$$

$$r \sin \theta = c. \qquad (3)$$

First solution. Dividing the second equation by the first we obtain

$$\tan \phi = \frac{b}{a}$$

from which ϕ is obtained. ϕ being known, $r\cos\theta$ is obtained from either equation (1) or (2)

$$r\cos\theta = \frac{a}{\cos\phi} = \frac{b}{\sin\phi}. \qquad (4)$$

From (3) and (4) r and θ may be found as in Example 1.

Second solution. Squaring each of the equations (1), (2) and (3) and adding the results gives

$$r^2 = a^2 + b^2 + c^2$$

from which r may be found.

r being known, θ is found from the equation (3), and, r and θ being known, ϕ is found from either equation (1) or (2).

The first solution is preferable if logarithms are to be used throughout.

EXAMPLE 3. Solve the equations

$$r\sin(\alpha+\theta) = a$$
$$r\sin(\beta+\theta) = b$$

for r and θ.

Solution. Applying the addition theorem for the sine, we have

$$r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = a$$
$$r\sin\beta\cos\theta + r\cos\beta\sin\theta = b.$$

Put $r\cos\theta = x$, $r\sin\theta = y$, the equations then become

$$x\sin\alpha + y\cos\alpha = a$$
$$x\sin\beta + y\cos\beta = b.$$

Solving these equations for x and y, we obtain

$$x = r\cos\theta = \frac{a\cos\beta - b\cos\alpha}{\sin\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{a\cos\beta - b\cos\alpha}{\sin(\alpha-\beta)}.$$

$$y = r\sin\theta = \frac{a\sin\beta - b\sin\alpha}{\cos\alpha\sin\beta - \sin\alpha\cos\beta} = \frac{b\sin\alpha - a\sin\beta}{\sin(\alpha-\beta)}.$$

x and y, that is, $r\cos\theta$ and $r\sin\theta$ being known, r and θ are found as in Example 1.

EXERCISE 54

In the following equations, consider r positive.

1. Given $r\sin\theta = 8.219$, $r\cos\theta = 12.88$, find r and θ.

Ans. $r = 15.28$, $\theta = 32° 33'$.

2. Given $r\sin\theta = 3$, $r\cos\theta = 4$, find r and θ.

3. Given $r \sin\theta = 27.138$, $r\cos\theta = -92.692$, find r and θ.

Ans. $r = 96.583$, $\theta = 163^\circ\,40'\,52''$.

4. Given
$$\left.\begin{aligned} r\cos\theta\cos\phi &= 59.953, \\ r\cos\theta\sin\phi &= 197.207, \\ r\sin\phi &= 39.062. \end{aligned}\right\} \text{ Find } r,\ \theta \text{ and } \phi.$$

Ans. $r = 208.16$, $\theta = 10^\circ\,49'\,00''$, $\phi = 74^\circ\,42'\,00''$.

5. Given
$$\left.\begin{aligned} r\sin\theta\cos\phi &= 5, \\ r\cos\theta\sin\phi &= 12, \\ r\sin\theta &= 84. \end{aligned}\right\} \text{ Find } r,\ \theta \text{ and } \phi.$$

Ans. By natural functions, $r = 85$, $\theta = 81^\circ\,12'$, $\phi = 86^\circ\,36'$.

6. Find r, λ and μ from the equations

$$r\cos\lambda\cos\mu = 4, \quad r\cos\lambda\sin\mu = 5, \quad r\sin\lambda = \sqrt{59}.$$

7. $r\sin\left(\frac{\pi}{3}+x\right) = \sqrt{3}$, $r\sin\left(\frac{\pi}{6}+x\right) = 1$. Find r and x.

Ans. $r = 2$, $x = 0^\circ$.

8. Show how to solve the equations

$$r\cos(x-\alpha) = a,\ r\sin(x+\beta) = b, \text{ for } r \text{ and } x.$$

9. Solve the equations

$$\cos(x-y) = \tfrac{1}{2}\sqrt{2},\ \sin(x+y) = \tfrac{1}{2}\sqrt{3},$$

giving both the principal and general values.

Ans. Principal values, $x = \frac{7\pi}{24}$ or $\frac{\pi}{24}$, $y = \frac{\pi}{24}$ or $\frac{7\pi}{24}$.

General values, $x = (m+2n)\frac{\pi}{2} + (-1)^m\frac{\pi}{6} \pm \frac{\pi}{8}$,

$y = (m-2n)\frac{\pi}{2} + (-1)^m\frac{\pi}{6} \mp \frac{\pi}{8}$,

where m and n are any two integers.

10. Solve the equations

$$\cos 2x - \cos 2y = a, \qquad \cos x - \cos y = b.$$

Ans. $x = 2n\pi \pm \cos^{-1}\left(\frac{a+2b^2}{4b}\right)$, $y = 2n\pi \pm \cos^{-1}\left(\frac{a-2b^2}{4b}\right)$.

11. Show how to solve the equations

$$\tan(x+y) = a, \qquad \tan x \cdot \tan y = b.$$

123. Solutions Adapted to Logarithmic Computation. The solution of a problem in trigonometry is not considered completed until it can be effected by the use of logarithms, in fact the adaptation of formulas to the use of logarithms forms an important part in trigonometric investigations. Many trigonometric equations whose algebraic solution is exceedingly simple require further treatment from the trigonometric point of view. Equations 15 to 19, Exercise 52, are typical equations of this kind. We will show now how each of these equations may be solved by logarithms.

EXAMPLE 1. To solve the equation

$$a \sin x + b \cos x = c. \qquad (1)$$

Solution. Divide each term of the equation by $\sqrt{a^2 + b^2}$, then the coefficients $\dfrac{a}{\sqrt{a^2+b^2}}$ and $\dfrac{b}{\sqrt{a^2+b^2}}$ are fractions the sum of whose squares equals 1, we may therefore put

$$\frac{a}{\sqrt{a^2+b^2}} = \cos\phi, \qquad \frac{b}{\sqrt{a^2+b^2}} = \sin\phi, \qquad (2)$$

and the given equation becomes

$$\sin x \cos\phi + \cos x \sin\phi = \frac{c}{\sqrt{a^2+b^2}},$$

or

$$\sin(x+\phi) = \frac{c}{\sqrt{a^2+b^2}} = \frac{c}{a}\cos\phi = \frac{c}{b}\sin\phi, \qquad (3)$$

and from (2)

$$\tan\phi = \frac{b}{a}. \qquad (4)$$

Having found ϕ from (4), x is found from (3).

An angle like the angle ϕ, introduced to facilitate the solution of a problem, is called an *auxiliary* angle.

NUMERICAL ILLUSTRATION

Suppose the given equation is

$$3.4537 \sin x - 0.9328 \cos x = -1.3794,$$

then $\quad a = 3.4537, \quad b = -0.9328, \quad c = -1.3794.$

Solution.

$$\begin{array}{rlrl} & \text{By (4)} & & \text{By (3)} \\ \log b &= 9.96979\,n\ ^{*} & \log c &= 0.13969\,n \\ \log a &= \underline{0.53828} & \operatorname{colog} a &= 9.46172 \\ \log\ \tan\phi &= 9.43151\,n & \log\cos\phi &= \underline{9.98471} \\ \phi^{\dagger} &= -\ 15^\circ\ 06'\ 52'', & \log\sin(x+\phi) &= 9.58612\,n \end{array}$$

$$x+\phi = n\pi + (-1)^n\ (22^\circ\ 40'\ 48'')$$

$$x = n\pi + 15^\circ\ 06'\ 52'' + (-1)^{n+1}\ (22^\circ\ 40'\ 48'')$$

The two smallest positive values, ($n = 1, 2$), are

$$x = 217^\circ\ 47'\ 40'',\ 352^\circ\ 26'\ 04''.$$

Check. By (3)

$$\begin{aligned} \log c &= 0.13969\,n \\ \operatorname{colog} b &= 0.03021\,n \\ \log\sin\phi &= \underline{9.41621\,n} \\ \log\sin(x+\phi) &= 9.58611\,n \end{aligned}$$

Example 2. To solve the equation

$$a\tan x + b\cot x = c.$$

Solution. Expressing the tangent and cotangent each in terms of the sine and cosine, the equation reduces to (Problem 19, Exercise 50)

$$c\sin 2x + (a-b)\cos 2x = a+b.$$

This equation is of the form

$$a\sin x + b\cos x = c,$$

which has been solved in Example 1.

Example 3. Solve the equation

$$\sin(\alpha + x) = m\sin x. \qquad (1)$$

Solution. By composition and division the proportion

$$\frac{\sin(\alpha+x)}{\sin x} = \frac{m}{1}$$

gives rise to

$$\frac{\sin(\alpha+x)+\sin x}{\sin(\alpha+x)-\sin x} = \frac{2\sin(\alpha+\frac{1}{2}x)\cos\frac{1}{2}\alpha}{2\cos(\alpha+\frac{1}{2}x)\sin\frac{1}{2}\alpha}$$

$$=\tan(\alpha+\tfrac{1}{2}x)\cot\tfrac{1}{2}\alpha = \frac{m+1}{m-1},$$

---

[*] This n indicates that the number to which the logarithm belongs is negative.

[†] Since ϕ is an auxiliary angle which is not retained in the end, any one of its values may be used.

or
$$\tan(\alpha+\tfrac{1}{2}x)=\frac{m+1}{m-1}\tan\tfrac{1}{2}\alpha\cdot \tag{2}$$

From (2) $\alpha+\frac{1}{2}x$ and hence x can be found.

If $m=\tan\phi$

$$\frac{m+1}{m-1}=-\frac{\tan\phi+\tan\frac{1}{4}\pi}{1-\tan\frac{1}{4}\pi\tan\phi}=-\tan(\phi+\tfrac{1}{4}\pi)=\cot(\phi-\tfrac{1}{4}\pi),$$

hence

$$\tan(\alpha+\tfrac{1}{2}x)=\cot(\phi-\tfrac{1}{4}\pi)\tan\tfrac{1}{2}\alpha,\text{ where }\tan\phi=m. \tag{3}$$

Many computers prefer (3) to (2) in solving equation (1).

Example 4. Solve the equation

$$\tan(\alpha+x)=m\tan x.$$

Solution. Taking the proportion

$$\frac{\tan(\alpha+x)}{\tan x}=\frac{m}{1}$$

by composition and division, we obtain

$$\frac{\tan(\alpha+x)+\tan x}{\tan(\alpha+x)-\tan x}=\frac{m+1}{m-1}.$$

But by Problem 20, Exercise 51,

$$\frac{\tan(\alpha+x)+\tan x}{\tan(\alpha+x)-\tan x}=\frac{\sin(\alpha+2x)}{\sin\alpha},$$

hence

$$\sin(\alpha+2x)=\frac{m+1}{m-1}\sin\alpha, \tag{1}$$

or we may write the result in the form

$$\sin(\alpha+2x)=\cot(\phi-\tfrac{1}{4}\pi)\sin\alpha,\text{ where }\tan\phi=m. \tag{2}$$

Either (1) or (2) may be used to find $\alpha+2x$ and hence x.

Example 5. Solve the equation

$$\tan(\alpha+x)\tan x=m.$$

Solution. Expressing the tangents in terms of sines and cosines, we find

$$\sin(\alpha+x)\sin x=m\cos(\alpha+x)\cos x,$$

which by Problem 10, Exercise 51, may be written

$$\cos\alpha-\cos(\alpha+2x)=m[\cos(\alpha+2x)+\cos\alpha],$$

from which

$$\cos(\alpha + 2x) = \frac{1-m}{1+m}\cos\alpha. \qquad (1)$$

By Example 3, $\dfrac{m+1}{m-1} = \cot(\phi - \tfrac{1}{4}\pi)$, where $\tan\phi = m$,

hence $\dfrac{m-1}{m+1} = \tan(\phi - \tfrac{1}{4}\pi)$,

and $\dfrac{1-m}{1+m} = -\tan(\phi - \tfrac{1}{4}\pi) = \tan(\tfrac{1}{4}\pi - \phi)$,

so that equation (1) may be written

$$\cos(\alpha + 2x) = \tan(\tfrac{1}{4}\pi - \phi)\cos\alpha, \text{ where } \tan\phi = m. \qquad (2)$$

From (1) or from (2) $\alpha + 2x$ may be found and hence x.

EXERCISE 55

1. If $\cos(x+\alpha) = m\cos x$, show that x is given by the formula
$\tan(\alpha + \tfrac{1}{2}x) = \tan(\tfrac{1}{4}\pi - \phi)\cot\tfrac{1}{2}\alpha$, where $\tan\phi = m$.

2. If $\sin(x+\alpha) = m\cos x$, show that x is given by the formula
$\tan(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi + x) = \cot(\tfrac{1}{4}\pi - \phi)\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha)$, where $\tan\phi = m$.

3. If $\cos(x+\alpha) = m\sin x$, show that x is given by the formula
$\tan(\tfrac{1}{2}\alpha - \tfrac{1}{4}\pi + x) = \tan(\tfrac{1}{4}\pi - \phi)\cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha)$, where $\tan\phi = m$.

4. If $\tan(x+\alpha) = m\cot x$, show that
$\cos(\alpha + 2x) = \tan(\tfrac{1}{4}\pi - \phi)\cos\alpha$, where $\tan\phi = m$.

5. If $\cot(x+\alpha) = m\cot(x-\alpha)$, then
$\sin 2x = \cot(\phi - \tfrac{1}{4}\pi)\sin 2\alpha$, where $\tan\phi = m$.

6. If $\tan(\alpha + x)\cot x = m$, show that
$\sin(\alpha + 2x) = \cot(\phi - \tfrac{1}{4}\pi)\sin\alpha$, where $\tan\phi = m$.

7. If $\tan(\alpha + x)\tan(\alpha - x) = m$, show that
$\cos 2x = \cot(\tfrac{1}{4}\pi - \phi)\cos 2\alpha$.

8. Find the angles between 0° and 360° which satisfy the equation

$$4\sin x + 3\cos x = 5.$$

Ans. $x = 53^\circ\, 07'\, 45''$.

9. Find θ from the equation

$$2.76\cos\theta - 2.32\sin\theta = 1.91.$$

Ans. $\theta = 17^\circ\, 59.6'$, $261^\circ\, 55'$.

10. Find the general solution of the equation

$$\sqrt{3}\sin x - \cos x = \sqrt{2}.$$

Ans. $x = 2n\pi + \frac{5}{12}\pi$, or $(2n+1)\pi - \dfrac{\pi}{12}$.

11. Find the general solution of the equation

$$(1+\sqrt{3})\tan x + (1-\sqrt{3})\cot x = 2.$$

$$x = n\pi + \tfrac{1}{4}\pi,\ n\pi - \frac{\pi}{12}.$$

12. Find r and x from the equations

$$r\sin(\alpha + x) = m,$$
$$r\sin(\beta + x) = n.$$

Suggestion. Form the sum and difference of these equations, and change each into a product by the formulas in Article 113. Dividing the first result by the second gives

$$\tan\left(\frac{\alpha+\beta}{2} + x\right) = \frac{m+n}{m-n}\tan\frac{\alpha-\beta}{2}$$

or

$$\tan\left(\frac{\alpha+\beta}{2} + x\right) = \tan(\tfrac{1}{4}\pi + \phi)\tan\frac{\alpha-\beta}{2},\ \text{where } \tan\phi = \frac{n}{m}.$$

13. If $r\cos(\alpha + x) = m$, $r\cos(\beta + x) = n$, then x is given by

$$\cot\left(\frac{\alpha+\beta}{2} + x\right) = \tan(\tfrac{1}{4}\pi + \phi)\tan\frac{\alpha-\beta}{2},\ \text{where } \tan\phi = \frac{m}{n}.$$

14. Adapt the formula

$$\tan x = \frac{m\sin\alpha}{1 + m\cos\alpha}$$

to computation by logarithms.

Suggestion. Express the tangent in terms of the sine and cosine and clear fractions, the result may be written

$$\sin x = m\sin(\alpha - x),$$

which by Example 3 may be transformed into

$$\tan(x - \tfrac{1}{2}\alpha) = \frac{m-1}{m+1}\tan\tfrac{1}{2}\alpha = \tan(\phi - \tfrac{1}{4}\pi)\tan\tfrac{1}{2}\alpha,\ \text{where } \tan\phi = m.$$

15. Show that

$$a + b = \frac{a}{\cos^2\phi},\quad \text{where}\quad \tan^2\phi = \frac{b}{a},$$

$$a - b = a\cdot\cos^2\phi,\ \text{where } \sin^2\phi = \frac{b}{a}.$$

These formulas enable us to find the logarithm of any sum or difference. Thus,

$$\log(a+b) = \log a + 2 \operatorname{colog} \cos\phi, \quad \log\tan\phi = \tfrac{1}{2}(\log b - \log a).$$
$$\log(a-b) = \log a + 2 \log\cos\phi, \quad \log\sin\phi = \tfrac{1}{2}(\log b - \log a).$$

16. Establish the following transformations:

$$\left.\begin{aligned} a\sin\alpha + b\sin\alpha &= \frac{a\cos(\alpha-\theta)}{\cos\theta} \\ a\cos\alpha - b\sin\alpha &= \frac{a\cos(\alpha+\theta)}{\cos\theta} \end{aligned}\right\}, \text{ where } \tan\theta = \frac{b}{a}.$$

124. Inverse Functions. The two expressions

$$y = \sin x \qquad (1)$$
$$x = \sin^{-1} y \qquad (2)$$

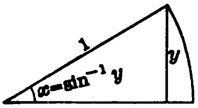

Fig. 151.

represent different views of the same relation. The first one states that y is the sine of x, the second that x is the angle or arc (measuring the angle) whose sine is y. (1) expresses y in terms of x, (2) expresses x in terms of y.

Taking the sine of each member of (2) gives us

$$\sin x = \sin(\sin^{-1} y) = y, \text{ by (1)}, \qquad (3)$$

and taking the inverse sine of each member of (1) gives us

$$\sin^{-1} y = \sin^{-1}(\sin x) = x, \text{ by (2)}. \qquad (4)$$

In precisely the same way it may be shown that

$$\cos(\cos^{-1} x) = x, \quad \cos^{-1}(\cos x) = x,$$
$$\tan(\tan^{-1} x) = x, \quad \tan^{-1}(\tan x) = x,$$
$$\text{etc.,} \qquad\qquad \text{etc.}$$

From these relations it appears that of each pair of operations, say for example that of taking the sine and that of taking the arc-sine, either undoes the other, that is, if the two operations are performed in succession, the result is that the quantity operated on is left unchanged. This explains why $\sin^{-1}x$ is called the inverse sine of x, $\tan^{-1}x$ the inverse tangent of x, etc., in fact in any pair of such functions each is the inverse of the other. Viewed as operations, the relation of the members of each pair is like that of addition to subtraction, of multiplication to division, of involution to evolution.

In general, if $y = f(x)$ represents any function of x, the inverse function is represented by $f^{-1}(x)$, and the relation between the two is always such that, considered as operations on x, each undoes the other, that is,

$$f[f^{-1}(x)] = x, \quad \text{and} \quad f^{-1}[f(x)] = x. \tag{5}$$

If $y = f(x)$ is known, $f^{-1}(x)$ is found by solving (provided we can solve) $y = f(x)$ for x, and by substituting in the result x for y. Thus, if

$$y = \frac{x^2 - 4}{3} = f(x), \tag{6}$$

we find $$x = \sqrt{3y + 4} = f^{-1}(y),$$

and $$\sqrt{3x + 4} = f^{-1}(x). \tag{7}$$

To verify the relations (5) we substitute for x in (6) the expression (7), and for x in (7) the expression (6); thus,

$$f[f^{-1}(x)] = \frac{(\sqrt{3x + 4})^2 - 4}{3} = x,$$

$$f^{-1}[f(x)] = \sqrt{3\left(\frac{x^2 - 4}{3}\right) + 4} = x.$$

There is this important difference between the trigonometric functions and their inverses, — while each of the former has a single value, each of the latter has an indefinite number of values. Thus, if $x = \frac{\pi}{6}$, $\sin x$ has a single value, namely $\frac{1}{2}$, but if $x = \frac{1}{2}$, $\sin^{-1} x$ may have any of the values $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, $\frac{17\pi}{6}$, etc. In general,

$$\sin^{-1} x = n\pi + (-1)^n\alpha,$$
$$\cos^{-1} x = 2n\pi \pm \alpha,$$
$$\tan^{-1} x = n\pi + \alpha,$$

where α is the principal value of the angle.

Any relation between trigonometric functions may be expressed by means of inverse functions. We shall illustrate the method by some examples.

EXAMPLE 1. Express the relation

$$\cos 2A = 1 - 2\sin^2 A$$

in terms of inverse functions.

Solution. Let $\sin A = m$, then $A = \sin^{-1} m$, and the given equation becomes

$$\cos(2\sin^{-1} m) = 1 - 2m^2,$$

or

$$2\sin^{-1} m = \cos^{-1}(1 - 2m^2).$$

Example 2. Express the formula

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

in terms of inverse functions.

Solution. Let $\sin A = m,\quad \sin B = n,$

then $\cos A = \sqrt{1 - m^2},\quad \cos B = \sqrt{1 - n^2},$

and the given formula becomes

$$\sin(\sin^{-1} m + \sin^{-1} n) = m\sqrt{1 - n^2} + n\sqrt{1 - m^2},$$

or

$$\sin^{-1} m + \sin^{-1} n = \sin^{-1}(m\sqrt{1 - n^2} + n\sqrt{1 - m^2}).$$

Example 3. Express the formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

in terms of inverse functions.

Solution. Let $\tan A = m,\quad \tan B = n,$

then $A = \tan^{-1} m,\quad B = \tan^{-1} n.$

Substituting

$$\tan(\tan^{-1} m + \tan^{-1} n) = \frac{m + n}{1 - mn},$$

or

$$\tan^{-1} m + \tan^{-1} n = \tan^{-1}\frac{m + n}{1 - mn}.$$

Formulas involving inverse functions may be verified by reversing the process illustrated in the above examples.

Example 4. Show that

$$\tan^{-1} m + \tan^{-1} n = \tan^{-1}\frac{m + n}{1 - mn}.$$

Solution. Put $\tan^{-1} m = A,\quad \tan^{-1} n = B,$

then $m = \tan A,\quad n = \tan B.$

Substituting, we find

$$A + B = \tan^{-1}\frac{\tan A + \tan B}{1 - \tan A \tan B},$$

or

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

This latter expression we know is true, hence the original expression is also true.

EXAMPLE 5. Find the value of

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}.$$

Solution. Put $\tan^{-1}\tfrac{1}{2} = A, \quad \tan^{-1}\tfrac{1}{3} = B,$

then $\tan A = \tfrac{1}{2}, \quad \tan B = \tfrac{1}{3},$

and the given expression becomes $A + B$.

Now $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1,$

therefore $A + B = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = n\pi + \dfrac{\pi}{4}.$

EXAMPLE 6. Solve the equation

$$\sin^{-1} 2x + \sin^{-1} 3x = \cos^{-1}(-\tfrac{2}{3}).$$

Solution. Put $\sin^{-1} 2x = A, \ \sin^{-1} 3x = B,$

then
$$\sin A = 2x, \ \sin B = 3x,$$
$$\cos A = \sqrt{1 - 4x^2}, \ \cos B = \sqrt{1 - 9x^2},$$

and the given expression becomes

$$A + B = \cos^{-1}(-\tfrac{2}{3}).$$

Take the cosine of both sides,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B = -\tfrac{2}{3},$$

and expressing this in terms of x,

$$\sqrt{1 - 4x^2} \cdot \sqrt{1 - 9x^2} - 2x \cdot 3x = -\tfrac{2}{3}.$$

Solving for x,

$$x = \pm\tfrac{1}{3}.$$

EXERCISE 56

1. Find the general value of each of the following angles:

$\sin^{-1}\tfrac{1}{2}, \ \cos^{-1}\dfrac{\sqrt{2}}{2}, \ \tan^{-1}\sqrt{3}, \ \cos^{-1} 0, \ \sec^{-1} 1, \ \tan^{-1}\infty.$

Ans. $n\pi + (-1)^n\dfrac{\pi}{6}, \quad 2n\pi \pm \dfrac{\pi}{4}, \quad n\pi + \dfrac{\pi}{3},$ etc.

Considering principal values only, verify the following:

2. $\tan^{-1}\frac{3}{2}-\tan^{-1}\frac{1}{5}=\frac{\pi}{4}.$

3. $\tan^{-1}m+\tan^{-1}\frac{1}{m}=\frac{\pi}{2}.$

4. $\cos^{-1}\frac{8}{17}+\cos^{-1}\frac{15}{17}=\frac{\pi}{2}.$

5. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{13}=\frac{\pi}{4}.$

Express the following formulas in terms of inverse functions:

6. $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$

Ans. $2\tan^{-1}m=\tan^{-1}\left(\frac{2m}{1-m^2}\right)$, where $m=\tan\theta$.

7. $\cos(A-B)=\cos A\cos B+\sin A\sin B.$

Ans. $\sin^{-1}m-\sin^{-1}n=\cos^{-1}\left(mn+\sqrt{(1-m^2)(1-n^2)}\right)$, where $m=\sin A$, $n=\sin B$.

8. $\sin 2x=2\sin x\cos x.$

Ans. $2\sin^{-1}m=\sin^{-1}(2m\sqrt{1-m^2})$, where $m=\sin x$.

9. $\sin 3x=3\sin x-4\sin^3 x.$

Ans. $3\sin^{-1}m=\sin^{-1}(3m-4m^3)$, where $m=\sin x$.

10. Show that

$$\sin^{-1}m=\cos^{-1}\sqrt{1-m^2}=\tan^{-1}\frac{m}{\sqrt{1-m^2}}=\sec^{-1}\frac{1}{\sqrt{1-m^2}}$$
$$=\csc^{-1}\frac{1}{m}=\cot^{-1}\frac{\sqrt{1-m^2}}{m}.$$

11. Show that

$$\sin^{-1}\tfrac{3}{5}+\cos^{-1}\tfrac{12}{13}=\tan^{-1}\tfrac{56}{33}.$$

12. Show that

$$2\tan^{-1}\tfrac{1}{2}+3\tan^{-1}\tfrac{1}{3}=\tan^{-1}(-3).$$

Find x in the following equations:

13. $\sin^{-1}2x+\sin^{-1}x=\frac{\pi}{3}.$ *Ans.* $x=\pm\frac{\sqrt{21}}{14}.$

14. $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\tan^{-1}\frac{2}{25}.$ *Ans.* $x=\pm 5.$

15. $\sin^{-1}x+2\cos^{-1}x=\tan^{-1}\sqrt{3}.$ *Ans.* $x=\pm\frac{\sqrt{3}}{2}.$

16. Show that if $f(x) = \frac{x+1}{x}$, then $f^{-1}(x) = \frac{1}{x-1}$.

17. Prove that the inverse of $x^2 + 4$ is $\sqrt{x-4}$.

18. What is the inverse of $\log_{10}x$? Of $1 - x$?

Ans. 10^x, $1 - x$.

19. Prove that $\frac{x+1}{x-1}$ is its own inverse.

Find the inverse functions of each of the following, and verify the results:

20. $f(x) = \frac{x^3}{3}$.

21. $f(y) = \frac{y^2 + 2y}{5}$.

22. $f(\theta) = \frac{\theta^2 + 4}{\theta^2 - 4}$. *Ans.* $f^{-1}(\theta) = 2\sqrt{\frac{\theta + 1}{\theta - 1}}$.

125. Review. 1. (*a*) Define the sine, cosine and tangent of any angle. (*b*) Give the signs of the principal functions in each of the four quadrants. (*c*) Give the formula which expresses the periodicity of the sine. (*d*) Prove that $\tan(\theta + n\pi) = \tan\theta$. (*e*) Which other function has the same period as the tangent?

2. (*a*) Follow the changes in the sine of an angle as the angle increases from 0 to 2π. (*b*) Do the same for the cosine. (*c*) Do the same for the tangent. (*d*) Follow the changes in the reciprocal of the tangent.

3. (*a*) Draw the lines which represent the various functions of an arc of a circle (or of its angle at the center of the circle) when the radius of the circle is taken as the unit of measure. (*b*) Draw these lines for an angle in the third quadrant. (*c*) Explain the derivation of the words secant, tangent, sine and cosine.

4. (*a*) Prove geometrically that $\sin(R + \theta) = \cos\theta$, $\tan(3R - \theta) = \cot\theta$, θ being an angle in the first quadrant. (*b*) Prove that $\sin(-\theta) = -\sin\theta$, $\cos(2R + \theta) = -\cos\theta$ for every value of θ. (*c*) Give from memory the principal functions of 30°, 150°, 210°, 330°. (*d*) Give from memory the principal functions of 45°, 135°, 225°, 315°.

5. (*a*) Show that

$$\frac{\sin(-A) + \cos(-A)}{\tan(-A) - \cot(-A)} = \frac{\sin(90^\circ + A) + \cos(270^\circ - A)}{\cot(180^\circ + A) + \tan(360^\circ - A)}.$$

(*b*) Simplify $\dfrac{\sin(R+x)\cos(R-x)}{\cos(2R+x)}+\dfrac{\sin(R-x)\cos(R+x)}{\sin(2R+x)}$.

(*c*) Find from the tables sin 234°, cos 342°, tan 134° 54′, sin 967° 45′, $\tan\left(n\pi+\dfrac{\pi}{10}\right)$.

6. (*a*) Prove the addition theorem for the sine, for the cosine and for the tangent. (*b*) Give the formulas for sin 2 θ, cos 2 θ and tan 2 θ. (*c*) Show that $\sin(x+y)\sin(x-y)=(\sin x+\sin y)(\sin x-\sin y)$. (*d*) Given $\cos x=\frac{1}{8}$, find $\sin\dfrac{x}{2}$, $\cos\dfrac{x}{2}$ and $\tan\dfrac{x}{2}$ without the use of tables.

7. Given the law of sines, prove the law of tangents.

8. Express each of the following as a product:

$$\sin 7^\circ+\sin 15^\circ,\ \cos\frac{x}{2}-\cos\frac{3x}{2},\ \sin A+\sin 3A+\sin 4A.$$

9. By multiplying each side of the expression

$$S=1+\cos x+\cos 2x+\cos 3x+\ldots+\cos nx,\text{ by } 2\sin\frac{x}{2}$$

and expressing each product on the right as a difference of two sines show that

$$S=\frac{\sin\dfrac{(2n+1)x}{2}+\sin\dfrac{x}{2}}{2\sin\left(\dfrac{x}{2}\right)}=\frac{\sin\dfrac{(n+1)x}{2}\cos\dfrac{(nx)}{2}}{\sin\left(\dfrac{x}{2}\right)}.$$

10. (*a*) What is meant by the principal value of an angle? (*b*) Give the principal values of $\sin^{-1}\frac{1}{2}$, $\sin^{-1}-\frac{1}{2}$, $\cos^{-1}\frac{1}{2}$, $\tan^{-1}-1$. (*c*) Give the general values of the angles under (*b*).

11. (*a*) What is meant by a trigonometric equation? (*b*) Solve the following equations, $\sin x=-\cos x$, $2\sin^2 x+3\cos^2 x=5$, $a\sin x=b\tan x$.

12. Solve the equations:

(*a*) $\sin(x+c)-\cos x\sin c=\cos c$.

(*b*) $\sin(a-x)=\cos(a+x)$.

(*c*) $\sin(x+y)=\cos(x-y)=\dfrac{\sqrt{2}}{2}$.

13. Solve the equations:

(*a*) $\sin 2y + \sin 3y = 3 \sin y$. (*b*) $1 + \cos x = \cos \frac{x}{2}$.

(*c*) $\sin x \cos x = \frac{1}{4}$. (*d*) $\sin x + \cos x = \sqrt{2}$.

14. (*a*) Show how to adapt the equation $a \sin x + b \cos x = c$ to solution by means of logarithms. (*b*) Solve the simultaneous equations: $x \cos \alpha + y \sin \alpha = a$, $x \sin \alpha + y \cos \alpha = b$.

15. (*a*) Define the inverse trigonometric functions.

(*b*) Complete the following equalities, $\sin^{-1} x = \cos^{-1} (\quad) = \tan^{-1} (\quad) = \sec^{-1} (\quad) = \csc^{-1} (\quad) = \cot^{-1} (\quad)$.

(*c*) Find $\tan (\sin^{-1} \frac{4}{5})$, $\sin (\tan^{-1} x)$.

(*d*) Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 45°$.

16. (*a*) Prove that $\sin (2 \sin^{-1} x) = 2x\sqrt{1 - x^2}$.

(*b*) $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{13}{85} = \frac{\pi}{2}$.

17. (*a*) If $f(x) = \frac{x^2 + 1}{x^2 - 1}$, find $f^{-1}(x)$.

(*b*) Solve the equation,

$$\cos^{-1} x + \cos^{-1} (1 - x) = \cos^{-1} (-x).$$

CHAPTER XIII

TRIGONOMETRIC CURVES

126. Functions Represented by Curves. The student is probably already familiar with the fact that for every function of $x, f(x)$, a curve or graph may be constructed which is said to represent that function. This curve is merely the totality of all the points whose coördinates x, y, satisfy the relation $y = f(x)$. The actual construction of the curve representing a given function $y = f(x)$ consists in plotting a limited number of points, using for abscissas properly chosen values of x, and for ordinates the corresponding values of y determined by the relation $y = f(x)$. The smooth curve connecting the points thus plotted is said to be the curve or graph representing the function $y = f(x)$.

127. The Straight Line, $y = mx + c$.

Suppose the given function is

$$y = 2x + 1.$$

We give x certain values and compute the corresponding values of y. Thus, if

$x = 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5,$ etc.,
$y = 11, 9, 7, 5, 3, 1, -1, -3, -5, -7, -9,$ etc.

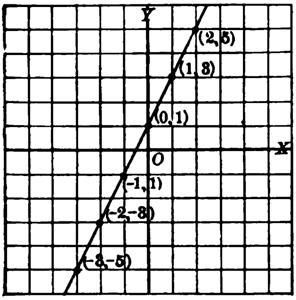

Fig. 152. $y = 2x + 1$.

We now locate the point whose abscissa is 5 and whose ordinate is 11, another point whose abscissa is 4 and whose ordinate is 9, and similarly each of the points (3, 7), (2, 5), (1, 3), (0, 1), (− 1, − 1), etc.

We then connect the points thus located by a smooth curve, in this case a straight line, Fig. 152. This line is said to be the curve or graph representing the function $y = 2x + 1$, or, in short, the straight line $y = 2x + 1$.

In the example just given, x may have one value as well as another, say 1,000,000 as well as 2 or 3, consequently the line representing the equation $y = 2x + 1$ will be indefinite in length and we must content ourselves with drawing only a portion of it. What portion this is to be depends on the purpose in view, but unless there is a special reason to the contrary, it is customary to construct the portion nearest the origin.

In a similar manner every equation of the form

$$y = mx + c,$$

where m and c are known numbers, is represented by some straight line, m and c determining the direction and position of the line with respect to the coördinate axes.

128. The Circle, $x^2 + y^2 = a^2$.

Suppose the given equation is

$$x^2 + y^2 = 25,$$

then

$$y = \pm\sqrt{25 - x^2},$$

and we have for corresponding values of x and y,

$x =$	5,	4,	3,	2,	1,	0,
$y =$	0,	± 3,	± 4,	± 4.58,	± 4.89,	± 5,
$x =$	-1,	-2,	-3,	-4,	-5,	etc.,
$y =$	± 4.89,	± 4.58,	± 4,	± 3,	0,	etc.

If, as before, we construct the separate points $(5, 0)$, $(4, 3)$, $(4, -3)$, $(3, 4)$, $(3, -4)$, $(2, 4.58)$, $(2, -4.58)$, etc., and connect the points thus obtained by a smooth curve, we obtain the circle, Fig. 153, which is said to be the curve or graph representing the equation $x^2 + y^2 = 25$, or, in short, the circle $x^2 + y^2 = 25$. In this case x cannot be numerically greater than 5, for then y would be imaginary.

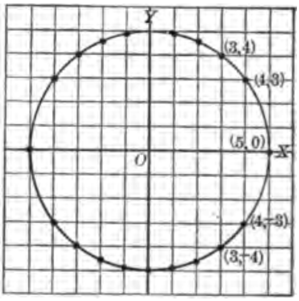

Fig. 153. $x^2 + y^2 = 25$.

In like manner every equation of the form

$$x^2 + y^2 = a^2,$$

where a is some known number, is represented by a circle whose center is at the origin and whose radius is a.

129. The Hyperbola, $x^2 - y^2 = a^2$.

As another example, let us construct the curve whose equation is

$$x^2 - y^2 = 25.$$

Solving for y,

$$y = \pm\sqrt{x^2 - 25}.$$

Corresponding values of x and y are

$x = 5$,	6,	7,	8,	9,	10
$y = 0$,	± 3.32,	± 4.90,	± 6.25,	± 7.48,	± 8.66,
$x = -5$,	-6,	-7,	-8,	-9,	etc.
$y = +0$,	± 3.32,	± 4.90,	± 6.25,	± 7.48,	etc.,

In this case x cannot be numerically less than 5, for otherwise y is imaginary.

If we construct the separate points (5, 0), (6, 3.32), (6, − 3.32), etc., and draw a smooth curve connecting them, we obtain the two curves PQ, $P'Q'$, Fig. 154. These curves constitute the two branches of a single curve, known as the hyperbola, more specifically as the *equilateral hyperbola.*

It is easy to see from the equation that the larger x is, the more nearly will x and y be equal, that is, the branches of the equilateral hyperbola approach the straight lines PP' and QQ' drawn through the origin and making angles of 45° with the two directions of the x-axis respectively.

In like manner it will be found that every equation of the form

$$x^2 - y^2 = a^2,$$

where a is some known number, is represented by an equilateral hyperbola. a determines the distance from the origin at which the hyperbola crosses the x-axis. This is known as the *semimajor axis* of the hyperbola.

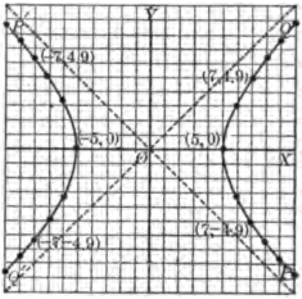

Fig. 154. $x^2 - y^2 = 25$.

130. The Sine Curve, $y = \sin x$.

Let us now construct the curve representing $y = \sin x$.

Since x may have any value either positive or negative, the curve representing the sine function will extend indefinitely in both directions (right and left). Let us first construct that portion of the curve which corresponds to values of x between 0 and $\frac{1}{2}\pi$, that is, to angles in the first quadrant. Referring to the table of natural sines, we find the following corresponding values of x and y:

$x =$	In degrees,	0°,	10°,	20°,	30°,	40°,	50°,	60°,	70°,	80°,	90°.
	In radians,	0,	$\frac{\pi}{18}$,	$\frac{\pi}{9}$,	$\frac{\pi}{6}$,	$\frac{2\pi}{9}$,	$\frac{5\pi}{18}$,	$\frac{\pi}{3}$,	$\frac{7\pi}{18}$,	$\frac{4\pi}{9}$,	$\frac{\pi}{2}$.
$y =$		0,	0.17,	0.34,	0.5,	0.64,	0.77,	0.87,	0.94,	0.98,	1.

In order to avoid awkward fractions, we will use $\frac{1}{3}\pi$ as the unit along the x-axis.*

With $\frac{\pi}{3}$ for a unit, $\frac{\pi}{18} = \frac{1}{6}$ unit, $\frac{\pi}{9} = \frac{1}{3}$ unit, etc., and we now readily locate the points

$$O = (0, 0),\ P_1 = \left(\frac{\pi}{18}, 0.17\right),\ P_2 = \left(\frac{\pi}{9}, 0.34\right), \ldots, R_1 = \left(\frac{\pi}{2}, 1\right).$$

Connecting these points by a smooth curve we obtain the curve $OP_1P_2P_3 \ldots R_1$ (Fig. 155), which is the sine curve for the first quadrant.

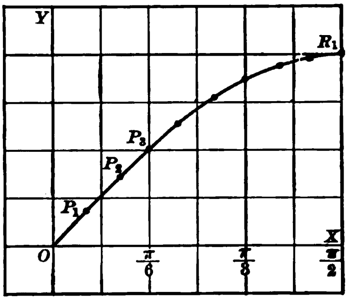

Fig. 155. $y = \sin x$.

We may now easily continue the sine curve through as many additional quadrants as we choose.

Second quadrant. While x varies from $\frac{1}{2}\pi$ to π, $\sin x$ varies from 1 to 0. Moreover, since sines of supplementary angles are equal, ordinates equally distant from R_1 will be equal, that is, the curve will be symmetrical with respect to the ordinate at R_1. Hence, continuing from R_1, the curve will approach the x-axis, meeting it at R_2 (Fig. 156), whose distance from the origin is π.

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* This will distort the curve slightly, since $\frac{\pi}{3} = \frac{3.14159+}{3} = 1.0471+$.

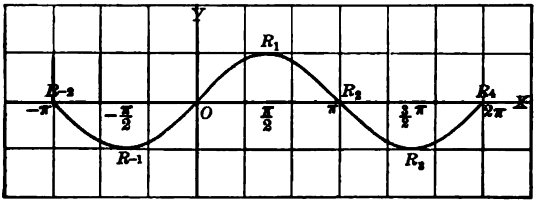

Fig. 156. $y = \sin x$.

Third and fourth quadrants. While x varies from π to 2π, $\sin x$ is negative, the numerical value being the same as when x varies from 0 to π. Hence, continuing from R_2 the curve will descend below the x-axis, reaching the lowest point at R_3, where $x = \frac{3}{2}\pi$, and meeting the x-axis again at R_4, where $x = 2\pi$. The form of the portion of the curve below the x-axis will be like that above the x-axis when revolved about this axis through an angle of 180°.

When x is increased or diminished by 2π, $\sin x$ has the same value as before, hence extending from R_4 to the right or from O to the left the curve repeats itself indefinitely, that is, the complete sine curve consists of an infinite number of waves or undulations of which $OR_1R_2R_3R_4$ is one.

OR_4, the distance between two consecutive points at which the curve crosses the x-axis in the same direction, is called the ***wave length*** of the curve. The greatest height of the curve, represented by the ordinate at R_1, is called the ***amplitude*** of the curve.

A curve like the sine curve, which repeats itself at definite intervals, is called a ***periodic curve;*** the interval at which the repetition takes place is called the ***period.*** Likewise the function which such a curve represents is called a ***periodic function.***

The sine function is a periodic function whose period is 2π.

131. The Tangent Curve, $y = \tan x$.

To construct the tangent curve for the first quadrant, we compute by means of a table of natural functions the corresponding values of x and y, as follows:

$x =$	In degrees,	0°,	10°,	20°,	30°,	40°,	50°,	60°,	70°,	80°,	90°.
	In radians,	0,	$\frac{\pi}{18}$,	$\frac{\pi}{9}$,	$\frac{\pi}{6}$,	$\frac{2\pi}{9}$,	$\frac{5\pi}{18}$,	$\frac{\pi}{3}$,	$\frac{7\pi}{18}$,	$\frac{4\pi}{9}$,	$\frac{\pi}{2}$.
$y =$		0,	0.18,	0.36,	0.58,	0.84,	1.19,	1.73,	2.75,	5.67,	∞.

Plotting the points obtained by using the x's for abscissas and the corresponding values of y for ordinates, and connecting these points by a smooth curve, we obtain the curve OR, Fig. 157, which represents the equation $y = \tan x$ for values of x in the first quadrant. Since $\tan \frac{1}{2}\pi = \infty$, the curve will not intersect the perpendicular at $\frac{1}{2}\pi$, but will approach it indefinitely.

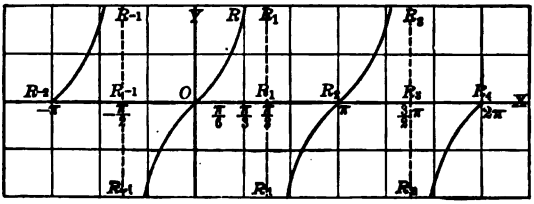

Fig. 157. $y = \tan x$.

Second quadrant. While x varies from $\frac{1}{2}\pi$ to π, the tangent varies from $-\infty$ to 0, hence between R_1 and R_2 the tangent curve will be below the x-axis, the numerical values of the ordinates being equal to those between O and R_1 taken in the reverse order.

Tan $(\pi + x) = \tan x$, hence beginning with R_2 the curve repeats itself. The tangent curve, therefore, consists of an infinite number of disconnected branches.

The tangent curve is a periodic curve, the tangent function is a periodic function, the period in each case is π.

A line like R_1R_1 or $R_{-1}R_{-1}$, Fig. 157, to which the curve approaches indefinitely near without ever reaching it, is called an *asymptote* to the curve. The lines PP' and QQ', Fig. 154, are asymptotes to the hyperbola. An asymptote is a tangent to the curve at infinity.

When the student has become familiar with the forms of the sine and tangent curves, he can readily sketch them from a very few points whose coördinates are known from memory. Thus, for the values

$$x = 0, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}$$

the corresponding values of the functions are known without consulting a table, namely

$$\sin x = 0, \quad 0.5, \quad \frac{\sqrt{2}}{2} = 0.71, \quad \frac{\sqrt{3}}{2} = 0.87, \quad 1,$$

$$\tan x = 0, \quad \frac{\sqrt{3}}{3} = 0.58, \quad 1, \quad \sqrt{3} = 1.73, \quad \infty.$$

The same remark applies to the sketching of each of the remaining trigonometric functions.

Exercise 57

1. Construct the cosine curve.

2. Construct the cosecant curve.

(Suggestion. This curve is most readily sketched from the sine curve by remembering that $y = \csc x = \frac{1}{\sin x}$.)

3. Construct the secant curve. $\left(y = \sec x = \frac{1}{\cos x}\right)$.

4. Construct the cotangent curve. $\left(y = \cot x = \frac{1}{\tan x}\right)$.

5. Construct the curves whose equations are $y = \sin^{-1} x$, $y = \tan^{-1} x$.

(Suggestion. If $y = \sin^{-1} x$, then $x = \sin y$, etc.)

6. $y = \cos x = \sin\left(\frac{\pi}{2} + x\right)$. From this relation it follows that for every ordinate on the sine curve there is an equal ordinate on the cosine curve whose abscissa is the abscissa of the former diminished by $\frac{1}{2}\pi$, that is, for every point on the sine curve there is a point on the cosine curve, the latter being $\frac{1}{2}\pi$ to the left of the former. Thus we see that the cosine curve is merely the sine curve shifted a distance $\frac{1}{2}\pi$ to the left. By a similar reasoning show that the secant curve is the cosecant curve shifted a distance $\frac{1}{2}\pi$ to the left.

7. $y = \cot x = -\tan\left(\frac{\pi}{2} + x\right)$. From this relation show that the cotangent curve may be obtained by shifting the tangent curve a distance of $\frac{1}{2}\pi$ to the left, and revolving it about the x-axis through an angle of 180°.

8. $y = \cos x = \sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right)$. From this relation show that the cosine curve may be obtained by shifting the sine

curve a distance of $\frac{1}{2}\pi$ to the right, and revolving it about the x-axis through an angle of 180°.

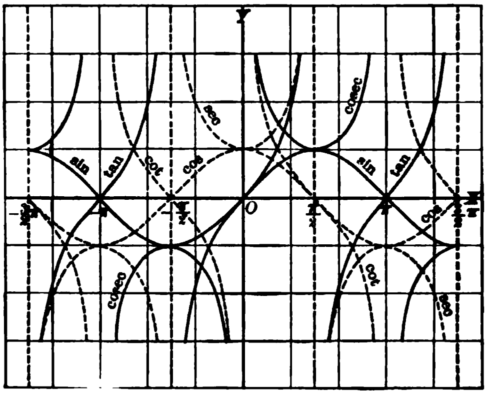

Fig. 158. The Six Trigonometric Curves.

132. The Sinusoidal or Simple Harmonic Curves,

$$y = a \sin (kx + \epsilon).$$

(*a*) $y = a \sin x$, a being constant. Each ordinate of the curve representing this equation is a times the corresponding ordinate of the curve $y = \sin x$. The required curve is the curve obtained by lengthening or shortening (according as a is greater or less than

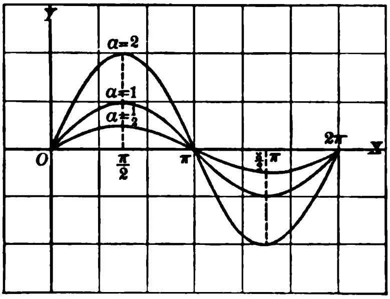

Fig. 159. $y = a \sin x$.

unity) the sine curve in the direction of the y-axis, leaving the wave length unchanged. The curve is a sinelike curve (sinusoid) whose amplitude is a. Fig. 159 shows three sinusoids of equal wave lengths and amplitudes $\frac{1}{2}$, 1, 2 respectively.

(*b*) $y = a \sin kx$, a and k being constants.

The curve representing this equation has the same amplitude as the curve $y = a \sin x$, but the wave length differs, for it crosses the x-axis when

$$kx = 0, \quad \pi, \quad 2\pi, \text{ etc.},$$

that is, when
$$x = 0, \quad \frac{\pi}{k}, \quad \frac{2\pi}{k}, \text{ etc.}$$

The distance between two consecutive crossings of the x-axis in the same direction is $2\pi/k$, that is, the required curve is a sinusoid whose wave length is $2\pi/k$. If we denote this wave length by λ, we have

$$\lambda = \frac{2\pi}{k}, \text{ from which } k = \frac{2\pi}{\lambda},$$

and the equation $y = a \sin kx$ may be written

$$y = a \sin \frac{2\pi x}{\lambda}, \ \lambda \text{ being the wave length.}$$

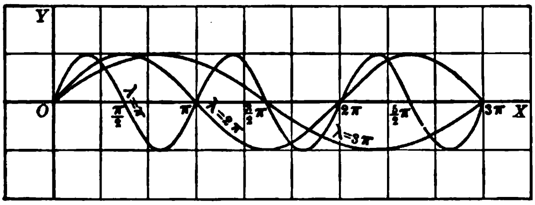

Fig. 160. $y = a \sin (2\pi x/\lambda)$.

Fig. 160 shows three sinusoids of equal amplitudes and wave lengths π, 2π, 3π respectively.

(*c*) $y = a \sin (kx + \epsilon)$, a, k and ϵ being constants.

The curve representing this equation crosses the x-axis where

$$kx + \epsilon = \quad 0, \quad \pi, \quad 2\pi, \quad \text{etc.},$$

that is, where
$$x = -\frac{\epsilon}{k}, \quad \frac{\pi - \epsilon}{k}, \quad \frac{2\pi - \epsilon}{k}, \text{ etc.}$$

The ordinates of the highest and lowest points are a and $-a$ respectively, and the distance between two points where the curve crosses the x-axis in the same direction is $2\pi/k$. When $x = 0$, $y = a \sin \epsilon$. The curve is a sinusoid, amplitude a, wave length $2\pi/k$, but instead of passing through the origin, it crosses the y-axis at a distance $a \sin \epsilon$ above the origin. Fig. 161 shows three sinusoids of

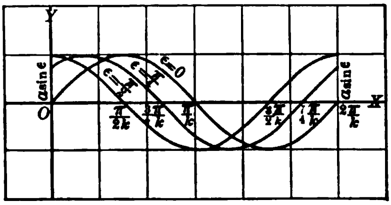

Fig. 161.

equal amplitudes and wave lengths and $\epsilon = 0, \frac{1}{4}\pi$ and $\frac{1}{2}\pi$ respectively. The third of these curves has for its equation

$$y = a \sin\left(kx + \frac{\pi}{2}\right) = a \cos kx,$$

which shows that every cosine curve is a sinusoid.

Sinusoids are extensively used in physics to represent the motion of vibrating strings, tuning forks, and other vibrating bodies emitting musical sounds. For this reason they are often referred to as *harmonic*, or, more strictly, as *simple harmonic curves.*

133. Angles as Functions of Time. In many physical problems which give rise to equations of the form $y = a \sin(kx + \epsilon)$, the independent variable represents not an angle, but the time of an action or motion.

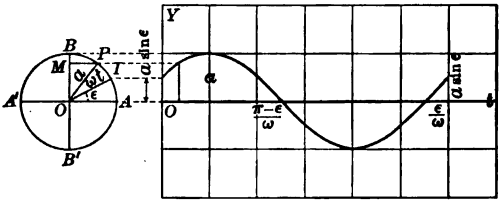

Fig. 162.

Let I, Fig. 162, be the initial position of a point moving in the circumference of a circle (radius $OP = a$) with a constant velocity. Let P be the position occupied by the moving point, t seconds after leaving I, and let ω represent the angular velocity of P, that is, the angle described by OP in one second of time. Then angle $IOP = \omega t$, and angle $AOP = \omega t + \epsilon$, where ϵ is the angle which OI makes with some fixed diameter as AOA'. Furthermore, let M represent the projection of P on BB', the diameter perpendicular to AA', then $OM = a \sin(\omega t + \epsilon)$. The curve

$$y = a \sin(\omega t + \epsilon) \tag{1}$$

obtained by using equal distances on the axis of abscissas to represent equal intervals of time, and using for ordinates the distances OM, which correspond to various values of t, enables us to see at a glance the position of M at any given time t.

If T represents the time required by P to complete one revolution we have

$$\omega T = 2\pi, \text{ from which } \omega = \frac{2\pi}{T},$$

so that (1) may also be written

$$y = a \sin\left(\frac{2\pi t}{T} + \epsilon\right), \tag{2}$$

where T is the *periodic time*, or *period of oscillation* of P and M.

Since T represents the time required to complete one revolution, its reciprocal $1/T$ will represent the number of revolutions completed in a unit of time. This value $1/T$ is known as the *frequency of the oscillation*. If the frequency is denoted by ν, equation (2) assumes the form

$$y = a \sin(2\nu\pi t + \epsilon). \tag{3}$$

Motion like that of the point M in Fig. 162, that is, any motion which can be expressed by an equation of the form $y = a \sin(bt + c)$ is called *simple harmonic*. The motion of vibrating tuning forks, of water waves, of an oscillating pendulum, of a galvanometer needle, of alternating currents, of sound and light and magnetic electric waves, are familiar examples of simple harmonic motions.

EXERCISE 58

Plot the following curves:

1. $y = \sin 2x$.
2. $y = \sin\frac{x}{2}$.
3. $y = 3 \sin\frac{x}{3}$.
4. $y = \frac{1}{3} \sin 3x$.
5. $y = \sin\left(x + \frac{\pi}{4}\right)$.
6. $y = 2 \sin\left(\frac{3\pi}{2}x - \frac{\pi}{6}\right)$.

7. Construct sinusoids having the following amplitudes and wave lengths:

(*a*) Amplitude = 1.5, wave length = 3.5.
(*b*) Amplitude = 0.25, wave length = $\frac{\pi}{2}$.
(*c*) Amplitude = 1, wave length = 8π.

Write the equation for each of these curves.

Ans. (*a*) $y = 1.5 \sin \frac{4\pi x}{7}$, (*b*) $4y = \sin 4x$, (*c*) $y = \sin \frac{x}{4}$.

8. Show that the equation

$$y = a \cos (bt + c)$$

may be used to represent harmonic motion as well as the equation

$$y = a \sin (bt + c).$$

9. Plot the curves (*a*) $y = \sin^2 x$, (*b*) $y = \tan^2 x$.

10. Plot the curves (*a*) $y^2 = \sin x$, (*b*) $y^2 = \tan x$.

11. A point moves in the circumference of a circle whose radius is 8.5, from an initial position whose angular distance from the right-hand extremity of a horizontal diameter is 15°, with a uniform velocity such as to complete a single revolution in 54 seconds. Write the equation between y and t, where y represents the vertical distance of the point from the horizontal diameter at any given time t.

12. A piece of paper is wrapped around a wooden cylinder and then an oblique section is made by sawing the cylinder in two. If the paper is now unrolled and laid flat, its edge will form a sinusoidal curve. Prove it.

Ans. The equation of the curve is $y = a \sin \frac{x}{r}$, where $PR = y$, $OR = x$, $AT = a$, and $OC = r =$ the radius of the cylinder.

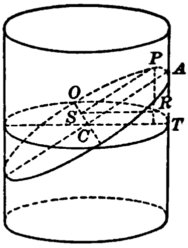

Fig. 163.

134. Composition of Sinusoidal Curves. Curves representing an equation of the form

$$y = a_1 \sin (b_1 x + c_1) + a_2 \sin (b_2 x + c_2) + \text{ etc.},$$

may be readily constructed from the component curves

$$y_1 = a_1 \sin (b_1 x + c_1), \quad y_2 = a_2 \sin (b_2 x + c_2), \text{ etc.},$$

for since

$$y = y_1 + y_2$$

the ordinate of any point on the required curve is found by adding the corresponding ordinates of the component curves.

Example 1. Plot the curve $y = \sin x + \cos x$.

Solution. Plot separately the two curves

$$y_1 = \sin x, \text{ [curve (1), Fig. 164]},$$
$$y_2 = \cos x, \text{ [curve (2), Fig. 164]}.$$

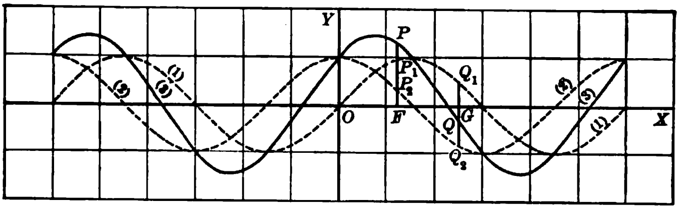

Fig. 164.

The required curve $y = y_1 + y_2$ [curve (3), Fig. 164] is then obtained by adding the two ordinates corresponding to any given value of x. Thus, if $x = OF$, $y = FP = FP_1 + FP_2$. It is important to remember that for points below the x-axis the ordinates are negative; thus, if $x = OG$, $y = GQ = GQ_1 + GQ_2$. Now GQ_1 is positive but GQ_2 is negative. GQ_2 being the longer of the two, their algebraic sum, that is, GQ, will be negative.

It should be noticed that certain points, as, for instance, those for which $y = \pm 1, \pm 2$, or 0, may be located at sight. After the shape of the curve is known, these points suffice to sketch the required curve.

Each of the sine curves $y_1 = \sin x$ and $y_2 = \cos x$ has the period 2π and the resultant curve $y = \sin x + \cos x$ is another sine curve with the period 2π. This may be shown analytically as follows:

$$y = \sin x + \cos x = \sin x + \sin\left(\frac{\pi}{2} - x\right),$$

$$= 2 \sin\frac{\pi}{4} \cos\left(\frac{\pi}{4} - x\right), \text{ by formulas in Article 113,}$$

$$= \sqrt{2} \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right),$$

that is, the curve $y = \sin x + \cos x$ is a sinusoid having an amplitude $\sqrt{2}$, period 2π, and crossing the x-axis at the point $x = -\frac{1}{4}\pi$.

The method employed in constructing the curve in the preceding example applies equally well to the compounding of any number of curves. The ordinates of the resultant curve are always the algebraic sums of the ordinates of the component curves.

Example 2. Plot the curve $y = \sin x + \sin 2x$.

Solution. Construct separately the two curves

$$y_1 = \sin x, \quad \text{[curve (1)]},$$
$$y_2 = \sin 2x, \ \text{[curve (2)]},$$

then

$$y = y_1 + y_2 = \sin x + \sin 2x$$

yields the curve (3), Fig. 165. This curve has the period 2π.

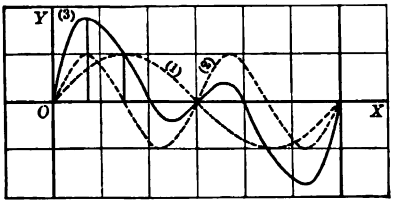

Fig. 165.

Example 3. Plot the curve $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

Solution.

$y_1 = \sin x$ gives curve (1).
$y_2 = \frac{1}{3}\sin 3x$ gives curve (2),
$y_3 = \frac{1}{5}\sin 5x$ gives curve (3),
$y = y_1 + y_2 + y_3$ gives curve (4).

This curve also has the period 2π.

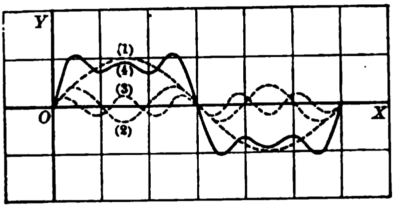

Fig. 166.

135. Theorem. *The resultant of two simple harmonic curves having equal wave lengths is another simple harmonic curve having the same wave length.*

Proof. Let the given curves be

$$y_1 = a_1 \sin\left(\frac{2\pi}{\lambda}x + c_1\right), \quad y_2 = a_2 \sin\left(\frac{2\pi}{\lambda}x + c_2\right),$$

λ being the common wave length. (Art. 132.)
Now

$$y_1 = a_1 \sin\left(\frac{2\pi}{\lambda}x\right)\cos c_1 + a_1 \cos\left(\frac{2\pi}{\lambda}x\right)\sin c_1,$$

$$y_2 = a_2 \sin\left(\frac{2\pi}{\lambda}x\right)\cos c_2 + a_2 \cos\left(\frac{2\pi}{\lambda}x\right)\sin c_2,$$

therefore

$$\begin{aligned} y &= y_1 + y_2 \\ &= (a_1 \cos c_1 + a_2 \cos c_2) \sin\left(\frac{2\pi}{\lambda}x\right) + (a_1 \sin c_1 + a_2 \sin c_2) \cos\left(\frac{2\pi}{\lambda}x\right) \\ &= a \cos c \sin\left(\frac{2\pi}{\lambda}x\right) + a \sin c \cos\left(\frac{2\pi}{\lambda}x\right) \\ &= a \sin\left(\frac{2\pi}{\lambda}x + c\right), \qquad (1) \end{aligned}$$

where

$$\begin{aligned} a \cos c &= a_1 \cos c_1 + a_2 \cos c_2, \\ a \sin c &= a_1 \sin c_1 + a_2 \sin c_2. \end{aligned} \qquad (2)$$

Dividing the second of the equations (2) by the first, gives

$$\tan c = \frac{a_1 \sin c_1 + a_2 \sin c_2}{a_1 \cos c_1 + a_2 \cos c_2}, \qquad (3)$$

and taking the sum of their squares

$$\begin{aligned} a^2(\cos^2 c + \sin^2 c) = a_1^2(\cos^2 c_1 + \sin^2 c_1) + 2\,a_1a_2(\cos c_1 \cos c_2 \\ + \sin c_1 \sin c_2) + a_2^2(\cos^2 c_2 + \sin^2 c_2), \end{aligned}$$

or

$$a^2 = a_1^2 + a_2^2 + 2\,a_1a_2 \cos(c_1 - c_2). \qquad (4)$$

Equation (1) shows that the resultant curve is a simple harmonic curve whose wave length is λ. The amplitude a of the resultant curve is given by (4), and the constant c by (3).

136. Fourier's Theorem. Before leaving the subject of harmonic curves, we will state in simple language and without proof a most famous theorem, which in the language of Thomson and Tait[*] "is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along telegraph wires, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."

Any arbitrary periodic curve can be considered the resultant of a sum of simple harmonic curves, and can therefore be expressed by an equation of the form

$$y = a_1 \sin (b_1 x + c_1) + a_2 \sin (b_2 x + c_2) + \text{etc.}$$

The importance of the theorem lies in this, that every periodic phenomenon whose changes can be measured can be represented by a periodic curve. Fourier's theorem shows how every such phenomenon can be resolved into a series of simple harmonic motions.

EXERCISE 59

1. Plot the curve $y = \sin x - \cos x$.

Show analytically that this curve is a sine curve, whose amplitude is $\sqrt{2}$, and which crosses the x-axis at the distance $\frac{1}{4}\pi$ to the right of the origin.

2. Construct the resultant curve of which

$$y_1 = \cos\left(x + \frac{\pi}{3}\right), \quad \text{and} \quad y_2 = \cos\left(x - \frac{\pi}{3}\right),$$

are the components, and show that the equation of the resultant curve may be written $y = \cos x$.

Construct the following curves:

3. $y = \sin x + \frac{1}{3} \sin 3x$.
4. $y = \sin 2x + \sin 3x$.
5. $y = \sin x + \frac{1}{2} \sin 4x$.
6. $y = 2 \sin x - \sin 2x$.
7. $y = \cos 2x + 4 \cos x$.
8. $y = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x$.
9. $x = \tan^{-1} (1 + \sqrt{y}) + \tan^{-1} (1 - \sqrt{y})$.

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[*] Elements of Natural Philosophy, Second Edition, Chapter I.

10. Plot the curve resulting from compounding the two curves

$$y_1 = 2\sin\left(x + \frac{\pi}{6}\right), \qquad y_2 = 3\sin\left(x - \frac{\pi}{6}\right),$$

and show that the equation of the resultant curve is

$$y = a\sin(x + c),$$

where $a = \sqrt{19}$, $c = \tan^{-1}\left(-\frac{\sqrt{3}}{15}\right)$.

11. Show that

$$a\sin\phi + b\cos\phi = \sqrt{a^2 + b^2}\sin\left(\phi + \tan^{-1}\frac{b}{a}\right).$$

12. The force (inertia force) on the piston of an engine is given by the equation

$$F = F_0\left(\cos\theta + \frac{R}{L}\cos 2\theta\right),$$

where θ is the angle which the crank arm makes with a fixed direction, R the length of the crank arm, L the length of the connecting rod, and F_0 a constant (the ideal centrifugal force at the crank pin center). Plot the curve, showing the value of F for any given position of the crank, when $F_0 = 600$, $R = 15$, $L = 45$.

13. s, the distance of the piston of a steam engine from its extreme position corresponding to $\theta = 0$, is given approximately by the equation

$$s = R\left(1 - \cos\theta + \frac{R}{2L}\sin^2\theta\right),$$

where R, L and θ have the same meaning as in Example 12. Plot the curve showing the relation between s and θ, when $R = 10$ and $L = 20$.

137. The Logarithmic Curve, $y = \log_{10}x$.

With the aid of a table of common logarithms and a knowledge of the fundamental properties of logarithms, we find the following corresponding values of x and y:

$x =$	1,	2,	3,	4,	5,	10,	100,	..,	..,	∞.
$y =$	0,	0.30,	0.48,	0.60,	0.70,	1,	2,	..,	..,	∞.
$x =$		$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$,	0.1,	0.01,	..,	..,	0.
$y =$		-0.30,	-0.48,	-0.60,	-0.70,	-1,	-2,	..,	..,	$-\infty$.

In any case, if $x = \frac{1}{n}$, $y = -\log n$.

These values enable us to plot the portion of the curve shown in Fig. 167. They also give us a definite idea of the shape of the curve beyond the limits of the figure. To the right the curve diverges more and more from the x-axis as x increases, below the x-axis the curve approaches the y-axis more and more nearly as x approaches 0; that is, the logarithmic curve $y = \log_{10}x$ has the y-axis for an asymptote.

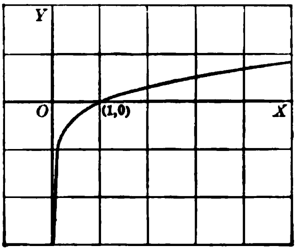

Fig. 167. $y = \log_{10}x$.

138. The Exponential Curve, $y = 10^x$.

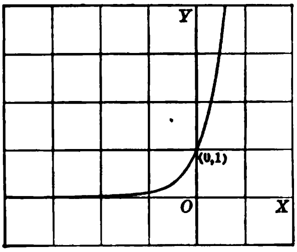

Fig. 168. $y = 10^x$.

Taking the common logarithm of each side of the equation $y = 10^x$, we obtain

$$\log_{10} y = x, \text{ or } x = \log_{10} y.$$

This shows that the exponential curve $y = 10^x$ may be obtained from the curve $y = \log_{10}x$ by interchanging the x's and y's, that is, by using the ordinates of the logarithmic curve for abscissas and the abscissas for ordinates. The resulting curve is shown in Fig. 168.

139. The Exponential Curves, $y = e^{kx}$, where k is any constant, and $e = 2.718+$, the base of the natural system of logarithms.

(*a*) $y = e^x$. Let us first consider the special case for $k = 1$.

A general idea of the shape of the curve $y = e^x$ may be gathered from the following sets of corresponding values of x and y:

$$\begin{array}{llllll}
x = 0, & \frac{1}{2}, & 1, & 2, & .., & \infty, \\
y = 1, & (2.7)^{\frac{1}{2}} = 1.6, & 2.7, & (2.7)^2 = 7.4, & .., & \infty, \\
x = -\frac{1}{2}, & & -1, & -2, & .., & -\infty. \\
y = & \frac{1}{1.6} = 0.61, & \frac{1}{2.7} = 0.38, & \frac{1}{7.39} = 0.14, & .., & -0.
\end{array}$$

From these values the curve may be sketched as in Fig. 168.

If the curve is to be plotted with greater accuracy than the above figures will permit, it is best to employ logarithms. For, on taking the logarithm of each member of the equation $y = e^x$, we have

$$\log_{10} y = x \log_{10} e = x \log_{10} 2.718+ = 0.4343\, x.$$

Assigning to x in succession the values 0, 0.1, 0.2, ., ., 1, 2, 3, etc. we obtain by means of a table of logarithms,

$x =$	0,	0.1,	0.2,	.,	.,	1,	2,	3, etc.,
$\log y =$	0,	0.04343,	0.08686,	.,	.,	0.4343,	0.8686,	1.3029, etc.,
$y =$	1,	1.1052,	1.2214,	.,	.,	2.718,	7.390,	20.085, etc.

The intervals for x may be chosen as small as we please, and the table extended at will, depending on the accuracy desired and the extent of the region for which the curve is to be plotted. The values of y corresponding to negative values of x are the reciprocals of the values of y when x is positive.

(*b*) $y = e^{kx}$, k positive.

When $x = 0$, $y = 1$, therefore, no matter what value k has, the curve passes through the point (0, 1) Fig. 169.

Let $k > 1$. So long as x is positive, $e^x > 1$, and therefore $e^{kx} = (e^x)^k > e^x$, that is, to the right of the y-axis the curve $y = e^{kx}$, $k > 1$, lies above the curve $y = e^x$, and it will diverge from it the more, the greater the value of k. For negative values of x, $k > 1$, $e^{kx} = (e^x)^k < e^x$, that is, to the left of the y-axis the curve $y = e^{kx}$, $k > 1$, will lie between the curve $y = e^x$ and the x-axis and will converge the more rapidly to this axis the greater the value of k. For any given value of k the curve may be roughly sketched by means of a few properly chosen points. If greater accuracy is required, we first compute a table of values from the relation $\log_{10} y = 0.4343\, kx$.

Considering in like manner the cases when $k < 1$, we find that the curves $y = e^{kx}$, $k < 1$, lie between the curve $y = e^x$ and the straight line drawn parallel to the x-axis through the point (0, 1) and that the curve will approach this straight line more nearly the smaller the value of k. Fig. 169 shows the curves for the values $k = \frac{1}{3}$, $k = \frac{1}{2}$, $k = 1$, $k = 2$ and $k = 3$.

(*c*) $y = e^{kx}$, k negative, or $y = e^{-kx}$, k positive.

The curve $y = e^{-kx}$ (1) is most readily obtained from the curve $y = e^{kx}$ (2). For $y = e^{-kx} = (e^k)^{-x}$, from which it is plain that for a positive value of x, the ordinate y of (1) will be the same as the value of y in (2) for the corresponding negative value of x, and vice versa, the y in (1) when x is negative will be equal to the y in (2) when x is positive. This means that the two curves $y = e^{-kx}$ and $y = e^{kx}$, k being the same in both cases, are symmetrical with respect to the y-axis, so that either one being given, the other may be traced from it without computing anew the coördinates of its points. Two such curves have the same relation to each other as an object and its reflection in a mirror. For this reason either of the two is said to be the reflection on the y-axis of the other. Fig. 169 shows the curves $y = e^{kx}$ for the values $k = -3, -2, -1, -\frac{1}{2}, -\frac{1}{3}$.

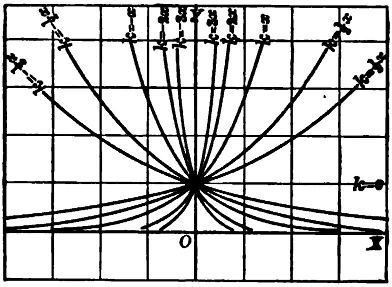

Fig. 169. $y = e^{kx}$.

140. The Compound Interest Law. In physics, chemistry and various branches of engineering, related quantities occur which are subject to laws which may be expressed by the formula

$$y = ae^{kx}, \tag{1}$$

or by the equivalent formula

$$x = \frac{1}{k} \log \frac{y}{a}. \tag{2}$$

In fact, it can be shown that whenever two quantities are so related, that the ratio between their changes is always proportional to one of the quantities, the relation between them may be expressed by either (1) or (2), where x and y are the quantities in question, and k and a constants which depend on the rate of change and the initial values of x and y. The amount of money due at any time on a sum of money put out at compound interest, the interest being added to the principal not at stated intervals but as fast as it accrues, varies with the time according to this law. For this reason the general law expressed by (1) or (2) is commonly known as the ***compound interest law***. The student of science will meet numerous examples of the compound interest law. A few simple examples are given in the set of problems which follows.

EXERCISE 60

Plot the following curves:

1. $y = \log_e x$. 2. $y = 10^{\frac{x}{2}}$. 3. $y = 10^{-x}$.

4. $y = 10^{-\frac{x}{2}}$. 5. $y = \frac{1}{2}\log_e \frac{x}{2}$. 6. $y = 3 \cdot 10^{\frac{x}{3}}$.

7. $y = 3^x$. 8. $y = 3^{-x}$. 9. $y = 2^{2x}$.

10. $y = 2^{-2x}$. 11. $y = 2\,e^x$ 12. $y = -2\,e^x$.

13. Given the curve $y = e^{kx}$, trace the curve $y = -e^{kx}$ without computing the coördinates of its points.

(Suggestion. The required curve is the reflection on the x-axis of the given curve.)

14. The amount A due on a principal of P dollars, put out at compound interest at r per cent for t years, the interest being added to the principal as fast as it accrues, is given by the formula

$$A = Pe^{\frac{rt}{100}}.$$

Plot the curve showing the amount due at any given time, when $P = 100$ and $r = 5$.

(Suggestion. Use different values of t for abscissas and the corresponding values of A for ordinates.)

15. The work W due to the expansion of steam in the cylinder of a steam engine, while expanding from a given volume to a volume V, is given by the formula

$$W = a \log_e V - b,$$

where a and b are two constants depending upon the initial volume and pressure. Plot the curve showing the relation between V and W for any volume from $V = 1$ to $V = 4$, when $a = 15{,}000$, and $b = 0$.

16. Newton's law of cooling. The difference θ between the temperature of a body and the temperature of the medium surrounding it is given by the formula

$$\theta = be^{-at},$$

where a and b are constants depending upon the nature of the body and the initial temperatures. If when $t = 0$, $\theta = 50°$, and when $t = 60$ seconds, $\theta = 25°$, plot a curve showing the temperature θ at any time t up to 60 seconds.

141. The Catenary, $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ in which e is the base of the natural system of logarithms and c any positive constant.

We first consider the special case $c = 1$, $y = \frac{1}{2}(e^x + e^{-x})$.

Put $y_1 = e^x$ and $y_2 = e^{-x}$, then $y = \frac{1}{2}(y_1 + y_2)$, which shows that any ordinate of the required curve is equal to half the sum of the corresponding ordinates of two curves $y_1 = e^x$ and $y_2 = e^{-x}$. If, therefore, these two curves are drawn first, as in Fig. 170, the required curve is easily drawn by taking each ordinate, as AP, equal to half the sum of the ordinates AP_1 and AP_2.

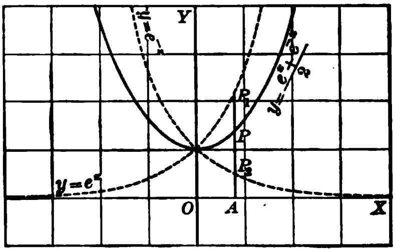

Fig. 170. The Catenary.

It should be observed that

$$y = \frac{y_1 + y_2}{2} = y_2 + \frac{y_1 - y_2}{2} = AP_2 + \frac{P_2P_1}{2},$$

that is, each point P bisects the line joining two corresponding points P_1 and P_2.

We now pass to the general case. $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ may be written

$$\frac{y}{c} = \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2} \qquad (1)$$

which becomes

$$y' = \frac{e^{x'} + e^{-x'}}{2}, \qquad (2)$$

where

$$y' = \frac{y}{c}, \qquad x' = \frac{x}{c}.$$

From this we see that if the coördinates x', y' of any point on the curve (2) be multiplied by c, the resulting numbers are the coördinates x, y of some point on the curve (1). The curve (1) is therefore merely some magnification of the curve (2). The curve in Fig. 170 may therefore be taken to represent any equation of the form (1), provided the proper scale is employed in the construction. It is only necessary to let each unit of length along the coördinate axis represent c units.

The curve $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ is called the catenary. It is the curve formed by a rope or flexible cable suspended between two points. c is the ratio of the horizontal tension to the weight of a unit length of the rope or cable.

142. The Curve of Damped Vibrations, $y = ae^{-kx}\sin(cx + d)$.

Put

$$y_1 = ae^{-kx}, \qquad (1), \qquad\qquad y_2 = \sin(cx + d), \qquad (2)$$

then

$$y = y_1y_2 = ae^{-kx}\sin(cx + d), \qquad (3)$$

that is, any ordinate of the required curve is equal to the product of the corresponding ordinates of the two curves (1) and (2).

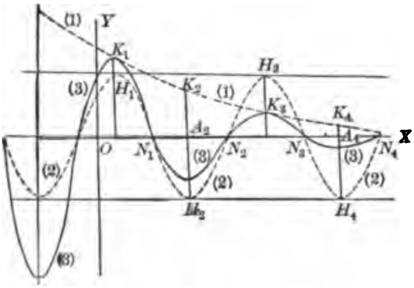

Fig. 171. The Curve of Damped Vibrations.

Let the curves (1) and (2) first be constructed separately, as in Fig. 171. Now observe:

(*a*) Since y_2 is less or at most equal to 1, y of (3) is less or at most equal to y_1, that is, the required curve lies below the curve (1).

(*b*) Whenever y_1 or y_2 equals zero, y of (3) equals zero also, hence the required curve crosses the x-axis at the points N_1, N_2, N_3, N_4, etc., at which the curve (2) crosses this axis.

(*c*) Whenever $y_2 = 1$, as at H_1, H_3, etc., y of (3) equals y_1, hence the required curve touches the curve (1) (since it cannot cross it) at the points K_1, K_3, etc.

(*d*) Whenever $y_2 = -1$, as at H_2, H_4, etc., $y = -y_1$. This enables us to locate the points K_2, K_4, of the required curve.

The shape of the curve (3) is now apparent. It is a wave curve of constant wave length, but the amplitude of the successive waves diminishes. The rapidity with which the amplitude decreases depends on the value of the constant k in (1). This constant k is known as the ***logarithmic decrement*** of the curve.

Like the exponential curve and the sine curve, this curve finds frequent applications in science. While the sine curve represents free vibrations, that is, vibrations not retarded by friction or otherwise, the curve $y = ae^{-kx} \sin (cx + d)$ represents damped vibrations, that is, vibrations suffering resistance of some kind. A pendulum vibrating in air or water, waves propagated in a viscous fluid and oscillatory discharges from an electric condenser, are familiar examples of damped vibratory motion.

EXERCISE 61

Plot the curves:

1. $y = e^{-x} \cos x$. 2. $y = e^{x} \sin x$. 3. $y = \dfrac{e^{x} - e^{-x}}{2}$.

4. $y = \dfrac{2}{e^{x} - e^{-x}}$. (Suggestion. The given equation may be written $y = \dfrac{1}{y'}$, where $y' = \dfrac{e^{x} - e^{-x}}{2}$.)

5. $y = \dfrac{2}{e^{x} + e^{-x}}$. See suggestion under Problem 4.

6. $y = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$. (Suggestion. The given equation may be written $y = \dfrac{y_1}{y_2}$, where $y_1 = \dfrac{e^{x} - e^{-x}}{2}$, $y_2 = \dfrac{e^{x} + e^{-x}}{2}$.)

7. $y = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$. (Suggestion. This equation is the reciprocal of that in Problem 6.)

8. The six functions,

$$y_1 = \frac{e^{x} - e^{-x}}{2}, \qquad y_2 = \frac{e^{x} + e^{-x}}{2}, \qquad y_3 = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}},$$

$$y_4 = \frac{2}{e^x - e^{-x}}, \qquad y_5 = \frac{2}{e^x + e^{-x}}, \qquad y_6 = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

are known as the ***hyperbolic functions***. Show that

$$y_2^2 - y_1^2 = 1,$$
$$y_5^2 + y_3^2 = 1,$$
$$y_6^2 - y_4^2 = 1.$$

Compare the graphs of these functions, Fig. 172. The graphs are numbered in correspondence with the suffixes of the y's in their equations above.

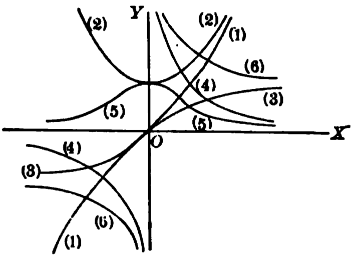

Fig. 172. The Six Hyperbolic Curves.

9. A cable, weighing one pound to the foot, is suspended between two piers under a tension of 100 pounds. Plot the curve which it forms.

10. The displacement of the end of a spring, from its position of equilibrium, at the time t seconds is given by the equation

$$s = ae^{-kt} \sin \frac{2\pi}{T} t,$$

where a is the amplitude of the vibration of the string if there were no friction, k represents the effect of friction retarding the vibration, and T is the time of a single vibration. Plot a curve showing the displacement at any moment during the first ten seconds, if

$$a = 5, \quad k = \tfrac{1}{2}, \quad T = 2.$$

CHAPTER XIV

TRIGONOMETRIC REPRESENTATION OF COMPLEX QUANTITIES

143. Imaginary Numbers. If we solve the equation

$$x^2 + 1 = 0,$$

we obtain $x = \pm\sqrt{-1}$. Similarly, every equation of the form

$$x^2 + a^2 = 0,$$

where a is any real number, has for its solution

$$x = \pm\sqrt{-a^2} = \pm\, a\sqrt{-1},$$

which contains $\sqrt{-1}$ as a factor.

This new number $\sqrt{-1}$, whose square is -1, is commonly denoted by the letter i and is called the *imaginary unit.*

Since $i^2 = -1$, it follows that

$$i^3 = -i,\quad i^4 = 1,\quad i^5 = i,\quad i^6 = -1,\quad i^7 = -i,\quad i^8 = 1,\ \text{etc.},$$

that is,

Every integral power of i is equal to 1, i, -1, or $-i$, according as the exponent of the power when divided by 4 leaves the remainder 0, 1, 2, or 3.

In symbols

$$i^{4n} = 1,\qquad i^{4n+1} = i,\qquad i^{4n+2} = -1,\qquad i^{4n+3} = -i.$$

Numbers like i, $2\,i$, $-5\,i$, $\frac{i}{2}$, $\sqrt{a}\cdot i$, etc., are called imaginary numbers or quantities. Every imaginary number consists of the imaginary unit i multiplied by some real coefficient. From the rule for the powers of i it follows that every even power of an imaginary number is a real number, every odd power of an imaginary number is again an imaginary number.

144. Geometrical Representation of Imaginary Numbers. Every positive or negative real number x may be represented geometrically by a distance on the x axis measured to the right or left according as x is positive or negative. Equal distances measured in the same direction represent equal numbers, but equal distances measured in opposite directions represent numbers which are equal in magnitude but opposite in sign. Thus, in Fig. 173, if the points X_{-2}, X_{-1}, O, X_1, X_2 are taken at equal intervals, and OX_1 represents unity, then each of the segments

$$OX_1,\ X_1X_2,\ X_{-2}X_{-1},\ X_{-1}O \text{ represents } +1,$$

and each of the segments

$$OX_{-1},\ X_{-1}X_{-2},\ X_2X_1,\ X_1O \text{ represents } -1.$$

In general, if the segment PQ represents the number a, QP represents the number $-a$, and therefore $PQ + QP = a + (-a) = 0$. It follows that a line segment PQ on the x-axis will continue to represent the same real number if it is moved along the x-axis from one position to another, so long as its direction remains unchanged.

If all the line segments have the same initial point O, as OX_1, OX_2, OX_{-1}, OX_{-2}, Fig. 173, the terminal points X_1, X_2, X_{-1}, X_{-2} will represent the numbers quite as well as the segments themselves. Thus, if $OX_1 = 1$, the numbers 1, 2, -1, -2 are represented equally well by the segments OX_1, OX_2, OX_{-1}, OX_{-2}, and by the points X_1, X_2, X_{-1}, X_{-2}, respectively.

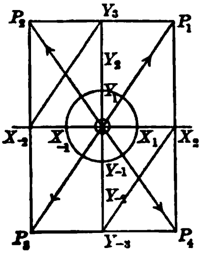

Fig. 173.

Likewise every positive or negative imaginary number iy may be represented geometrically by a distance on the y-axis measured upwards or downwards according as y is positive or negative. Equal distances on the y-axis measured in the same direction represent equal imaginary numbers, equal distances on the y-axis measured in opposite directions represent imaginary numbers which are equal but opposite in sign. Thus, in Fig. 173, if the points Y_{-2}, Y_{-1}, O, Y_1, Y_2 are taken at equal intervals, and OY_1 is taken in length equal to OX_1, then each of the segments OY_1, Y_1Y_2, $Y_{-2}Y_{-1}$, $Y_{-1}O$ represents $+i$, and each of the segments OY_{-1}, $Y_{-1}Y_{-2}$, Y_2Y_1, Y_1O represents $-i$. In general, if the segment RS represents the number ai, SR represents the number $-ai$, and therefore $RS + SR = ai + (-ai) = 0$. It follows that any line segment on the y-axis will continue to represent the same imaginary number if it is moved along the y-axis from one position to another so long as its direction remains unchanged.

If all the line segments have the same initial point O, as OY_1, OY_2, OY_{-1}, OY_{-2}, the terminal points Y_1, Y_2, Y_{-1}, Y_{-2} will represent the imaginary numbers quite as well as the segments themselves. Thus, if OY_1 represents i, the numbers i, $2i$, $-i$, $-2i$ are equally well represented by the segments OY_1, OY_2, OY_{-1}, OY_{-2}, and by the points Y_1, Y_2, Y_{-1}, Y_{-2}, respectively.

When the points on the x-axis and y-axis are used to represent real and imaginary numbers respectively, these axes are referred to as the *axis of reals* and the *axis of imaginaries* respectively.

The reason for choosing the axis of imaginaries at right angles to the axis of reals is found in the following considerations:

The equation $i^2 = -1$ may be put in the following form,

$$+1 : i = i : -1,$$

that is, i is a mean proportional between $+1$ and -1. Now the geometric construction for a mean proportional gives the perpendicular OY_1, erected at O, to the semicircle constructed on $X_{-1}X_1$ as a diameter. Hence, if OX_1 represents $+1$, and OX_{-1} represents -1, OY_1 will represent $\sqrt{+1 \times -1}$ or i.

Or, we may reason as follows: $i^2 = i \times i = -1$, that is, two successive multiplications by i have the same effect as multiplication by -1. Now multiplying any line segment OX_1 by -1 gives $-OX_1 = X_1O = OX_{-1}$, that is, multiplying by -1 has the effect of turning OX_1 through an angle of 180°, hence multiplying by i should have the effect of turning OX_1 through half this angle or 90°, that is, $i \times OX_1 = OY_1$, so that if OX_1 represents 1, OY_1 must represent i.

145. Geometrical Representation of Complex Numbers.

If we solve the equation $x^2 - 4x + 13 = 0$,

we obtain $$x = 2 \pm 3\sqrt{-1},$$

that is, $$x = 2 + 3i, \text{ or } 2 - 3i.$$

Each of these numbers consists of two parts, a real part $+2$, and an imaginary part $+3i$ or $-3i$. A number like $2 + 3i$, or $2 - 3i$, which consists of two parts, one which is real and the other imaginary, is called a *complex number*. The general form of a complex number is $a + bi$, where a and b are real numbers. a and b may be positive or negative, integral or fractional, rational or irrational.

A complex number $a + bi$ is represented geometrically by a line segment joining the origin O to the point whose coördinates are a and b. Thus, the complex number $2 + 3i$ is represented by the line segment OP_1, Fig. 173, obtained by joining the origin O to the point P_1, whose coördinates are 2, 3. Here also the direction of the line segment as well as its length is to be considered. The line-segments OP_1, OP_2, OP_3, OP_4 are equal in length, but they do not represent the same numbers. OP_1 represents the number $2 + 3i$, OP_2 represents $-2 + 3i$, OP_3 represents $-2 - 3i$, and OP_4 represents $2 - 3i$.

Two directed line segments which have the same length and the same direction are considered equal, and may be taken to represent the same number. It follows that a line segment will continue to represent the same number if it is moved parallel to itself. Thus, OP_1, $X_{-2}Y_3$, P_3O, $Y_{-3}X_2$ being parallel and equal in length, all represent the same complex number $2 + 3i$.

If the directed line segments all have the same initial point O, as OP_1, OP_2, OP_3, OP_4, the terminal points P_1, P_2, P_3, P_4 may be taken to represent the complex numbers as well as the directed segments. In this sense we may speak of the points P_1, P_2, P_3, P_4 as representing the numbers $2 + 3i$, $-2 + 3i$, $-2 - 3i$, $2 - 3i$, respectively.

If in $a + bi$ we put $b = 0$, we get all possible real numbers a; if $a = 0$, we get all possible imaginary numbers bi, so that the complex numbers $a + bi$ include as special cases all real and imaginary numbers.

146. Trigonometric Representation of Complex Numbers.

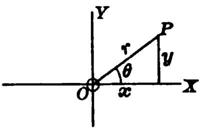

Fig. 174.

Let P (Fig. 174) represent the point $x + iy$. Let r denote the length OP and θ the angle which OP makes with the x-axis. Then

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad (1)$$

and

$$x + iy = r(\cos\theta + i\sin\theta). \qquad (2)$$

$r(\cos\theta + i\sin\theta)$ is called the *trigonometric form* of the complex number $x + iy$; r, which is always positive, being the distance of the point P from the origin θ, is called the ***modulus*** or ***absolute value***, and θ is called the ***argument*** or ***amplitude*** of the complex number $x + iy$.

From (1) we obtain

$$\cos\theta = \frac{x}{r},\quad \sin\theta = \frac{y}{r},\quad \tan\theta = \frac{y}{x},\quad r = \sqrt{x^2+y^2}. \qquad (3)$$

The equations (3) enable us to express any complex number in the trigonometric form. Thus, if $x + iy = 3 + 4i$, we find

$$r = \sqrt{3^2+4^2} = 5,\quad \cos\theta = \tfrac{3}{5},\quad \sin\theta = \tfrac{4}{5},\quad \tan\theta = \tfrac{4}{3},$$

so that

$$3 + 4i = 5(\cos 53^\circ 8' + i \sin 53^\circ 8').$$

If $x + iy = 3 - 4i$, $x = 3$, $y = -4$, and we have

$$r = \sqrt{3^2 + (-4)^2} = 5,\quad \cos\theta = \tfrac{3}{5},\quad \sin\theta = -\tfrac{4}{5},\quad \tan\theta = -\tfrac{4}{3},$$

from which $\theta = 306^\circ 52'$ or $-53^\circ 8'$,

so that $3 - 4i = 5(\cos 306^\circ 52' + i \sin 306^\circ 52')$,

or

$$\begin{aligned} 3 - 4i &= 5(\cos - 53^\circ 8' + i \sin - 53^\circ 8'),\\ &= 5(\cos 53^\circ 8' - i \sin 53^\circ 8'). \end{aligned}$$

Conversely, if a complex number is given in the trigonometric form, equations (1) enable us to express it in the form $x + iy$. Thus, if the given number is $2(\cos 30^\circ + i \sin 30^\circ)$, $r = 2$, $\theta = 30^\circ$, and we have from (1)

$$x = 2\cos 30^\circ = \sqrt{3},\ y = 2\sin 30^\circ = 1,$$

so that

$$2(\cos 30^\circ + i \sin 30^\circ) = \sqrt{3} + i.$$

147. Geometric Addition and Subtraction of Complex Numbers. Let OP and OP' (Fig. 175) represent any two complex numbers $x + iy$ and $x' + iy'$ respectively. Complete the parallelogram $OPQP'$, having OP and OP' for two of its sides. Draw QC perpendicular and PD parallel to OX. The right triangles PDQ and OBP' are equal (Why?), hence

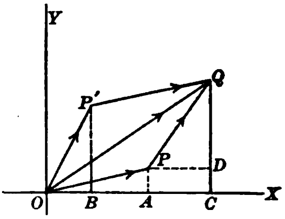

Fig. 175.

$$PD = OB = x', \qquad DQ = BP' = y',$$

and therefore

$$OC = OA + PD = x + x',\ CQ = AP + DQ = y + y',$$

so that the directed line OQ represents the complex number

$$(x + x') + i(y + y') = (x + iy) + (x' + iy').$$

The sum of two complex numbers represented by OP and OP′ respectively is represented by the diagonal drawn from O of the parallelogram formed with OP and OP′ as sides; or,

To find geometrically the sum of two complex numbers represented by OP and OP′ respectively, move OP′ parallel to itself so that its initial point falls on the terminal point P of OP. The directed line, drawn to connect the initial point O of OP to the terminal point of OP′ in its new position, represents the required sum.

In its second form this rule may be easily extended to the sum of any number of complex numbers. To add geometrically the numbers represented by OP, OP_1, OP_2, OP_3, etc., we first add any two of them, as OP and OP_1; to their sum we add any third, as OP_2; to the sum of these three we add a fourth, as OP_3, etc. Leaving out the lines which are not needed to obtain the final result, we obtain the following rule:

Construct a broken line $OPQRS$ . . . such that OP coincides with OP, PQ is equal and parallel to OP_1, QR is equal and parallel to OP_2, RS is equal and parallel to OP_3, etc. The directed line joining O to the terminal point of this broken line represents the required sum.

The length of the line representing the sum of two or more complex numbers will be less, or at most equal, to the sum of the lengths of the several lines representing the numbers added, hence

The modulus of the sum of any number of complex numbers is less than, or at most equal to, the sum of their moduli.

Geometric subtraction follows from geometric addition. Let OQ and OP (Fig. 175) represent any two complex numbers whose difference $OQ - OP$ is to be found geometrically. Since $OP + OP' = OQ$, $OQ - OP = OP' = PQ$. PQ, or its equal OP', represents then the required difference, in words,—

The difference OQ − OP between two complex numbers, represented by OQ and OP respectively, is represented by the third side PQ of the triangle of which the other two sides are OP and OQ.

148. Application of Complex Quantities to Physics. It is shown in physics that if OP and OP' (Fig. 175) represent in length two forces acting in the directions indicated by the arrows, then OQ, the diagonal of the parallelogram of which OP and OP' form the sides, will represent their resultant, both as regards magnitude and direction. It follows that if OQ represents a given force in magnitude and direction, and OP its component in the direction OP, $PQ = OP'$ must represent the other component. The laws of composition and resolution of forces are then precisely those which govern the geometrical addition and subtraction of complex numbers; we may therefore compound or resolve forces by adding or subtracting the complex numbers which represent them.

Suppose n forces f_1, f_2, f_3, . . . f_n (Fig. 176) act in the same plane on the same point P with the respective intensities r_1, r_2, r_3, . . . r_n pounds, and at angles θ_1, θ_2, θ_3, . . . θ_n with the x-axis. The forces are then represented by the complex numbers: —

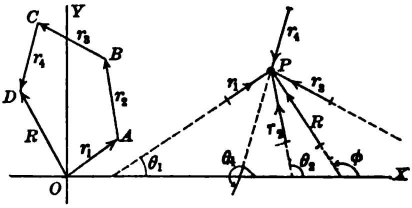

Fig. 176.

$$f_1 = r_1 (\cos\theta_1 + i \sin\theta_1),$$
$$f_2 = r_2 (\cos\theta_2 + i \sin\theta_2),$$
$$f_3 = r_3 (\cos\theta_3 + i \sin\theta_3),$$
$$. \quad . \quad . \quad . \quad .$$
$$f_n = r_n (\cos\theta_n + i \sin\theta_n).$$

Adding all these quantities we obtain for the resultant:

$$\begin{aligned} F &= r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 + \ . \ . \ . + r_n \cos\theta \\ &\qquad + i\,(r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 + \ . \ . \ . + r_n \sin\theta_n) \\ &= R\,(\cos\phi + i \sin\phi), \end{aligned}$$

where R is the modulus and ϕ the argument of the complex numbers representing the resultant force F.

The resultant may be found geometrically by constructing OA equal and parallel to r_1, AB equal and parallel to r_2, BC equal and parallel to r_3, and so on. Then OD, the directed line joining the origin O to the terminal extremity of the broken line thus constructed, represents the resultant force F both as regards magnitude and direction.

We will apply the method just explained to the solution of Problem 16, Exercise 26. Taking AP for the direction of the x-axis we have

$$r_1=15,\ r_2=6,\quad r_3=5.7,\quad r_4=7.9,\quad r_5=12.3,\quad r_6=10.$$

$$\theta_1=0^\circ,\ \theta_2=12^\circ\,30',\theta_3=31^\circ\,21',\theta_4=47^\circ\,46',\theta_5=58^\circ\,10',\theta_6=72^\circ\,18'.$$

$$\begin{array}{lll}
f_1 = & 15\,(\cos 0^\circ + i\sin 0^\circ) & = 15.0000 + i\ \ 0.0000 \\
f_2 = & 6\,(\cos 12^\circ\,30' + i\sin 12^\circ\,30') & = \ \ 5.8578 + i\ \ 1.2984 \\
f_3 = & 5.7\,(\cos 31^\circ\,21' + i\sin 31^\circ\,21') & = \ \ 4.8678 + i\ \ 2.9657 \\
f_4 = & 7.9\,(\cos 47^\circ\,46' + i\sin 47^\circ\,46') & = \ \ 5.3104 + i\ \ 5.8492 \\
f_5 = & 12.3\,(\cos 58^\circ\,10' + i\sin 58^\circ\,10') & = \ \ 6.4882 + i\,10.4501 \\
\underline{f_6 =} & \underline{10\,(\cos 72^\circ\,18' + i\sin 72^\circ\,18')} & = \ \ \underline{3.0400 + i\ \ 9.5270}
\end{array}$$

Adding

$$F = R\,(\cos\phi + i\sin\phi) \qquad = 40.5642 + i\,30.0904$$

$$R = \sqrt{40.5642^2 + 30.0904^2} = 50.51,$$

$$\phi = \cos^{-1}\frac{40.56}{50.51} = \sin^{-1}\frac{30.09}{50.51} = 36^\circ\,34',$$

whence $$F = 50.51\,(\cos 36^\circ\,34' + i\sin 36^\circ\,34'),$$

that is, the resultant force is 50.51 pounds and acts at an angle $36^\circ\,34'$ with AP.

149. Historical Note. The method of representing complex numbers by points in a plane is often referred to as Argand's representation, after J. R. Argand, a French mathematician, who gave a discussion of the method in 1806. This is another misnomer, for it is now known that Caspar Wessel, a German, published the same method as early as 1799. The method was, however, completely forgotten until in 1831 it was rediscovered and applied by the great Gauss, to whom is generally conceded the credit of having established the true theory of imaginary numbers. To Gauss we owe the term "complex numbers" and the symbol i to represent the imaginary unit. The term "imaginary" was first used by Descartes. The choice of the term was unfortunate, for imaginary numbers as now understood are no more "imaginary" in the ordinary meaning of that term than are negative numbers. In view of their geometrical interpretation the name "lateral numbers" has been suggested in the place of the name "imaginary." The trigonometric form of complex numbers was first used by the great French mathematician Cauchy. The concept of complex numbers and their geometrical representation forms the basis of many of the branches of higher mathematics, and is indispensable as well to the study of theoretical physics.

EXERCISE 62

1. Represent geometrically each of the numbers,

$$2,\ -3,\ 4i,\ -i,\ 1+i,\ 1-i,\ 2+3i,\ -\tfrac{1}{2}-i.$$

2. Represent geometrically each of the numbers,

$$2(\cos 30^\circ + i \sin 30^\circ),$$

$$\cos 60^\circ + i \sin 60^\circ,\ 3(\cos 120^\circ + i \sin 120^\circ),\ \tfrac{1}{2}(\cos 240^\circ + i \sin 240^\circ),$$

$$\cos 0^\circ + i \sin 0^\circ,\ \cos 90^\circ + i \sin 90^\circ,\ \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right).$$

3. Express the following numbers in the trigonometric form:

$\tfrac{1}{2}\sqrt{2} + i\tfrac{1}{2}\sqrt{2}$, $\tfrac{1}{2} + i\tfrac{1}{2}\sqrt{3}$, $\tfrac{1}{2} - i\tfrac{1}{2}\sqrt{3}$, $\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i$, $-\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i$, i, -1, 1.

Ans. $\cos 45^\circ + i \sin 45^\circ$, $\cos 60^\circ + i \sin 60^\circ$, . . . , $\cos 0^\circ + i \sin 0^\circ$.

4. Express each of the numbers in Problem 2 in the form $x + iy$.

Ans. $\sqrt{3} + i$, $\tfrac{1}{2} + i\tfrac{1}{2}\sqrt{3}$, . . . , $1 + i$.

5. Compare the moduli of the following numbers:

$\sqrt{3} + i$, $\sqrt{3} - i$, $-\sqrt{3} + i$, $-\sqrt{3} - i$, $\sqrt{2} + i\sqrt{2}$, $\sqrt{2} - i\sqrt{2}$, $-\sqrt{2} + i\sqrt{2}$, $-\sqrt{2} - i\sqrt{2}$, $1 + i\sqrt{3}$, $1 - i\sqrt{3}$, $-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$, 2, $2i$, -2, $-2i$.

What conclusion can you draw with reference to the location of the points representing these numbers?

6. Add geometrically $1 + i$ and $2 + i$;
$1 + 2i$, $2 + i$, and $1 - i$; $1 + 2i$, $-2 + 3i$, $1 - i$, $-3 - i$, and $3 - 3i$;
1, i, 2, and $1 - i$.

7. Add $1 + 2i$, $2 + i$, and $1 - i$ geometrically, in three different ways:

(*a*) By adding the first and second and then the third,
(*b*) By adding the first and third and then the second,
(*c*) By adding the second and third and then the first.

8. Subtract geometrically $2 + 3i$ from $3 + 4i$; $2 - i$ from $1 + i$; $3 + 2i$ from $1 - i$.

9. Prove the rule for geometric subtraction separately by means of the relation

$$(x + iy) - (x' + iy') = (x - x') + i(y - y').$$

10. Prove that the sum of the complex numbers representing the sides of a polygon taken in order equals zero.

11. Three forces of 151, 106, and 61 horse power respectively, make angles of 50° 04′ 30″, 211° 20′ 305″, and − 96° respectively with the x-axis Show that the resultant is zero, that is, that the forces are in equilibrium.

150. Multiplication and Division of Complex Numbers. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

represent any two complex numbers. Their product is

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \qquad (1) \end{aligned}$$

Dividing z_1 by z_2 we obtain

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}, \\ &= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos^2\theta - i^2\sin^2\theta) = 1}, \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \qquad (2) \end{aligned}$$

From equations (1) and (2) it appears that,

The product of two complex numbers is another complex number whose modulus is the product of the moduli and whose argument is the sum of the arguments of the numbers.

The quotient of two complex numbers is another complex number whose modulus is the modulus of the dividend divided by the modulus of the divisor, and whose argument is the argument of the dividend diminished by the argument of the divisor.

Corollary 1. Since

$$\begin{aligned} 1 &= \cos 0° + i\sin 0°, \\ i &= \cos 90° + i\sin 90°, \\ -1 &= \cos 180° + i\sin 180°, \end{aligned}$$

the modulus in each case being unity, it follows that:

multiplying any complex number by 1 leaves it unchanged,
multiplying any complex number by i increases its argument by 90°,
multiplying any complex number by − 1 increases its argument by 180°.

In the first case the line segment representing the complex number is left unchanged, in the second case the line segment is turned in the positive direction through an angle of 90°, in the third case the line segment is reversed.

Corollary 2. $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$; therefore $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ are reciprocals, and in general the reciprocal of $r(\cos\theta + i\sin\theta)$ is $\frac{1}{r}(\cos\theta - i\sin\theta)$.

Two numbers such as $r(\cos\theta + i\sin\theta)$ and $r(\cos\theta - i\sin\theta)$ are said to be *conjugate* to each other. Each is called the *conjugate* of the other.

151. Powers of Complex Numbers. By (1), Article 150, we have

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Let us multiply this result by a third complex number z_3, thus:

$$\begin{aligned} z_1z_2z_3 &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \times r_3(\cos\theta_3 + i\sin\theta_3) \\ &= r_1r_2r_3[\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)]. \end{aligned}$$

Similarly we obtain for the product of n factors

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2),\ \ldots\ z_n = r_n(\cos\theta_n + i\sin\theta_n)$$

$$z_1z_2\ldots z_n = r_1r_2\ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)] \quad (1)$$

From (1) we see that,

The modulus of the product of any number of complex numbers is equal to the product of the moduli of the factors, and the argument of the product is equal to the sum of the arguments of the factors.

Now let us suppose that the n factors in (1) are all equal, each factor being

$$z = r(\cos\theta + i\sin\theta),$$

we then have

$$z^n = r^n(\cos n\theta + i\sin n\theta). \quad (2)$$

In particular, if $r = 1$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \quad (3)$$

Equation (3) embodies one of the most famous theorems of modern analysis. It is known as *DeMoivre's** *theorem*, after its discoverer, and may be stated thus:

The argument of the nth power of any complex number is equal to n times the argument of the number. The theorem may be shown to hold for any value of n, negative, fractional, irrational or even imaginary, but unless n is integral $\cos n\theta + i \sin n\theta$ represents but one of the several values which $(\cos\theta + i \sin\theta)^n$ may have.

To illustrate the use of DeMoivre's theorem, we will employ it to raise $\frac{1}{2} + \frac{i}{2}\sqrt{3}$ to the 9th power. Changing $\frac{1}{2} + \frac{i}{2}\sqrt{3}$ to the trigonometric form we have

$$r = 1, \quad \cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}, \quad \text{hence} \quad \theta = \frac{\pi}{3},$$

and

$$\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)^9 = \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)^9 = \cos\frac{9\pi}{3} + i \sin\frac{9\pi}{3} = -1.$$

Similarly

$$(1+i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)\right]^5 = 4\sqrt{2}\left[\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right]$$
$$= -4(1+i).$$

152. Roots of Complex Numbers. Suppose it is required to find the nth root of the complex number $z = r(\cos\theta + i \sin\theta)$. Denote the root $\sqrt[n]{z}$ by

$$z' = r'(\cos\theta' + i \sin\theta').$$

Then $(z')^n = z$, and we have by DeMoivre's theorem

$$[r'(\cos\theta' + i \sin\theta')]^n = r'^n(\cos n\theta' + i \sin n\theta') = r(\cos\theta + i \sin\theta),$$

from which

$$r' = \sqrt[n]{r}, \quad n\theta' = \theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \ldots \theta + 2k\pi,$$

since when an angle is increased by any number of times 2π both the sine and cosine remain unchanged. It follows that

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* DeMoivre (1667–1754) created a large part of that portion of trigonometry which deals with complex numbers. His death has a curious psychological interest. Shortly before his death he slept a little longer each day, until when the limit of twenty-four hours was reached, he died in his sleep.

$$r' = \sqrt[n]{r},\ \theta' = \frac{\theta}{n}, \frac{\theta+2\pi}{n}, \frac{\theta+4\pi}{n}, \frac{\theta+6\pi}{n}, \ldots \frac{\theta+k\pi}{n}$$

and

$$\begin{aligned} z' = \sqrt[n]{z} &= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n}\right) \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), \end{aligned}$$

where k is any integer.

These values are not all different, for when $k = n$ we have

$$\begin{aligned} r^{n}\left(\frac{\cos\theta+2n\pi}{n} + i\sin\frac{\theta+2n\pi}{n}\right) &= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+2\pi\right) + i\sin\left(\frac{\theta}{n}+2\pi\right)\right] \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right), \end{aligned}$$

and similarly

$$r^{\frac{1}{n}}\left(\cos\frac{\theta+2(n+1)\pi}{n} + i\sin\frac{\theta+2(n+1)\pi}{n}\right) = r^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right),$$

and generally

$$\begin{aligned} & r^{\frac{1}{n}}\left(\cos\frac{\theta+2(n+\mu)\pi}{n} + i\sin\frac{\theta+2(n+\mu)\pi}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta+2\mu\pi}{n} + \sin\frac{\theta+2\mu\pi}{n}\right), \end{aligned}$$

so that z' has only n distinct values corresponding to the values

$$k = 0, 1, 2, \ldots n-1,$$

that is,

Every complex number $r(\cos\theta + i\sin\theta)$ has n nth roots given by the formula

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right], \qquad (1)$$

where k has the values $0, 1, 2, \ldots n-1$.

Example 1 To find the fourth roots of 1.

Solution. In the trigonometric form $1 = \cos 0° + i \sin 0°$, hence by the formula (1) above

$$\sqrt[4]{1} = \cos\left(\frac{0 + 2k\pi}{4} + i \sin \frac{0 + 2k\pi}{4}\right), \text{ where } k = 0, 1, 2, \text{ or } 3.$$

Denoting the four roots by z_0, z_1, z_2, z_3 respectively, we have

$$z_0 = \cos\frac{0}{4} + i \sin\frac{0}{4} = 1, \qquad z_1 = \cos\frac{2\pi}{4} + i \sin\frac{2\pi}{4} = i,$$

$$z_2 = \cos\frac{4\pi}{4} + i \sin\frac{4\pi}{4} = -1, \qquad z_3 = \cos\frac{6\pi}{4} + i \sin\frac{6\pi}{4} = -i.$$

Example 2. To find the fifth roots of -1.

Solution. In the trigonometric form $-1 = \cos\pi + i \sin\pi$, hence

$$\sqrt[5]{-1} = \cos\frac{\pi + 2k\pi}{5} + i \sin\frac{\pi + 2k\pi}{5}, \text{ where } k = 0, 1, 2, 3, 4.$$

This gives the five values

$$z_0 = \cos\frac{\pi}{5} + i \sin\frac{\pi}{5},\ z_1 = \cos\frac{3\pi}{5} + i \sin\frac{3\pi}{5},\ z_2 = \cos\pi + i \sin\pi,$$

$$z_3 = \cos\frac{7\pi}{5} + i \sin\frac{7\pi}{5},\ z_4 = \cos\frac{9\pi}{5} + i \sin\frac{9\pi}{5}.$$

The last two roots may be written equally well

$$z_3 = \cos\frac{3\pi}{5} - i \sin\frac{3\pi}{5},\ z_4 = \cos\frac{\pi}{5} - i \sin\frac{\pi}{5},$$

for

$$\cos\frac{7\pi}{5} = \cos\left(2\pi - \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5},\ \sin\frac{7\pi}{5} = \sin\left(2\pi - \frac{3\pi}{5}\right) = -\sin\frac{3\pi}{5},$$

and

$$\cos\frac{9\pi}{5} = \cos\left(2\pi - \frac{\pi}{5}\right) = \cos\frac{\pi}{5},\ \sin\frac{9\pi}{5} = \sin\left(2\pi - \frac{\pi}{5}\right) = -\sin\frac{\pi}{5},$$

so that finally

$$\sqrt[5]{-1} = \cos\frac{\pi}{5} \pm i \sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} \pm i \sin\frac{3\pi}{5},\ \cos\pi + i \sin\pi = -1.$$

153. To Solve the Equation $z^n - 1 = 0$. If $z^n - 1 = 0$, then

$$z^n = 1 = \cos 0 + i \sin 0,$$

and the n roots are

$$z_0 = \cos\frac{0}{n} + i\sin\frac{0}{n} = 1,$$

$$z_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

$$z_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$z_{n-2} = \cos\frac{2(n-2)\pi}{n} + i\sin\frac{2(n-2)\pi}{n},$$

$$z_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}.$$

Now

$$\cos\frac{2(n-1)\pi}{n} = \cos\left(2\pi - \frac{2\pi}{n}\right) = \cos\frac{2\pi}{n},$$

$$\sin\frac{2(n-1)\pi}{n} = \sin\left(2\pi - \frac{2\pi}{n}\right) = -\sin\frac{2\pi}{n},$$

hence

$$z_{n-1} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}, \text{ and similarly } z_{n-2} = \cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n},$$

that is, z_1 and z_{n-1} are conjugate complex numbers, and likewise z_2 and z_{n-2}, z_3 and z_{n-3}, etc., are each pairs of conjugate numbers.

(*a*) Let $n = 2m + 1$, an odd number. Then besides the first root z_0 there are m pairs of conjugate roots, that is, the roots are

$$1, \cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n}, \ldots \cos\frac{2m\pi}{n} \pm i\sin\frac{2m\pi}{n}.$$

(*b*) Let $n = 2m$, an even number. Then

$$z_n = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n} = \cos\pi + i\sin\pi = -1,$$

and the roots are

$$1, \cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n}, \ldots,$$

$$\cos\frac{(m-1)2\pi}{n} \pm i\sin\frac{(m-1)2\pi}{n}, -1.$$

The roots of the equation $z^n - 1 = 0$ are represented geometrically by the n lines (or by their terminal points) drawn from the origin as center to the circumference of a circle, radius unity, so as to divide the circumference into n equal parts, one of these lines coinciding with the positive direction of the x-axis.

154. To Solve the Equation $z^n + 1 = 0$. If $z^n + 1 = 0$, then

$$z^n = -1 = \cos\pi + i\sin\pi,$$

hence the n roots are

$$\begin{aligned}
z_0 &= \cos\frac{\pi}{n} + i\sin\frac{\pi}{n},\\
z_1 &= \cos\frac{3\pi}{n} + i\sin\frac{3\pi}{n},\\
z_2 &= \cos\frac{5\pi}{n} + i\sin\frac{5\pi}{n},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
z_{n-2} &= \cos\frac{(2n-3)\pi}{n} + i\sin\frac{(2n-3)\pi}{n},\\
z_{n-1} &= \cos\frac{(2n-1)\pi}{n} + i\sin\frac{(2n-1)\pi}{n},
\end{aligned}$$

where z_0 and z_{n-1}, z_1 and z_{n-2}, etc., are pairs of conjugate roots.

(*a*) Let $n = 2m + 1$, an odd number. The middle root is

$$z_m = \cos\frac{(2m+1)\pi}{n} + i\sin\frac{(2m+1)\pi}{n} = \cos\pi + i\sin\pi = -1,$$

so that the n roots are

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$
$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n},\ -1.$$

(*b*) Let $n = 2m$, an even number. Then the roots are

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$
$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n}.$$

The roots of the equation $z^n + 1 = 0$ are represented geometrically by the n lines (or by their terminal points) drawn from the origin as center to the circumference of a circle, radius unity, so as to divide the circumference into n equal parts, the positive x-axis being taken to bisect the angle between a pair of consecutive lines.

EXERCISE 63

Compute the following expressions by DeMoivre's theorem, then verify your results by expanding the binomials by the binomial theorem.

1. $(1 + i)^2$. *Ans.* $2i$.

2. $(1 - i)^4$. *Ans.* -4.

3. $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^3$.

4. $\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)^2$. *Ans.* $-i$.

5. $\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$.

6. Show that each of the following expressions equals -1,

$$\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right)^5, \quad \left(\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}\right)^5, \quad (\cos\pi \pm i\sin\pi)^5.$$

7. Construct the points representing the expressions

$\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$, $\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$, etc., in Problem 6.

8. Find the seven seventh roots of 1 and plot the points representing these roots.

Ans. 1, $\cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7}$, $\cos\frac{4\pi}{7} \pm i\sin\frac{4\pi}{7}$, $\cos\frac{6\pi}{7} \pm i\sin\frac{6\pi}{7}$.

9. Find the seven seventh roots of -1 and plot the points representing them.

10. If $z_0, z_1, z_2, \ldots z_{n-1}$ are the roots of the equation $z^n - 1 = 0$, show that

$$z_2 = z_1^2,\ z_3 = z_1^3,\ z_4 = z_1^4,\ \ldots\ z_{n-1} = z_1^{n-1}.$$

11. Find all the values of $\sqrt[5]{1 + i}$ and represent them geometrically. *Ans.* $\sqrt[10]{2}\,(\cos 9^\circ + i\sin 9^\circ)$, $\sqrt[10]{2}\,(\cos 81^\circ + i\sin 81^\circ)$, etc.

155. The Cube Roots of Unity. Let u_0, u_1, u_2 represent the three cube roots of 1, then by the preceding article

$$\begin{aligned}
u_0 &= \cos 0 + i \sin 0 = 1,\\
u_1 &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2},\\
u_2 &= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.
\end{aligned}$$

Now

$$\begin{aligned}
u_1^2 &= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = u_2,\\
u_2^2 &= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = u_1,
\end{aligned}$$

also

$$u_1 u_2 = u_1^3 = u_2^3 = 1,$$

that is,

The square of either of the imaginary cube roots of unity equals the other, and their product equals unity.

We may then denote the cube roots of unity by

$$1,\ \omega,\ \omega^2,$$

where ω is either one of the roots u_1, u_2.

156. The Cube Roots of Any Real or Complex Number. Let $r(\cos\theta + i\sin\theta)$ represent any number and z_0, z_1, z_2 its cube roots.

We have

$$\left.\begin{aligned}
z_0 &= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),\\
z_1 &= r^{\frac{1}{3}}\left(\cos\frac{\theta+2\pi}{3} + i\sin\frac{\theta+2\pi}{3}\right)\\
&= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \omega z_0,\\
z_2 &= r^{\frac{1}{3}}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right)\\
&= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \omega^2 z_0,
\end{aligned}\right\}\qquad(1)$$

where ω has the meaning given it in Article 155.

Furthermore, by applying the results just obtained

$$\omega z_1 = \omega^2 z_0 = z_2, \qquad \omega^2 z_1 = \omega^3 z_0 = z_0, \tag{2}$$

$$\omega z_2 = \omega^3 z_0 = z_0, \qquad \omega^2 z_2 = \omega^4 z_0 = \omega z_0 = z_1. \tag{3}$$

From (1), (2) and (3) it appears that any two cube roots of a number may be obtained by multiplying the third by ω and ω^2 respectively. Thus:

from (1),	from (2)	from (3),
$z_1 = \omega z_0,$	$z_2 = \omega z_1,$	$z_0 = \omega z_2,$
$z_2 = \omega^2 z_0,$	$z_0 = \omega^2 z_1,$	$z_1 = \omega^2 z_2,$

where $\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$

157. Solution of Cubic Equations. Every cubic equation can be expressed in the form

$$a_0x^3 + 3\,a_1x^2 + 3\,a_2x + a_3 = 0, \tag{1}$$

where a_0, a_1, a_2, a_3 are given numbers.

Equation (1) can be replaced by another in which the second term is missing by putting

$$x = \frac{z - a_1}{a_0}. \tag{2}$$

On making this substitution, equation (1) reduces to

$$z^3 + 3\,Hz + G = 0, \tag{3}$$

where

$$H = a_0a_2 - a_1^2,\ G = a_0^2a_3 - 3\,a_0a_1a_2 + 2\,a_1^3. \tag{4}$$

Now put

$$z = u + v, \tag{5}$$

then (3) becomes

$$(u + v)^3 + 3\,H(u + v) + G = u^3 + v^3 + 3(u + v)(H + uv) + G = 0. \tag{6}$$

If furthermore we put

$$H + uv = 0, \quad \text{that is,} \quad uv = -H, \tag{7}$$

then (6) becomes

$$u^3 + v^3 = -G, \tag{8}$$

and substituting for v in (8) its value from (7)

$$u^3 - \frac{H^3}{u^3} = -G, \text{ or } u^6 + Gu^3 - H^3 = 0. \tag{9}$$

(9) is a quadratic equation in u^3. Solving

$$u^3 = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2},$$

$$v^3 = -G - u^3 = \frac{-G \mp \sqrt{G^2 + 4H^3}}{2},$$

so that for either sign we obtain

$$z = u + v = \sqrt{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \sqrt{\frac{-G - \sqrt{G^2 + 4H^3}}{2}}. \qquad (10)$$

From (7) $v = \frac{-H}{u}$, so that (10) may be written $z = u - \frac{H}{u}$,

where $$u = \sqrt{\frac{-G + \sqrt{G^2 + 4H^3}}{2}}.$$

But the cube root of every number has three values, which by Article 156 may be denoted by u, ωu, $\omega^2 u$ respectively, where u is any one of these roots. The three values of z which satisfy the equation (3) are therefore

$$\begin{aligned} z_0 &= u - \frac{H}{u}, \\ z_1 &= \omega u - \frac{H}{\omega u} = \omega u - \frac{\omega^2 H}{u}, \qquad (11) \\ z_2 &= \omega^2 u - \frac{H}{\omega^2 u} = \omega^2 u - \frac{\omega H}{u}, \end{aligned}$$

u being either one of the cube roots of $\frac{-G + \sqrt{G^2 + 4H^3}}{2}$. (12)

In applying this method to the solution of any equation of the form (1), we first find H and G from (4), then u from (12), then the three values of z from (11) and finally the three corresponding values of x from (2).

EXAMPLE 1. Solve the equation $8x^3 + 12x^2 - 42x - 95 = 0$.

Solution. Here $a_0 = 8$, $a_1 = 4$, $a_2 = -14$, $a_3 = -95$.

From (4),

$$H = a_0 a_2 - a_1^2 = -128 = -2^7,$$

$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = -4608 = -2^9 3^2.$$

From (12),

$$\sqrt{G^2+4H^3} = \sqrt{2^{18}\,3^4 - 4.2^{21}} = 7.2^9,$$

$$u = \sqrt[3]{\frac{-G+\sqrt{G^2+4H^3}}{2}} = \sqrt[3]{\frac{2^9 3^2 + 7.2^9}{2}} = 2^4.$$

From (11),

$$z_0 = u - \frac{H}{u} = 2^4 + 2^3 = 24,$$

$$z_1 = \omega u - \frac{\omega^2 H}{u} = 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -12 + 4i\sqrt{3},$$

$$z_2 = \omega^2 u - \frac{\omega H}{u} = 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{1}{2} = i\frac{\sqrt{3}}{2}\right) = -12 - 4i\sqrt{3}.$$

From (2)

$$x_0 = \frac{z_0 - a_1}{a_0} = \frac{24-4}{8} = 2.5,$$

$$x_1 = \frac{z_1 - a_1}{a_0} = \frac{-12 + 4i\sqrt{3} - 4}{8} = -2 + \frac{\sqrt{3}}{2}i,$$

$$x_2 = \frac{z_2 - a_1}{a_0} = \frac{-12 - 4i\sqrt{3} - 4}{8} = -2 - \frac{\sqrt{3}}{2}i.$$

Check. $\left(x + 2 - \frac{\sqrt{3}}{2}i\right)\left(x + 2 + \frac{\sqrt{3}}{2}i\right) = x^2 + 4x + \frac{19}{4} = 0,$

$$(x^2 + 4x + \tfrac{19}{4})(x - \tfrac{5}{2}) = x^3 + \tfrac{3}{2}x^2 - \tfrac{21}{4}x - \tfrac{95}{8} = 0,$$

or $$8x^3 + 12x^2 - 42x - 95 = 0.$$

158. The Irreducible Case. When G^2+4H^3 is positive, as in the preceding example, its square root is real, and u, which is the cube root of $\frac{-G+\sqrt{G^2+4H^3}}{2}$ can be found by the rules of arithmetic. But if G^2+4H^3 is negative, its square root will be imaginary, and we must employ DeMoivre's theorem to find u. This is the so-called "irreducible case" of the cubic equation. Its solution is as follows:

Since G^2+4H^3 is negative, $-(G^2+4H^3)$ will be positive, and we may put

$$u^3 = \frac{-G + i\sqrt{-(G^2+4H^3)}}{2} = r(\cos\theta + i\sin\theta),$$

whence, by Article 146, (3),

$$r = \sqrt{\frac{(-G)^2 + [-(G^2 + 4H^3)]}{4}} = \sqrt{-H^3}, \tag{1}$$

$$\cos\theta = \frac{-G}{2\sqrt{-H^3}}. \tag{2}$$

$$u = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right) = \sqrt{-H}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),$$

$$\frac{-H}{u} = \sqrt{-H}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right),$$

$$\omega u = \sqrt{-H}\left(\cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}\right),$$

$$\frac{-H}{\omega u} = \sqrt{-H}\left(\cos\frac{\theta + 2\pi}{3} - i\sin\frac{\theta + 2\pi}{3}\right),$$

$$\omega^2 u = \sqrt{-H}\left(\cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}\right),$$

$$\frac{-H}{\omega^2 u} = \sqrt{-H}\left(\cos\frac{\theta + 4\pi}{3} - i\sin\frac{\theta + 4\pi}{3}\right).$$

Hence,

$$\begin{aligned} z_0 &= u - \frac{H}{u} = 2\sqrt{-H}\cos\frac{\theta}{3}, \\ z_1 &= \omega u - \frac{H}{\omega u} = 2\sqrt{-H}\cos\frac{\theta + 2\pi}{3}, \\ z_2 &= \omega^2 u - \frac{H}{\omega^2 u} = 2\sqrt{-H}\cos\frac{\theta + 4\pi}{3}, \end{aligned} \tag{3}$$

and from Article 157, (2),

$$x = \frac{z - a_1}{a_0}. \tag{4}$$

EXAMPLE I. Solve the equation $3x^3 + 3x^2 - 3x - 2 = 0$.

Solution. $a_0 = 3$, $a_1 = 1$, $a_2 = -1$, $a_3 = -2$.

$H = a_0^2 a_2 - a_1^2 = -4$, $G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = -7$,

$G^2 + 4H^3 = -207$, hence we are dealing with an irreducible case.

From (2),

$$\cos\theta = \frac{-G}{2\sqrt{-H^3}} = \frac{7}{16} = 0.4375, \quad \theta = 64^\circ\, 3'\, 20''.$$

From (3),

$$z_0 = 2\sqrt{-H}\cos\frac{\theta}{3} = 4\cos 21^\circ\, 21'\, 7'' = 3.7256,$$

$$z_1 = 2\sqrt{-H}\cos\frac{\theta + 2\pi}{3} = 4\cos 141^\circ\, 21'\, 7'' = -3.1241,$$

$$z\ = 2\sqrt{-H}\cos\frac{\theta + 4\pi}{3} = 4\cos 261^\circ\, 21'\, 7'' = -0.6015.$$

From (4),

$$x_0 = \frac{z_0 - a_1}{a_0} \qquad x_1 = \frac{z_1 - a_1}{a_0} \qquad x_2 = \frac{z_2 - a_1}{a_0}$$

$$= 0.9085, \qquad = -1.3747, \qquad = -0.5338.$$

Exercise 64

1. $\omega_1 = \dfrac{-1 + i\sqrt{3}}{2}$, $\omega_2 = \dfrac{-1 - i\sqrt{3}}{2}$.

By actual multiplication, show that

$$\omega_1^2 = \omega_2,\ \omega_2^2 = \omega_1,\ \omega_1\omega_2 = 1,\ \omega_1^3 = 1,\ \omega_2^3 = 1.$$

Show also that $1 + \omega_1 + \omega_2 = 0$.

2. Compute all the cube roots of 27; of $27\,i$.

Ans. $3, \dfrac{-3 + 3i\sqrt{3}}{2}, \dfrac{-3 - 3i\sqrt{3}}{2}$;
$\dfrac{3\sqrt{3} + 3i}{2}, \dfrac{-3\sqrt{3} + 3i}{2}, -3i$.

3. Compute all the cube roots of $1 + i$.

Ans. $\sqrt[6]{2}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)$, $\sqrt[6]{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$,
$\sqrt[6]{2}\left(\cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12}\right)$.

Solve the following equations:

4. $x^3 - 6x - 9 = 0$. *Ans.* $3, \dfrac{-3 \pm i\sqrt{3}}{2}$.

5. $4x^3 - 24x^2 + 45x - 25 = 0$. *Ans.* 1, 2.5, 2.5.

6. $x^3 - 12x - 10 = 0$. *Ans.* 3.8232, − 2.9304, − 0.8928.

7. $x^3 + 9x^2 + 24x + 19 = 0$.
Ans. By use of 4-place tables, −1.4680, −4.8794, −2.6527.

8. A man invests \$5000 and two years later \$2000 more. The interest was added to the principal at the end of each year. At the end of the third year the total increase was found to be \$2058.16. Find the rate per cent of profit. *Ans.* 11 %.

(Suggestion. Let $x = 1 + \text{rate}$.)

9. The inside of a tank is 3 ft. wider than it is deep and 3 ft. longer than it is wide. Its volume is 100 cu. ft. Find the dimensions of the tank. *Ans.* 2.284, 5.284, 8.284.

10. A loan of \$500 is to be repaid in three equal annual payments of \$190 each without further interest. Find the rate per cent.

11. In determining the deflection of a beam, uniformly loaded and supported at its two ends and points of trisection, the following equation occurs:

$$20x^3 - 24x^2 + 3 = 0.$$

Find its roots. *Ans.* 0.4460, 1.0687, − 0.3147.

159. To Express sin *n*θ and cos *n*θ in Terms of sin θ and cos θ.

DeMoivre's theorem enables us to express the sine and cosine of any multiple angle $n\theta$ in terms of powers of the sine and cosine of θ. We need only compare separately the real parts and the imaginary parts of

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n \tag{1}$$

after expanding the right-hand member by the binomial theorem.

For shortness sake let us put

$$\cos\theta = c, \quad \sin\theta = s,$$

the right-hand side of (1) then becomes

$$\begin{aligned}(c+is)^n = c^n + nic^{n-1}s &+ \frac{n(n-1)}{1\cdot 2} i^2c^{n-2}s^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} i^3c^{n-3}s^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} i^4c^{n-4}s^4 + \text{etc.,} \\ = c^n + nic^{n-1}s &- \frac{n(n-1)}{1\cdot 2} c^{n-2}s^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} ic^{n-3}s^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} c^{n-4}s^4 + \text{etc.,}\end{aligned}$$

since $i^2 = -1,\ i^3 = -i,\ i^4 = 1$, etc.

The real part of this expression must equal $\cos n\theta$, since this is the real part of the equivalent left-hand member in (1), and for a like reason the imaginary part must equal $\sin n\theta$, hence we have

$$\cos n\theta = c^n - \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 - \text{etc.} \quad (2)$$

$$\sin n\theta = nc^{n-1}s - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c^{n-5}s^5 - \text{etc.} \quad (3)$$

Thus, if $n = 2$,

$$\cos 2\theta = c^2 - \frac{2(2-1)}{1\cdot 2}c^0 s^2 = c^2 - s^2 = \cos^2\theta - \sin^2\theta,$$

$$\sin 2\theta = 2cs = 2\sin\theta\cos\theta.$$

If $n = 3$,

$$\cos 3\theta = c^3 - \frac{3(3-1)}{1\cdot 2}cs^2 = c^3 - 3cs^2 = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= 4\cos^3\theta - 3\cos\theta,$$

$$\sin 3\theta = 3c^2s - \frac{3(3-1)(3-2)}{1\cdot 2\cdot 3}c^0 s^3 = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta,$$

results which agree with those obtained in Exercise 50, Problems 10 and 11.

160. To Express $\cos\theta$ and $\sin\theta$ in Terms of Sines and Cosines of Multiple Angles. By Article 150, cor. 2,

$\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ are reciprocals.

Put $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$,

$z^n = \cos n\theta + i\sin n\theta$, $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$,

hence

$$z + \frac{1}{z} = 2\cos\theta, \qquad z - \frac{1}{z} = 2i\sin\theta,$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Now let us expand $\left(z+\frac{1}{z}\right)^n$ and $\left(z-\frac{1}{z}\right)^n$ by the binomial theorem,

$$\left(z+\frac{1}{z}\right)^n = 2^n\cos^n\theta = z^n + nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} + \dots$$
$$+\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} + n\frac{1}{z^{n-2}} + \frac{1}{z^n}, \qquad (1)$$

$$\left(z-\frac{1}{z}\right)^n = 2^n i^n\sin^n\theta = z^n - nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} - \dots$$
$$\pm\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} \mp n\frac{1}{z^{n-2}} \pm \frac{1}{z^n}, \qquad (2)$$

where in (2) the upper signs in the last terms are to be used when n is even and the lower signs when n is odd.

Let us group together the first and last term in each of the expressions (1) and (2), also the second and second last, the third and third last, etc.; we may then write

$$2^n\cos^n\theta = \left(z^n+\frac{1}{z^n}\right) + n\left(z^{n-2}+\frac{1}{z^{n-2}}\right)$$
$$+\frac{n(n-1)}{1\cdot 2}\left(z^{n-4}+\frac{1}{z^{n-4}}\right) + \dots \qquad (3)$$

$$2^n i^n\sin^n\theta = \left(z^n \pm \frac{1}{z^n}\right) - n\left(z^{n-2}\pm\frac{1}{z^{n-2}}\right)$$
$$+\frac{n(n-1)}{1\cdot 2}\left(z^{n-4}\pm\frac{1}{z^{n-4}}\right) + \dots \qquad (4)$$

where in (4) the upper signs are to be used when n is even and the lower signs when n is odd. The total number of terms in each binomial expression is one more than the index n, and since we have grouped the terms in pairs, there will be one term left over in case n is even. This term will not contain z at all, for since the exponent of z diminishes by 2 for each successive term, it will be 0, and $z^0 = 1$.

Let us now substitute for $z^n + \frac{1}{z^n}$, $z^n - \frac{1}{z^n}$, etc., their values $2\cos n\theta$, $2i\sin n\theta$, etc., and divide out the common factor 2. This gives

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta$$
$$+\frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \text{etc.}, \qquad (5)$$

$$2^{n-1} i^n \sin^n \theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta - \text{etc.,} \quad (n \text{ even}) \qquad (6)$$

$$= i\left[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\sin(n-4)\theta - \text{etc.,}\right] \quad (n \text{ odd}). \qquad (6')$$

The factor i disappears in either case, for when n is even, say $2m$, we have $i^n = i^{2m} = (-1)^m$, which is $+1$ or -1 according as m is even or odd, and when n is odd, say $2m+1$, we can divide both sides of the equation (6′) by i and have left on the left side $i^{2m} = +1$ or -1 as before.

EXAMPLES. If $n = 2$,

$$2\cos^2\theta = \cos 2\theta + 1, \quad \text{or} \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

$$2i^2\sin^2\theta = \cos 2\theta - 1, \quad \text{or} \quad \sin^2\theta = \frac{1-\sin 2\theta}{2}.$$

If $n = 3$,

$$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta, \quad \cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}.$$

$$2^2 i^3 \sin^3\theta = i(\sin 3\theta - 3\sin\theta), \quad \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}.$$

If $n = 4$,

$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 6, \quad \cos^4\theta = \frac{6 + 4\cos 2\theta + \cos 4\theta}{8}.$$

$$2^3 i^4 \sin^4\theta = \cos 4\theta - 4\cos 2\theta + 6, \quad \sin^4\theta = \frac{6 - 4\cos 2\theta + \cos 4\theta}{8}.$$

EXERCISE 65

1. By the method of Article 159 express $\cos 4\theta$ and $\sin 4\theta$ each in powers of $\sin\theta$ and $\cos\theta$.

Ans. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$,
$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

2. Show that

$$\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}.$$

3. Show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$
$$= \cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5).$$
$$\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta.$$
$$= \sin\theta\,(16\sin^4\theta - 20\sin^2\theta + 5).$$
$$\tan 5\theta = \frac{\tan\theta\,(\tan^4\theta - 10\tan^2\theta + 5)}{5\tan^4\theta - 10\tan^2\theta + 1}.$$

4. By the method of Article 160 express cos θ and sin θ in terms of functions of multiple angles.

$$\cos^5\theta = \frac{20\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}.$$
$$\sin^5\theta = \frac{20\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}.$$

5. Show that

$$\cos^6\theta = \frac{20 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta}{32}.$$
$$\sin^6\theta = \frac{20 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}{32}.$$

CHAPTER XV

TRIGONOMETRIC SERIES AND THE CONSTRUCTION OF TABLES

161. Definition of Infinite Series. An infinite series is an indicated sum of an endless number of terms. Infinite series are of common occurrence in arithmetic and algebra. Thus in

$$\frac{1}{9} = 0.11111 \ldots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \ldots, \qquad (1)$$

the right-hand member is an infinite series. Similarly, every recurring decimal may be expressed as an infinite series. Again, if we divide 1 by $1 - x$, according to the rule for long division we obtain

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \qquad (2)$$

The right-hand member of this is an infinite series. The square root of every number which is not a perfect square and the cube root of every number which is not a perfect cube may be expressed in the form of an infinite series.

Since it is impossible to write down all the terms of an infinite series, the series is not completely determined until we know the law or rule according to which the various terms are formed. When this law is known we can write down any term that is needed. In the first series above, the law for the general term is $\frac{1}{10^n}$, and the series is completely expressed thus:

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^n} + \ldots$$

In like manner the series (2) is completely expressed by

$$1 + x + x^2 + \ldots + x^{n-1} + \ldots$$

In each case n stands for the number of the term.

When the law of formation is clearly apparent from the first few terms, as in the above examples, the general term is not always expressed.

A series is frequently expressed by writing the Greek letter Σ before the general term; thus the second series above may be written $\Sigma\, x^{n-1}$, which is read "summation of x^{n-1}."

The general expression for an infinite series is

$$\Sigma\, u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

where $u_1, u_2, u_3, \ldots u_n$, are the terms of the series.

162. Convergent and Non-Convergent Series. Let S_n stand for the first n terms of the series, thus

$$\begin{aligned} S_2 &= u_1 + u_2, \\ S_3 &= u_1 + u_2 + u_3, \\ S_4 &= u_1 + u_2 + u_3 + u_4, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ S_n &= u_1 + u_2 + u_3 + \cdots + u_n. \end{aligned}$$

As n increases indefinitely (approaches ∞), one of three things must happen, —

(*a*) S_n may approach some finite quantity as its limit.

(*b*) S_n may become larger than every assignable finite quantity.

(*c*) S_n may neither approach a finite limit nor become infinite, but fluctuate between two or more different values.

No other alternative is conceivable. In the first case the series is said to be *convergent*, in the second *divergent*, in the third *oscillating*. Divergent and oscillating series are grouped together under the term *non-convergent* series.

If a series contains a variable, as the series (2), Article 161, the series may be convergent for certain values of the variable and non-convergent for other values.

EXAMPLE 1. Consider the series

$$x + x^2 + x^3 + \cdots + x^n + \cdots$$

(*a*) If $x = \frac{1}{2}$, the series becomes

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4}, \quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8},$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

As n approaches ∞, $\frac{1}{2^n}$ approaches 0 and S_n approaches 1, hence the series comes under (*a*) and is convergent.

(*b*) Next let us put $x = 2$, then the series becomes

$$2 + 2^2 + 2^3 + \cdots + 2^n + \cdots$$

$$S_1 = 2, \quad S_2 = 2 + 4 = 6, \quad S_3 = 2 + 4 + 8 = 14,$$

$$S_n = 2 + 4 + 8 + \cdots + 2^n.$$

Plainly, as n approaches ∞, S_n approaches ∞, hence the series comes under (*b*) and is divergent.

(*c*) Finally, if $x = -1$, the series becomes

$$-1 + 1 - 1 + \cdots + (-1)^n + \cdots$$

$S_1 = -1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$, $S_n = -1$ or 0 according as n is odd or even. In this case S_n neither approaches a finite limit nor becomes infinite, hence the series is oscillating.

Divergent series frequently lead to absurdities, and must therefore be avoided. For instance, if in the series (2), Article 161, we put $x = 2$, the left member becomes $\frac{1}{1 - 2} = -1$, while the right member becomes

$$1 + 2 + 4 + 8 + \cdots;$$

hence for the value $x = 2$, which makes the series divergent, the two members of (2) can no longer be considered equal.

163. Absolutely Convergent Series. A series which remains convergent after all its terms are made positive is said to be *absolutely convergent.*

Convergent series which become divergent when all the terms are made positive are said to be *semi-convergent* or *conditionally convergent.*

For instance, if in (2), Article 161, we put $x = -\frac{1}{2}$, the series becomes

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots,$$

which is convergent, for it remains convergent when all its terms are taken with the positive sign. The series is therefore absolutely convergent.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, but when all the terms are made positive the resulting series is divergent. The given series is therefore a semi-convergent or conditionally convergent series.

Absolutely convergent series are subject to all the fundamental laws of algebra,* that is, they may be added, subtracted, multiplied and divided, like expressions consisting of a finite number of terms. This is not true of semi-convergent and divergent series. Curious results may be arrived at if this is not kept in mind. For example, take the divergent series

$$S = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + \cdots$$
$$0 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Adding
$$\begin{aligned} S &= 2 + 2 + 6 + 6 + 10 + 10 + 14 + 14 + \cdots \\ &= 4 + 12 + 20 + 28 + \cdots \\ &= 4\,(1 + 3 + 5 + 7 + \cdots) \\ &= 4\,S, \text{ which of course is absurd.} \end{aligned}$$

Or take the semi-convergent series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

which is known to be equal to log 2 = 0.69315 ⋯ We may write

$$\begin{aligned} S &= 1 + \frac{1}{2} - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} - \frac{2}{6} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots - \left(\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \cdots\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) \end{aligned}$$

= 0, that is, a constant 0.69312 ⋯ equal to zero, which is absurd. These examples show that we cannot treat an infinite series like we do other expressions until we know whether it is absolutely convergent or not.

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* For the proof of this statement we must refer the student to textbooks on higher algebra, such as Chrystal's Algebra, Chapter XXVI.

A series which is absolutely convergent will of course remain convergent when some or even all of the terms are made negative, but if a series is divergent when all its terms are positive it may become convergent when a certain proportion of its terms are made negative.

164. The Sum of an Infinite Series. When a series converges, not otherwise, the limit which S approaches as n approaches ∞ is called the sum of the series. Thus when we say that the sum of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots$$

is 1, we mean that the limit of

$$S_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

is 1 as n approaches ∞. A divergent series has no sum in the proper sense of that word.

165. The Limit of r^n as n Approaches ∞.

(*a*) Let $r = 1$. 1 multiplied by 1 equals 1 no matter how often the multiplication is repeated. Hence the limit of $r^n = 1$.

(*b*) Let $r > 1$. We may write $r = 1 + d$, where d is some positive quantity. By applying the binomial theorem we have

$$r^n = (1 + d)^n = 1 + nd + \cdots,$$

Now no matter how small d may be, if n is taken sufficiently large nd can be made larger than any assignable quantity, hence we see that as n approaches ∞, r^n approaches ∞ also.

(*c*) Let $r < 1$. Then $\frac{1}{r} > 1$, and since by (*a*) and (*b*) $\left(\frac{1}{r}\right)^n = \frac{1}{r^n}$ approaches ∞ as n approaches ∞, therefore r^n approaches $\frac{1}{\infty} = 0$ as n approaches ∞.

As n approaches ∞, r^n approaches 0, 1 or ∞ according as r is less than, equal to or greater than 1.

166. The Infinite Geometrical Series. Let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

then

$$rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting $S_n - rS_n = (1 - r)S_n = a - ar^n = a(1 - r^n)$,

from which $S_n = \frac{a(1 - r^n)}{1 - r}$.

(*a*) Let r be numerically less than 1. By Article 165 (*c*) as n approaches ∞ r^n approaches 0, hence S_n approaches the limit $\frac{a}{1 - r}$ and the series is convergent. Since the series converges when all its terms are positive, it is absolutely convergent.

(*b*) Let $r = 1$. Then $S_n = a + a + a + \cdots + a = na$, which approaches ∞ as n approaches ∞; hence in this case the series is divergent.

(*c*) Let $r = -1$. Then $S_n = a - a + a - \cdots + (-1)^{n-1}a = 0$ or a, according as n is even or odd. In this case the series is oscillating.

(*d*) Let r be numerically greater than 1. By Article 165 (*b*) as n approaches ∞ r^n approaches ∞, hence $S_n = \frac{a(1 - r^n)}{1 - r}$ approaches ∞ and the series is divergent.

The results may be summed up in the following theorem:

An infinite geometrical series is absolutely convergent if its ratio r is numerically less than 1, non-convergent if its ratio is numerically equal to or greater than 1. When convergent its sum

$$S = \frac{a}{1 - r},$$

where a is the first term and r the ratio.

167. Convergency Test. Let

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

represent any infinite series of positive terms. Let r_n represent the ratio of any term u_n to the preceding term u_{n-1}. Then

$$u_2 = u_1 r_2,\quad u_3 = u_2 r_3,\quad u_4 = u_3 r_4,\quad \cdots u_n = u_{n-1} r_n,\quad \cdots$$

and consequently

$$\begin{aligned}
u_1 &= u_1,\\
u_2 &= u_1 r_2,\\
u_3 &= u_2 r_3 = u_1 r_2 r_3,\\
u_4 &= u_3 r_4 = u_1 r_2 r_3 r_4,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
u_n &= u_{n-1} r_n = u_1 r_2 r_3 r_4 \cdots r_n,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

Adding $u_1 + u_2 + u_3 + u_4 + \cdots = u_1(1 + r_2 + r_2r_3 + r_2r_3r_4 + \cdots)$ (1)

$$< u_1(1 + R + R^2 + R^3 + \cdots) \quad (2)$$

$$> u_1(1 + r + r^2 + r^3 + \cdots) \quad (3)$$

where R is greater than the greatest, and r less than the least, of all the ratios $r_2, r_3, r_4, \ldots r_n, \ldots$

Now (2) is an infinite geometrical series which is convergent provided R is less than 1, and (3) is an infinite geometrical series which is divergent provided r is equal to or greater than 1, hence the infinite series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent provided R is less than 1, divergent provided r is equal to or greater than 1.

The ratio $r_n = \frac{u_n}{u_{n-1}}$ is called the *ratio of convergency* or *test ratio* of the series Σu_n. We have then the following theorems:

A series is absolutely convergent if the test ratio is always less than some number R which is itself less than 1.

A series is divergent if the test ratio is always greater than some number r which is itself equal to or greater than 1.

Nothing is settled in case the test ratio is ultimately equal to 1. In this case other tests must be applied.

These theorems remain true if, not from the first, but after some particular term, say the kth, the test ratio has the values stated. For the sum of k terms is finite, hence the whole series will be convergent or divergent, according as the series beginning with the kth term is convergent or divergent.

The convergency test established in this article is known as the *test-ratio test.* It is sufficient for all the series treated in the remaining portion of this chapter and the chapter following. In fact, the test-ratio test will answer most purposes of elementary mathematics; the cases in which the test fails, that is when the test ratio approaches 1 in the limit, form exceptional cases which can usually be avoided. There are a great many other convergency tests by which the convergency of a series can be settled in doubtful cases. The theory of series forms a separate subject of study, to the development of which many famous mathematicians have devoted their best efforts.

168. Convergency of Special Series.

(*a*) The *exponential series*,*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

where $n!$ (factorial n) stands for $1 \times 2 \times 3 \times \cdots \times n$.

Here $u_{n+1} = \frac{x^n}{n!}$, $u_n = \frac{x^{n-1}}{(n-1)!}$, hence the test ratio

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!} = \frac{x}{n}.$$

If n is taken sufficiently large, $\frac{x}{n}$ will become and remain less than 1 no matter how large x may be, provided only that it is finite; hence the series is absolutely convergent for every finite value of x.

(*b*) The *cosine series*

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \pm \frac{x^{2n}}{(2n)!} \mp \cdots$$

Here the general term is $\pm \frac{x^{2n}}{(2n)!}$, the preceding term $\mp \frac{x^{2n-2}}{(2n-2)!}$, hence the test ratio is

$$\frac{\pm x^{2n}}{(2n)!} \div \frac{\mp x^{2n-2}}{(2n-2)!} = \frac{-x^2}{2n(2n-1)}.$$

When n is taken sufficiently large, this ratio becomes and remains less than 1, therefore the series is absolutely convergent for all finite values of x.

(*c*) The *sine series*

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^{2n+1}}{(2n+1)!} \mp \cdots$$

---

* The reasons for the names given to the series in this article will appear shortly.

In this case the test ratio is

$$\frac{\pm x^{2n+1}}{(2n+1)!} \div \frac{\mp x^{2n-1}}{(2n-1)!} = \frac{-x^2}{2n(2n+1)}.$$

When n is taken sufficiently large this ratio becomes and remains less than unity, therefore the series is absolutely convergent for every finite value of x.

(*d*) The ***logarithmic series***

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots \ \pm \frac{x^n}{n} \mp \ldots$$

$u_n = \pm \frac{x^n}{n}$, $u_{n-1} = \mp \frac{x^{n-1}}{n-1}$, hence the test ratio is

$$\frac{\pm x^n}{n} \div \frac{\mp x^{n-1}}{n-1} = \frac{-(n-1)x}{n}.$$

$\frac{n-1}{n}$ is a proper fraction which approaches 1 as n approaches ∞.

The test ratio will therefore approach a number less than 1, equal to 1, or greater than 1, according as x is less than 1, equal to 1, or greater than 1.

The series is therefore absolutely convergent so long as x is less than 1. The test ratio fails to lead to any co clusion in the case x is equal to or greater than 1.

(*e*) The ***binomial series***

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

The general term is

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!}x^r.$$

The preceding term is

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{(r-1)!}x^{r-1}.$$

Dividing the general term by the preceding term and canceling the common factors, the test ratio reduces to

$$\frac{n-r+1}{r}x = \left(\frac{n+1}{r} - 1\right)x.$$

As r approaches ∞, $\frac{n+1}{r}$ approaches 0; the test ratio therefore approaches $-x$ as its limit, and we may conclude that the series is absolutely convergent so long as x is less than 1.

EXERCISE 66

Write down the first six terms of each of the following series:

1. $\sum n^2$. 2. $\sum \frac{n}{n+1}$. 3. $\sum \frac{nx^n}{n^2+1}$. 4. $\sum \frac{(-1)^n 2^n x^n}{n!}$.

Write down the general term of each of the following series:

5. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$

6. $\frac{x}{x+1} + \frac{x^2}{x+2} + \frac{x^3}{x+3} + \frac{x^4}{x+4} + \cdots$

7. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

8. $\frac{x}{1} - \frac{x^2}{1+x} + \frac{x^3}{1+2x^2} - \frac{x^4}{1+3x^3} + \frac{x^5}{1+4x^4} - \cdots$

Show that the following series are absolutely convergent:

9. $1 + \frac{2^2}{2} + \frac{3^2}{2^2} + \frac{4^2}{2^3} + \cdots + \frac{n^2}{2^{n-1}} + \cdots$

10. $1 + \frac{2 \cdot 2}{3} + \frac{3 \cdot 2^2}{3^2} + \frac{4 \cdot 2^3}{3^3} + \frac{5 \cdot 2^4}{3^4} + \cdots + \frac{n \cdot 2^{n-1}}{3^{n-1}} + \cdots$

11. $2^2 + \frac{3^2 x}{2!} + \frac{4^2 x^2}{3!} + \frac{5^2 x^3}{4!} + \cdots + \frac{(n+1)^2 x^{n-1}}{n!} + \cdots$

12. $1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \cdots + (-1)^n \frac{(2n+1)x^{2n}}{(2n)!} + \cdots$

Examine the following series as to convergency:

13. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots + \frac{n^2}{n!} + \cdots$

14. $1 + \frac{x}{1} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots + \frac{x^n}{n^2} + \cdots$

Ans. Conv. for $x < 1$.

15. $1 + 2!x + 3!x^2 + 4!x^3 + \cdots + n!x^{n-1} + \cdots$

Ans. Div.

16. $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$

Ans. Conv. for $x < 1$, div. for $x > 1$.

17. $\frac{1}{2} - \frac{7}{6} + \frac{17}{12} - \frac{31}{20} + \frac{49}{30} - \frac{71}{42} + \cdots$

$$+ (-1)^{n-1}\left(\frac{n-1}{n} + \frac{n}{n+1}\right) + \cdots$$

(Suggestion. Find $S_2, S_3, S_4, \ldots, S_n = (-1)\frac{n-1}{n}$. From this it is seen that the series is oscillating.)

18. What is wrong with the following reasoning:

$$\begin{aligned} \text{Let } S' &= 1 + 3 + 5 + 7 + 9 + \cdots \\ S'' &= 2 + 4 + 6 + 8 + 10 + \cdots \\ \text{Adding } S' + S'' &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + \cdots \\ 2S' + 2S'' &= 2 + 4 + 6 + 8 + 10 + \cdots = S'', \end{aligned}$$

hence $2S' + S'' = 0$.

19. Show that the series

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n + \cdots$$

where $u_1 > u_2 > u_3 > u_4 > \cdots > u_n$, and u_n approaches 0 as n approaches ∞, is certainly convergent.

169. The Number e. Consider the infinite series

$$1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots \qquad (1)$$

The test ratio is $\frac{1}{n!} \div \frac{1}{(n-1)!} = \frac{1}{n}$, which is less than 1 for all values of n greater than 1, hence the series is convergent.

Let the sum of the series be denoted by e. The sum of the first three terms alone is 2.5, hence e is certainly greater than 2.5. We will show that e is less than 2.75.

$$\begin{aligned} \frac{1}{4!} &= \frac{1}{3!\,4} < \frac{1}{3!\,3}, \\ \frac{1}{5!} &= \frac{1}{3!\,4 \cdot 5} < \frac{1}{3!\,3^2}, \\ \frac{1}{6!} &= \frac{1}{3!\,4 \cdot 5 \cdot 6} < \frac{1}{3!\,3^3}, \quad \text{etc.,} \end{aligned}$$

hence

$$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots<\frac{1}{3!}+\frac{1}{3!\,3}+\frac{1}{3!\,3^2}+\frac{1}{3!\,3^3}+\cdots$$

The series on the right is a geometric series whose first term is $\frac{1}{3!}$ and whose ratio is $\frac{1}{3}$; its sum by Article 166 is equal to

$$\frac{1}{3!}\cdot\frac{1}{1-\frac{1}{3}}=\frac{1}{3!}\cdot\frac{3}{2}=\frac{1}{4}=0.25,$$

therefore $$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots<0.25,$$

and since $$1+\frac{1}{1}+\frac{1}{2!}=2.5,$$

therefore $$e=1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots<2.75,$$

that is $$2.5<e<2.75.$$

To determine e more exactly, we proceed as follows: Denote the terms of the series e by u_1, u_2, u_3, etc.

$$u_1 = 1.000\,000\,00$$

$$u_2=\frac{u_1}{1}=1.000\,000\,00$$

$$u_3=\frac{u_2}{2}=0.500\,000\,00$$

$$u_4=\frac{u_3}{3}=0.166\,666\,67$$

$$u_5=\frac{u_4}{4}=0.041\,666\,67$$

$$u_6=\frac{u_5}{5}=0.008\,333\,33$$

$$u_7=\frac{u_6}{6}=0.001\,388\,89$$

$$u_8=\frac{u_7}{7}=0.000\,198\,41$$

$$u_9=\frac{u_8}{8}=0.000\,024\,80$$

$$u_{10}=\frac{u_9}{9}=0.000\,002\,76$$

$$u_{11} = \frac{u_{10}}{10} = 0.000\,000\,28$$

$$u_{12} = \frac{u_{11}}{11} = 0.000\,000\,03$$

Adding $$S_{12} \qquad = 2.718\,281\,84.$$

This is the sum of the first twelve terms of the series for e. There is an error in the last figure, owing to the neglected part of each of the decimal fractions added, but this error cannot exceed 10×0.5 or 5 in the last decimal place. Besides this error there is the neglected portion of the series, which is $u_{13} + u_{14} + u_{15} + \cdots$.

Now
$$u_{13} = \frac{1}{13\,!}$$

$$u_{14} = \frac{1}{14\,!} < \frac{1}{13\,!\,13}$$

$$u_{15} = \frac{1}{15\,!} < \frac{1}{13\,!\,13^2}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding $$u_{13} + u_{14} + u_{15} + \cdots < \frac{1}{13\,!}\left(1 + \frac{1}{13} + \frac{1}{13^2} + \cdots\right)$$

$$< \frac{1}{13\,!}\,\frac{1}{1 - \frac{1}{13}} = \frac{1}{12\,!\,12}.$$

From the computation above $u_{12} < 0.000\,000\,03$, hence the neglected portion of e is less than $\frac{u_{12}}{12} < 0.000\,000\,003$, that is, less than 3 in the ninth decimal place. Therefore

$$e = 2.718\,281\ldots \text{ correct to six places.}$$

170. The Exponential Series. By the binomial theorem

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots$$

$$= 1 + 1 + \frac{\left(1 - \frac{1}{n}\right)}{2!} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3!} + \cdots. \qquad (1)$$

If n is greater than 1, each term of (1) is numerically equal to or less than the corresponding term of

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad (2)$$

hence the series (1) is convergent for every value of $n > 1$. As n approaches ∞, each term after the second approaches as its limit the corresponding term of (2), hence

the limit of $\left(1 + \frac{1}{n}\right)^n$, as n approaches ∞, equals $e = 2.718 \cdots$. (3)

Next let us expand $\left(1 + \frac{1}{n}\right)^{nx}$ by the binomial theorem.

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + nx\,\frac{1}{n} + \frac{nx\,(nx - 1)}{2!}\,\frac{1}{n^2} + \frac{nx\,(nx - 1)(nx - 2)}{3!}\,\frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{2!} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{3!} + \cdots. \quad (4)$$

If n is greater than 1, each term of (4) is equal to or less than the corresponding term of the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad (5)$$

which in Article 168 was shown to be convergent for every value of x, hence (4) is convergent for every value of x. As n approaches ∞, each term of (4), beginning with the third, approaches as its limit the corresponding term of (5). Hence, as n approaches ∞,

the limit of $\left(1 + \frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ (6)

Finally, by the law of exponents

$$\left[\left(1 + \frac{1}{n}\right)^n\right]^x = \left(1 + \frac{1}{n}\right)^{nx} \quad (7)$$

no matter how large n and x may be. As n approaches ∞, the expression on the right approaches the series (6) as its limit, while the limit of the expression within the brackets on the left equals e, hence in the limit

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots. \quad (8)$$

The series (8) is known as the *exponential series* because it is the equivalent of the exponential function e^x.

The exponential series may be used to compute the number corresponding to any given natural logarithm. Assign to x any given value k, and let N be the number which is obtained by substituting k for x in the series (8). Then $e^k = N$, and since e is the base of the natural system of logarithms, Article 36, we have by the definition of a logarithm,

$$\log_e N = k,$$

that is, N is the number which has k for its natural logarithm.

171. The Logarithmic Series. Put $1+y=e^k$, then $k=\log_e(1+y)$, and

$$(1+y)^x = e^{kx} = e^{x \log_e(1+y)}.$$

By (8), Article 170,

$$(1+y)^x = e^{x\log_e(1+y)} = 1 + x\log_e(1+y) + \frac{[x\log_e(1+y)]^2}{2!} + \frac{[x\log_e(1+y)]^3}{3!} + \cdots$$

By the binomial theorem

$$(1+y)^x = 1 + xy + \frac{x(x-1)}{2!}y^2 + \frac{x(x-1)(x-2)}{3!}y^3 + \cdots$$

By the preceding article the first of these series is absolutely convergent, and so is the second provided $y < 1$, hence we may treat them like algebraic expressions containing a finite number of terms (Article 163). Equating the two series, subtracting 1 from each side and dividing out x, we obtain

$$\log_e(1+y)\left[1 + \frac{x}{2!}\log_e(1+y) + \frac{x^2}{3!}\log_e^2(1+y) + \cdots\right] = y + \frac{x-1}{2!}y^2 + \frac{(x-1)(x-2)}{3!}y^3 + \cdots$$

This equality holds for every value of x. As x approaches 0, the series within the brackets on the left approaches 1, and the series on the right approaches $y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots$, hence in the limit

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots + (-1)^{n-1}\frac{y^n}{n} + \cdots \quad (1)$$

Similarly, if y is negative, but numerically less than 1, we obtain

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots - \frac{y^n}{n} - \cdots \quad (2)$$

The series (1) and (2), which differ only in the sign of y, are known as the *logarithmic series* because they are equivalent to the logarithms of $1+y$ and $1-y$ respectively.

By the aid of (1) and (2) the natural logarithm of any given number between 0 and 1 may be computed, but the actual computation of logarithms will be very much shortened by means of the method explained in the following article.

172. Calculation of Logarithms. The series (1) and (2) of the preceding article have been shown to be absolutely convergent (Article 168, (*d*)); we may therefore subtract the second from the first and obtain

$$\log_e(1+y)-\log_e(1-y)=\log_e\frac{1+y}{1-y}=2\left(y+\frac{y^3}{3}+\frac{y^5}{5}+\frac{y^7}{7}+\cdots\right) \quad (1)$$

If in this series we put $y=\dfrac{1}{2n+1}$, we obtain

$$\log_e\frac{n+1}{n}=2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\cdots\right)$$

or

$$\log_e(n+1)=\log_e n+2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}\right.$$
$$\left.+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\cdots\right) \quad (2)$$

The series (2) is absolutely convergent so long as y is less than 1, that is, so long as n is greater than 0. If n is greater than 1 the series converges rapidly, so that but a few terms need be taken to obtain the first five or six decimal places of the number to which the series converges. The natural logarithm of any number may thus be readily computed provided we already know the logarithm of the next lower number. But the logarithm of 1 to any base is 0 (Article 26), hence the logarithm of 2 may be computed. Knowing $\log_e 2$ we may compute $\log_e 3$, thence $\log_e 4$, etc. Of course, only the logarithms of prime numbers need be computed by means of the series, for the logarithm of any composite number is equal to the sum of the logarithms of the factors of the number.

To compute the natural logarithm of 2 we put in (2) $n=1$, thus

$$\log_e 2=\log_e 1+\left(\frac{2}{3}+\frac{2}{3\cdot 3^3}+\frac{2}{5\cdot 3^5}+\frac{2}{7\cdot 3^7}+\cdots\right) \quad (3)$$

The actual calculation to five places of decimals may be conveniently arranged a follows. Denote the terms of the series in the parenthesis by u_1, u_2, u_3, etc., then

$$\begin{array}{r|l}
\multicolumn{2}{l}{\log_e 1 \qquad\qquad\qquad\qquad\qquad = 0.000\,000\,0} \\
3 & \underline{2.000\,000\,0} \\
3^2 & \underline{0.666\,666\,7} \div 1 = 0.666\,666\,7 = u_1 \\
3^2 & \underline{0.074\,074\,1} \div 3 = 0.024\,691\,4 = u_2 \\
3^2 & \underline{0.008\,230\,5} \div 5 = 0.001\,646\,1 = u_3 \\
3^2 & \underline{0.000\,914\,5} \div 7 = 0.000\,130\,6 = u_4 \\
3^2 & \underline{0.000\,101\,5} \div 9 = 0.000\,011\,3 = u_5 \\
3^2 & \underline{0.000\,011\,3} \div 11 = 0.000\,001\,0 = u_6 \\
\multicolumn{1}{r}{} & 0.000\,001\,3 \div 13 = \underline{0.000\,000\,1} = u_7
\end{array}$$

Adding

$$\log_e 1 + u_1 + u_2 + \cdots + u_7 = 0.693\,147\,2$$

There is an error in the last figure, due to the neglected parts of the fractions added, but this error cannot exceed 7×0.5 or 4 in the last decimal place. Besides, there is the neglected portion of the series, consisting of the terms

$$u_8 = \frac{2}{15 \cdot 3^{15}}$$

$$u_9 = \frac{2}{17 \cdot 3^{17}} < \frac{2}{15 \cdot 3^{15}} \cdot \frac{1}{3^2}$$

$$u_{10} = \frac{2}{19 \cdot 3^{19}} < \frac{2}{15 \cdot 3^{15}} \cdot \frac{1}{3^4}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding $u_8 + u_9 + u_{10} + \cdots < \dfrac{2}{15 \cdot 3^{15}}\left(1 + \dfrac{1}{3^2} + \dfrac{1}{3^4} + \cdots\right)$

The series within the brackets on the right is a geometric series whose ratio is $\dfrac{1}{3^2} = \dfrac{1}{9}$. The sum of this series is $\dfrac{1}{1 - \frac{1}{9}} = \dfrac{9}{8}$, hence the neglected portion of the series is less than $\dfrac{2}{15 \cdot 3^{15}} \cdot \dfrac{9}{8} = \dfrac{2}{120 \cdot 3^{13}}$.

From the computation above we already know that $\dfrac{2}{3^{13}} = 0.000\,001\,3$, hence $\frac{2}{120 \cdot 3^{18}} = 0.000\,000\,01$. The error, to which the above sum 0.693 147 2 is subject, may therefore cause a difference of at most 1 in the sixth decimal place. It follows that $\log_e 2 = 0.693\,15 \ldots$, correct to 5 places of decimals.

Similarly we obtain the following, each correct to 5 places of decimals:

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9}\right) = 1.098\,61 \ldots$$

$$\log_e 4 = 2 \log_e 2 = 1.386\,29 \ldots$$

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7}\right) = 1.609\,44 \ldots$$

$$\log_e 6 = \log_e 2 + \log_e 3 = 1.791\,76 \ldots$$

$$\log_e 7 = \log_e 6 + 2\left(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5}\right) = 1.945\,91 \ldots$$

$$\log_e 8 = 3 \log_e 2 = 2.079\,44 \ldots$$

$$\log_e 9 = 2 \log_e 3 = 2.197\,22 \ldots$$

$$\log_e 10 = \log_e 2 + \log_e 5 = 2.302\,59 \ldots$$

and so on.

It will be observed that the number of terms of the series, which need be computed to obtain the logarithm correct to a given number of decimal places, grows smaller as n grows larger. When n is 43 or more, the first term of the series suffices to give the first five places of the logarithm.

When the natural logarithm of a number is known, the common logarithm is easily obtained from it, for by Article 35

$$\log_{10} N = \frac{\log_e N}{\log_e 10}.$$

Now $\log_e 10$ was just found to be $2.302\,59 \ldots$, hence

$$\log_{10} N = \frac{\log_e N}{2.302\,59 \ldots} = 0.434\,29 \ldots \log_e N \qquad (4)$$

The number 0.434 29 . . . , more accurately 0.434 294 48 . . . , is called the ***modulus***[*] of the common system of logarithms. We have then the following simple rule.

---

[*] The base e, the modulus of the common system, and the natural logarithms of 2, 3 and 5, have been calculated to more than 250 places of decimals.

RULE: *To find the common logarithm of a number, multiply the corresponding natural logarithm by the modulus of the common system.*

Thus for the numbers from 2 to 10 inclusive we find the

Common Logarithms

$$\log_{10} 2 = 0.434\,294 \dots \times 0.693\,15 \dots = 0.301\,03 \dots$$
$$\log_{10} 3 = 0.434\,294 \dots \times 1.098\,61 \dots = 0.477\,12 \dots$$
$$\log_{10} 4 = 0.434\,294 \dots \times 1.386\,29 \dots = 0.602\,06 \dots$$
$$\log_{10} 5 = 0.434\,294 \dots \times 1.609\,44 \dots = 0.698\,97 \dots$$
$$\log_{10} 6 = 0.434\,294 \dots \times 1.791\,76 \dots = 0.778\,15 \dots$$
$$\log_{10} 7 = 0.434\,294 \dots \times 1.945\,91 \dots = 0.845\,10 \dots$$
$$\log_{10} 8 = 0.434\,294 \dots \times 2.079\,44 \dots = 0.903\,09 \dots$$
$$\log_{10} 9 = 0.434\,294 \dots \times 2.197\,22 \dots = 0.954\,24 \dots$$
$$\log_{10} 10 = 0.434\,294 \dots \times 2.302\,59 \dots = 1.000\,00$$

each correct to five places of decimals.

From (4) we have $\log_e N = 2.302\,59 \dots \cdot \log_{10} N.$ (5)

By means of (5) we can readily find any required natural logarithm from a table of common logarithms. It is therefore not very important to have separate tables of natural logarithms. Thus, if at any time we needed to know the natural logarithm of 237.3 we would look up the common logarithm of 237.3 and multiply the result by 2.30259 and obtain

$$\log_e 237.3 = 2.30259 \log_{10} 237.3 = 2.30259 \times 2.37530 = 5.46933 \dots$$

173. Errors Resulting from the Use of Logarithms. In Article 44 were given certain rules governing the accuracy to be expected in the results obtained by the use of logarithmic tables containing a certain number of decimals. Some of these rules we are now able to verify.

Let N be any number obtained by logarithmic computation, and δ the error in the number due to the use of logarithms, so that the true result is $N + \delta$, where of course δ may be negative as well as positive. The difference between the logarithms of the true result and the result N is

$$\log_{10}(N + \delta) - \log_{10} N = \log_{10}\left(\frac{N + \delta}{N}\right) = \log_{10}\left(1 + \frac{\delta}{N}\right)$$

$$= \mu \log\left(1 + \frac{\delta}{N}\right), \text{ where } \mu = 0.43429 \cdots$$

(By Article 172, (4))

$$= \mu\left(\frac{\delta}{N} - \frac{\delta^2}{2N^2} + \frac{\delta^3}{3N^3} - \cdots\right)$$

(By Article 169, (1))

$$= \mu\frac{\delta}{N}, \text{ approximately,}$$

provided δ is small as compared with N.

This error $\mu\frac{\delta}{N}$ in the logarithm of N is the result of the errors in the logarithms as given in the tables.

In a four-place table the value of a unit in the last place is $0.0001 = \frac{1}{10^4}$, in a five-place table it is $0.00001 = \frac{1}{10^5}$, in an n-place table the value of the unit in the last place is $\frac{1}{10^n}$. The neglected part of any one logarithm does not exceed one-half of a unit in the last place; when several logarithms are added the positive and negative errors will tend to offset each other, but in special cases the errors may be accumulative. We will assume that the combined errors do not exceed a unit in the last place. This gives for an n-place table

$$\log_{10}(N + \delta) - \log_{10} N = \mu\frac{\delta}{N} \leqq \frac{1}{10^n},$$

from which

$$\delta \leqq \frac{N}{10^n \mu} = \frac{2.30259}{10^n} N.$$

Putting n successively equal to 3, 4, 5, 6, 7, we find:

In a 3-pl. table, $\delta \leqq 0.002\,3\,N$, or less than $\frac{1}{4}$ of 1 % of N,

In a 4-pl. table, $\delta \leqq 0.000\,23\,N$, or less than $\frac{1}{4}$ of $\frac{1}{10}$ % of N,

In a 5-pl. table, $\delta \leqq 0.000\,023\,N$, or less than $\frac{1}{4}$ of $\frac{1}{100}$ % of N,

In a 6-pl. table, $\delta \leqq 0.000\,002\,3\,N$, or less than $\frac{1}{4}$ of $\frac{1}{1000}$ % of N,

In a 7-pl. table, $\delta \leqq 0.000\,000\,23\,N$, or less than $\frac{1}{4}$ of $\frac{1}{10000}$ % of N.

It follows that the third figure of a number found from a three-place table, the fourth figure of a number found from a four-place table, the fifth figure of a number found from a five-place table, etc., cannot be relied upon with certainty.

EXERCISE 67

1. Compute the natural logarithms of 11, 101, 257, each to four places of decimals. *Ans.* 2.3979, 4.6151, 5.5491.

2. From the results already obtained compute the natural logarithms of 12, 15, 0.05, $\frac{1}{3}$, each to four places of decimals. *Ans.* 2.4849, 2.7081, −2.9957, −1.0986.

3. From the results of Problem 1 compute the common logarithms of 11, 101, 257, each to four places of decimals, and compare your results with those given in the table of common logarithms.

4. By means of a table of common logarithms, find the natural logarithms of 7, 341, 0.0473. *Ans.* $\log_e 341 = 5.4848$, $\log_e 0.0473 = -3.0513$.

5. Prove that

$$\frac{1}{e} = \frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \cdots + \frac{2n}{(2n+1)!} + \cdots$$

6. Compute $\frac{1}{e}$ to five places of decimals, using the series in Problem 5. *Ans.* 0.36788 . . .

7. Show that the compound amount A on P dollars for t years at r %, interest to be added to the principal as fast as it accrues, is

$$A = Pe^{rt}.$$

(Suggestion. Set up the expression for the compound amount when the period is the nth part of a year and find the limit which this expression approaches as n approaches ∞).

8. Sum the series $1 + \frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} + \cdots$

Ans. $\frac{e}{e-1}$.

174. Limiting Values of the Ratios $\frac{\sin x}{x}$, $\frac{\tan x}{x}$, x being the radian measure of the angle.

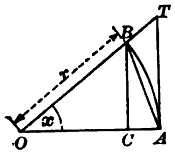

Fig. 177.

Let x be the radian measure of any angle less than $\pi/2$. With the vertex O as a center and any radius $OA = r$ describe an arc cutting the sides of the angle in A and B. Join A and B. From B draw a perpendicular to AO cutting AO in C. At A erect the perpendicular to AO cutting OB produced in T. Then

$$CB = r \sin x, \quad \text{arc}\, AB = rx, \quad AT = r \tan x.$$

The area of triangle $OAB = \dfrac{OA \times CB}{2} = \dfrac{r^2 \sin x}{2}$.

The area of sector $OAB = \dfrac{OA \times \text{arc}\, AB}{2} = \dfrac{r^2 x}{2}$.

The area of triangle $OAT = \dfrac{OA \times AT}{2} = \dfrac{r^2 \tan x}{2}$.

Also triangle $OAB <$ sector $OAB <$ triangle OAT,

that is,
$$\frac{r^2 \sin x}{2} < \frac{r^2 x}{2} < \frac{r^2 \tan x}{2},$$

from which
$$\sin x < x < \tan x \tag{1}$$

for every value of x less than $\frac{\pi}{2}$.

Dividing each term of (1) by the positive quantity $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

hence
$$1 > \frac{\sin x}{x} > \frac{\cos x}{1}. \tag{2}$$

Now let x approach 0, then $\cos x$ approaches 1, therefore $\frac{\sin x}{x}$, which always lies between $\cos x$ and 1, must approach 1 also.

Similarly, if we divide (1) by $\tan x$

$$\cos x < \frac{x}{\tan x} < 1, \tag{3}$$

from which it is seen that $\frac{\tan x}{x}$ approaches 1 as x approaches 0.

Summing up,—

$$\left.\begin{array}{l} \text{As } x \text{ approaches } 0, \text{ the limit of } \dfrac{\sin x}{x} \text{ equals } 1. \\[2ex] \text{As } x \text{ approaches } 0, \text{ the limit of } \dfrac{\tan x}{x} \text{ equals } 1. \end{array}\right\} \tag{4}$$

Corollary.

$$\left.\begin{array}{l} \text{As } n \text{ approaches } \infty, \text{ the limit of } \dfrac{\sin(x/n)}{x/n} \text{ equals } 1. \\[2ex] \text{As } n \text{ approaches } \infty, \text{ the limit of } \dfrac{\tan(x/n)}{x/n} \text{ equals } 1. \end{array}\right\} \tag{5}$$

175. Limiting Value of $\cos^n \frac{x}{n}$ and $\left(\frac{\sin (x/n)}{x/n}\right)^n$ as n approaches ∞.

Put

$$y = \cos^n \frac{x}{n} = \left(1 - \sin^2 \frac{x}{n}\right)^{\frac{n}{2}},$$

then

$$\log y = \frac{n}{2} \log\left(1 - \sin^2 \frac{x}{n}\right)$$

$$= -\frac{n}{2}\left(\sin^2 \frac{x}{n} + \frac{1}{2}\sin^4 \frac{x}{n} + \frac{1}{3}\sin^6 \frac{x}{n} + \cdots\right), \text{ by Art. 171, (2).}$$

The series in the parenthesis is less than

$$\sin^2 \frac{x}{n} + \sin^4 \frac{x}{n} + \sin^6 \frac{x}{n} + \cdots,$$

which is a geometrical series, ratio $\sin^2 \frac{x}{n}$, whose sum is (Art. 166)

$$\frac{\sin^2 (x/n)}{1 - \sin^2 (x/n)} = \frac{\sin^2 (x/n)}{\cos^2 (x/n)} = \tan^2 \frac{x}{n},$$

hence, $\log y$ is numerically less than $\frac{n}{2} \tan^2 \frac{x}{n} = \frac{x}{2} \tan \frac{x}{n} \frac{\tan (x/n)}{x/n}$.

Now let n approach ∞, then $\frac{x}{n}$ approaches 0, $\tan \frac{x}{n}$ approaches 0, and $\frac{\tan (x/n)}{x/n}$ approaches 1 (Article 174, (5)), hence in the limit

$$\log y \text{ approaches } 0,$$

and consequently

$$y = \cos^n \frac{x}{n} \text{ approaches } 1.$$

Also from Article 174, (2),

$$1 > \frac{\sin (x/n)}{x/n} > \cos \frac{x}{n},$$

and therefore

$$1 > \left(\frac{\sin (x/n)}{x/n}\right)^n > \cos^n \frac{x}{n},$$

provided n is taken so large that $\frac{x}{n} < \frac{\pi}{2}$, that is, if n is taken sufficiently large $\left(\frac{\sin(x/n)}{x/n}\right)^n$ lies always between 1 and $\cos^n \frac{x}{n}$, but the latter has just been shown to approach 1 as n approaches ∞, hence $\left(\frac{\sin(x/n)}{x/n}\right)^n$ also approaches 1. Summing up:

As n approaches ∞, the limit of $\cos^n \frac{x}{n}$ equals 1. (1)

As n approaches ∞, the limit of $\left(\frac{\sin(x/n)}{x/n}\right)^n$ equals 1. (2)

176. The Sine, Cosine and Tangent Series. In the equations (2) and (3), Article 159, namely,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\cdot\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta\sin^5\theta - \cdots$$

put $n\theta = x$, so that $\theta = \frac{x}{n}$, then

$$\cos x = \cos^n\frac{x}{n} - \frac{n(n-1)}{2!}\cos^{n-2}\frac{x}{n}\sin^2\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\frac{x}{n}\sin^4\frac{x}{n} - \cdots$$

$$\sin x = n\cos^{n-1}\frac{x}{n}\sin\frac{x}{n} - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\frac{x}{n}\sin^3\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\frac{x}{n}\sin^5\frac{x}{n} - \cdots.$$

An examination of the test ratio shows that both series are absolutely convergent for every value of n. As n approaches ∞,

$$\cos^n\frac{x}{n} \text{ approaches } 1,$$

$$n\cos^{n-1}\frac{x}{n}\sin\frac{x}{n} = x\cos^{n-1}\frac{x}{n}\,\frac{\sin(x/n)}{x/n} \text{ approaches } x,$$

$$\frac{n(n-1)}{2!}\cos^{n-2}\frac{x}{n}\sin^2\frac{x}{n}=\left(1-\frac{1}{n}\right)\frac{x^2}{2!}\cos^{n-2}\frac{x}{n}\left(\frac{\sin(x/n)}{x/n}\right)^2 \text{ appr. } \frac{x^2}{2!},$$

and similarly

$$\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\frac{x}{n}\sin^3\frac{x}{n} \text{ approaches } \frac{x^3}{3!},$$

$$\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\frac{x}{n}\sin^4\frac{x}{n} \text{ approaches } \frac{x^4}{4!}, \text{ etc.,}$$

Hence as n approaches ∞, the above series become

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots \qquad (1)$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots \qquad (2)$$

These series being absolutely convergent (Article 168, (*b*), (*c*)), we may divide (2) by (1) and obtain

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots \qquad (3)$$

The law of the exponents in each of the series (1), (2) and (3) is obvious, and so is the law of the coefficients in the series (1) and (2). The law of the coefficients in the series (3) is too complicated to be worked out by beginners.

177. Computation of Natural Function Tables. We know that the functions of any angle whatever may be expressed in terms of the functions of an angle less than 45°. Moreover, the sine and cosine of any angle between 30° and 45° may be expressed in terms of the sine and cosine of angles less than 30°, for by Article 113, (1),

$$\left.\begin{aligned}\sin x+\sin y&=2\sin\frac{x+y}{2}\cos\frac{x-y}{2},\\ \cos x-\cos y&=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.\end{aligned}\right\}\qquad(1)$$

If in (1) we put for x, $30°+\theta$, and for y, $30°-\theta$, and transpose the second terms on the left to the right-hand side of the equation, we obtain

$$\left.\begin{aligned}\sin(30°+\theta)&=2\sin 30°\cos\theta-\sin(30°-\theta)=\cos\theta-\sin(30°-\theta),\\ \cos(30°+\theta)&=-2\sin 30°\sin\theta+\cos(30°-\theta)=-\sin\theta+\cos(30°-\theta),\end{aligned}\right\}\;(2)$$

from which it appears that the functions of angles between 30° and 45° may be obtained from the functions of angles less than 30° by giving to θ successively the values from 0° to 15°. Thus, if $\theta = 5^\circ$,

$$\sin(30^\circ + \theta) = \sin 35^\circ = \cos 5^\circ - \sin 25^\circ,$$
$$\cos(30^\circ + \theta) = \cos 35^\circ = -\sin 5^\circ + \cos 25^\circ.$$

A complete table of natural functions may therefore be readily constructed provided we know the functions of angles between 0° and 30°.

The sine and cosine of any angle less than 30° may be easily computed by means of the series in Article 176.

Suppose we wish to compute the sine and cosine of 10° correct to four places of decimals.

The radian measure of 10° is $x = \frac{\pi}{18} = 0.174\,53\ldots$, and by means of logarithms we find

$$x^2 = 0.030\,46\ldots, \qquad x^3 = 0.005\,32\ldots,$$
$$x^4 = 0.000\,93\ldots, \qquad x^5 = 0.000\,16\ldots$$

Substituting these values in the series for the sine and cosine we have

$\cos x$	$\sin x$
$1.000\,00$	$x = 0.174\,53\ldots$
$-\frac{x^2}{2\,!} = -0.015\,23\ldots$	$-\frac{x^3}{3\,!} = -0.000\,89\ldots$
$+\frac{x^4}{4\,!} = +0.000\,04\ldots$	$+\frac{x^5}{5\,!} = +0.000\,00\ldots$
$\cos 10^\circ = 0.984\,81\ldots$	$\sin 10^\circ = 0.173\,64\ldots$

In either case the error due to the neglected parts of the decimals added, cannot exceed a unit in the fifth place. In either case the error due to the neglected terms of the series is less than

$$\frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots < \frac{x^6}{6!}(1 + x + x^2 + \cdots)$$

The series in parenthesis is a geometric series whose ratio is $x < 1$. Its sum is $\frac{1}{1 - x}$, hence the neglected terms of either series add up to less than

$$\frac{x^6}{6\,!}\cdot\frac{1}{1-x}=\frac{(0.174\ 53\ldots)^6}{6\,!}\cdot\frac{1}{0.825\ 47\ldots}=0.000\ 000\ 05\ldots$$

In neither case, therefore, can the combined errors affect the fourth decimal place, and we have

$$\cos 10^\circ = 0.9848\ldots,\quad \sin 10^\circ = 0.1736\ldots,$$

each correct to four places.

As a check we have

$$\cos^2 10^\circ + \sin^2 10^\circ = (0.9848)^2 + (0.1736)^2 = 1.0000.$$

It appears from the above computation that if the angle is less than 10° the sine and cosine are given correct to four places by the formulas

$$\cos x = 1 - \frac{x^2}{2\,!},\qquad \sin x = x - \frac{x^3}{3\,!}. \tag{3}$$

These approximation formulas are sufficiently important to be remembered.

We will next show how to compute a table of natural sines and cosines for intervals of 1′.

By means of the sine and cosine series we first compute the sine and cosine of 1′. We find

$$\sin 1' = 0.000\ 290\ 888\ 2\ldots,\quad \cos 1' = 0.999\ 999\ 957\ 7\ldots$$

If now we put in (1) $x = \theta + 1'$, $y = \theta - 1'$, we obtain

$$\begin{aligned}\sin(\theta+1') &= 2\sin\theta\cos 1' - \sin(\theta-1'),\\ \cos(\theta+1') &= -2\sin\theta\sin 1' + \cos(\theta-1').\end{aligned} \tag{4}$$

Putting now $\theta = 1'$ we find

$$\begin{aligned}\sin 2' &= 2\sin 1'\cos 1' - \sin 0' = 0.000\ 581\ 776\ldots\\ \cos 2' &= -2\sin 1'\sin 1' + \cos 0' = 0.999\ 999\ 831\ldots\end{aligned}$$

Next put $\theta = 2'$, then

$$\begin{aligned}\sin 3' &= 2\sin 2'\cos 1' - \sin 1' = 0.000\ 872\ 665\ldots\\ \cos 3' &= -2\sin 2'\sin 1' + \cos 1' = 0.999\ 999\ 619\ldots\end{aligned}$$

Similarly, if $\theta = 3'$

$$\begin{aligned}\sin 4' &= 2\sin 3'\cos 1' - \sin 2' = 0.001\ 163\ 553\ldots\\ \cos 4' &= -2\sin 3'\sin 1' + \cos 2' = 0.999\ 999\ 322\ldots,\ \text{etc.}\end{aligned}$$

To construct a table of sines and cosines for intervals of 10″, we should first compute the sine and cosine of 10″ and then make use of the formulas

$$\sin(\theta + 10'') = 2\sin\theta\cos 10'' - \sin(\theta - 10''),$$
$$\cos(\theta + 10'') = -2\sin\theta\sin 10'' + \cos(\theta - 10'').$$

178. Approximate Equality of Sine, Tangent and Radian Measure of very Small Angles. In Article 174 it was shown that the ratio of the sine to its angle expressed in radians, as well as the ratio of the tangent to its angle expressed in radians, approaches 1 as the angle approaches 0. This means that for very small angles the sine, the tangent and the angle expressed in radians are approximately equal. Thus, if we actually compute the sine and the tangent of 1″ by means of the series in Article 176, and compare the results with the radian measure of 1″, we shall find that the results agree to 15 places of decimals. The sine, tangent and radian measure of 1′ agree to 11 places, and even when the angle is as large as 1° the sine, tangent and radian measure are equal so far as the first five places of decimals are concerned. It follows that when the angle is small, say 1° or less,* we may replace either or both the sine and the tangent by the angle expressed in radians, without affecting the first five places.

EXAMPLE 1. Find the smallest value of x that will satisfy the equation

$$2\sin x + 3x = 0.0513.$$

Solution.

$$3x = 0.0513 - 2\sin x$$
$$< 0.0513, \text{ that is, } x < 0.0171 \text{ (less than } 1^\circ),$$

hence we may replace $\sin x$ by x. Then

$$2x + 3x = 5x = 0.0513,$$
$$x = 0.01026 \text{ radians} = 0^\circ\, 35'\, 17''.$$

NOTE. The methods described in this chapter are not the methods that were used in calculating the tables now in use. The methods actually used were clumsy and laborious as compared with those we have studied. If the tables had to be calculated anew still more refined methods would be used, methods based on the calculus of finite differences, a branch of higher mathematics which cannot well be explained at this point.

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* In fact, the first five places are the same up to 1° 59′, that is, practically 2°.

By means of the differential calculus the sine and cosine series can be much more easily derived than has been done in this chapter. In the differential calculus all the series of Article 168 and many others are derived by means of a single theorem known as Taylor's Theorem.

EXERCISE 68

1. Calculate the sine and cosine of 5° correct to five places.
Ans. $\sin 5^\circ = 0.08716$, $\cos 5^\circ = 0.9962$.

2. Using the results of problem 1, calculate the tangent, cotangent, secant and cosecant of 5°.
Ans. $\sec 5^\circ = 1.0038$, $\csc 5^\circ = 11.4737$.

3. By means of equations (4), Article 177, compute the sine and cosine of 5°.

4. Compute the sine and cosine of 10″ correct to 10 places.
Ans. $\sin 10'' = 0.000\,048\,481\,4$, $\cos 10'' = 0.999\,999\,998\,8$.

By means of the sine and cosine series verify the relations:

5. $\sin^2 x + \cos^2 x = 1$. 6. $\sin 2x = 2 \sin x \cos x$.

7. Find the first three terms of the series for sec x.

$$Ans.\quad \sec x = \frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

8. An angle is to be corrected by an amount δ which is known to be less than 1° and is known to satisfy the equation

$$2\delta = 1.001599 - \cos\delta + \sin\delta.$$

Find δ expressed in minutes and seconds. *Ans.* $\delta = 5' \, 30''$.

9. A straight rail AB, 1 mile long, whose extremities A and B are fixed, expands 1 inch, forming a curve ACB. Assuming the curve to be the arc of some circle, find the distance CD through which the middle point of the rail moved during the expansion.
Ans. 12 ft. 10.14 in

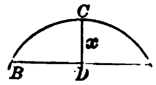

Fig. 178.

Suggestion. Let

$$CD = x, \qquad AD = c, \qquad \text{arc } AC = c + \delta,$$

r = radius of circle of which AB forms an arc,

θ = angle which AB subtends at the center.

Then

$$\frac{c}{r} = \sin\theta, \quad \frac{c+\delta}{r} = \theta, \quad x = r(1 - \cos\theta).$$

Eliminate r and solve for θ on the assumption that θ is very small, a natural assumption which may easily be verified.

CHAPTER XVI

HYPERBOLIC FUNCTIONS

179. Series with Complex Terms. The trigonometric series which we considered in the last chapter had all their terms real. We have now occasion to consider series whose terms are not all real.

Suppose that the terms of a series are complex numbers, the general term being of the form $u_n + iv_n$, then the sum of the first n terms may be written

$$\begin{aligned} S_n &= (u_1 + iv_1) + (u_2 + iv_2) + (u_3 + iv_3) + \cdots + (u_n + iv_n) \\ &= U_n + iV_n, \end{aligned}$$

where

$$\begin{aligned} U_n &= u_1 + u_2 + u_3 + \cdots + u_n, \\ V_n &= v_1 + v_2 + v_3 + \cdots + v_n. \end{aligned}$$

If now, as n approaches ∞, U_n approaches a finite limit U, and V_n approaches a finite limit V, then the series

$$(u_1 + iv_1) + (u_2 + iv_2) + (u_3 + iv_3) + \cdots + (u_n + iv_n) + \cdots,$$

it is said to be convergent, and by its sum S is understood the expression

$$S = U + iV.$$

It follows that in order that a series with complex terms may be convergent it is both necessary and sufficient that the real and the imaginary parts of the series are separately convergent.

If either the real part or the imaginary part of a series is divergent the series is considered divergent.

If the general term $u_n + iv_n$ be expressed in the trigonometric form $r_n (\cos \theta_n + i \sin \theta_n)$, then

$$\begin{aligned} S_n &= r_1 (\cos \theta_1 + i \sin \theta_1) + r_2 (\cos \theta_2 + i \sin \theta_2) \\ &\qquad + r_3 (\cos \theta_3 + i \sin \theta_3) + \cdots + r_n (\cos \theta_n + i \sin \theta_n) \\ &= U_n + iV_n, \end{aligned}$$

where

$$U_n = r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 + \cdots + r_n \cos\theta_n,$$
$$V_n = r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 + \cdots + r_n \sin\theta_n.$$

Each sine and cosine is less than or at most equal to 1, hence

$$U_n \leqq r_1 + r_2 + r_3 + \cdots + r_n,$$
$$V_n \leqq r_1 + r_2 + r_3 + \cdots + r_n,$$

and consequently, as n approaches ∞, U_n and V_n will each approach a finite limit, provided the infinite series

$$r_1 + r_2 + r_3 + \cdots + r_n + \cdots$$

is convergent, that is,

If the moduli of the terms of a series with complex terms form a convergent series, the series is convergent. Since every modulus is positive, the series will in this case be absolutely convergent.

It may be shown that series with complex terms which are absolutely convergent may be added and subtracted, multiplied and divided, by the same rules as polynomials, but this is not true of conditionally convergent series, that is, of convergent series which are not absolutely convergent.

180. Definition of the Trigonometric Functions of Complex Numbers.

If in the exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{1}$$

we place x by $z = x + iy = r(\cos\theta + i\sin\theta)$, the series of the moduli is

$$1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots + \frac{r^n}{n!} + \cdots, \tag{1'}$$

which we know to be convergent for every value of r (Article 168, (*a*)). Hence by Article 179 the series is absolutely convergent. Its sum may conveniently be denoted by $e^z = e^{x+iy} = e^{r(\cos\theta + i\sin\theta)}$, that is,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \tag{2}$$

Similarly we may define cos z and sin z by the series

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \tag{3}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \tag{4}$$

for in each case the series of the moduli consisting of alternate terms of the series (1′) is convergent.

The remaining trigonometric functions of complex numbers are defined by means of the relations

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z}, \quad \csc z = \frac{1}{\sin z}, \quad \sec z = \frac{1}{\cos z}. \tag{5}$$

With (2) as a definition of e^z, it may be shown that

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \text{ and } (e^{z_1})^{z_2} = e^{z_1 z_2},$$

that is, we may treat e^z, where z is a complex number, exactly as we treat e^x, where x is a real number.

Similarly, with (3), (4) and (5) as definitions of the trigonometric functions all the familiar fundamental laws may be shown to hold when z is a complex number as well as when x is a real number. z of course is a number and can be represented by an angle only when it is real.

181. Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$.

First Proof. If in (2), Article 180, we put $z = i\theta$,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned} \tag{1}$$

This result, known as Euler's theorem, is next to DeMoivre's theorem one of the most important results of modern analysis. Since every number real or complex can be expressed in the form $r(\cos\theta + i\sin\theta)$, the theorem may be stated thus:

Every number may be expressed in the form $re^{i\theta}$, where r is the modulus and θ the argument of the number in its trigonometric form.

Second Proof. Euler's theorem may be established by the aid of DeMoivre's theorem as follows. By DeMoivre's theorem

$$\cos\theta + i\sin\theta = \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n, \text{ for every value of } n.$$

Consider θ constant, and let n approach ∞. Then $\sin\frac{\theta}{n}$ approaches $\frac{\theta}{n}$, $\cos\frac{\theta}{n}$ approaches 1, that is,

$$\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n \text{ approaches the same limit as } \left(1 + \frac{i\theta}{n}\right)^n.$$

Put $\frac{i\theta}{n} = \frac{1}{m}$, that is, $n = mi\theta$, then

$$\left(1 + \frac{i\theta}{n}\right)^n = \left(1 + \frac{1}{m}\right)^{mi\theta} = \left[\left(1 + \frac{1}{m}\right)^m\right]^{i\theta}.$$

As n approaches ∞, m also approaches ∞, but

the limit of $\left(1 + \frac{1}{m}\right)^m$, as m approaches ∞, is e (Article 170, (3)),

hence

the limit of $\left(1 + \frac{i\theta}{n}\right)^n$, as n approaches ∞, is $e^{i\theta}$, and we have

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

182. Geometrical Representation of Euler's Theorem. That the limit of $\left(1 + \frac{i\theta}{n}\right)^n$, as n approaches ∞, is $\cos\theta + i\sin\theta$ may be seen from the following geometrical construction.

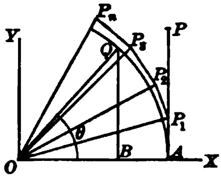

Fig. 179.

Let AOQ be any angle θ, AQ the subtended arc drawn with a unit radius $OA = 1$. At A construct $AP = \text{arc } AQ = \theta$ perpendicular to OA, and let AP_1 be one of the n equal parts of AP. Then $AP_1 = \frac{\theta}{n}$.

Now consider OX the axis of reals and OY the axis of imaginaries, then the directed line OP_1 represents the expression

$$OA + i \cdot AP_1 = 1 + \frac{i\theta}{n}.$$

Construct in succession the similar right triangles OP_1P_2, OP_2P_3, . . . , OP_3P_n, n in number including the triangle AOP_1, each having for its base the hypotenuse of the triangle immediately preceding, then from the proportionality of the homologous sides we have

$$OP_2 : OP_1 = OP_1 : OA\text{ , or } OP_2 = \overline{OP_1}^2 \text{ (since } OA = 1\text{),}$$
$$OP_3 : OP_2 = OP_2 : OP_1\text{, or } OP_3 = \overline{OP_1}^3.$$

. . . .

Similarly $$OP_n = \overline{OP_1}^n.$$

The directed line OP_1 represents $1 + \frac{i\theta}{n}$,

hence the directed line OP_2 represents $\left(1 + \frac{i\theta}{n}\right)^2$,

and the directed line OP_3 represents $\left(1 + \frac{i\theta}{n}\right)^3$,

and finally the directed line OP_n represents $\left(1 + \frac{i\theta}{n}\right)^n$.

Now let n be indefinitely increased, AP_1 becoming correspondingly smaller. As n approaches ∞, the straight lines AP_1, P_1P_2, P_2P_3, . . . approach equality, the broken line $AP_1P_2P_3P_n$ approaches the arc AQ, and the directed line OP_n approaches the directed line OQ. But the directed line OQ represents $OB + i \cdot BQ = \cos\theta + i\sin\theta$, since $OQ = OA = 1$. Hence the limit of $\left(1 + \frac{i\theta}{n}\right)^n$, as n approaches ∞, is $\cos\theta + i\sin\theta$.

183. Exponential Form of the Sine and Cosine. If in

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

we replace θ by $-\theta$, we get

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta, \tag{2}$$

and solving (1) and (2) for $\cos\theta$ and $\sin\theta$ respectively we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{3}$$

If in (1) we put θ successively equal to $\frac{\pi}{2}$, π and 2π, we obtain the important results

$$e^{\frac{i\pi}{2}} = i, \quad e^{i\pi} = -1, \quad e^{2i\pi} = 1, \tag{4}$$

that is, to increase the exponent of e by $\frac{i\pi}{2}$ is to multiply the expression by i; to increase the exponent of e by $i\pi$ is to change the sign of the expression; to increase the exponent of e by $2\,i\pi$ is to leave the expression unchanged.

The relations (3) can be shown to hold for any value of θ whether real or complex. In fact, assuming a knowledge of imaginaries and of *exponentials* (expressions of the form e^z), we might have defined the sine and cosine by means of these relations. With these relations as a basis, the whole of trigonometry becomes an easy application of algebra. We will show by some examples how the formulas of trigonometry could be derived from the relations (3).

Example 1. To prove that $\sin 2\,\theta = 2\sin\theta\cos\theta$.

Solution.

$$\sin 2\,\theta = \frac{e^{2\,i\theta} - e^{-2\,i\theta}}{2\,i} = 2\,\frac{e^{i\theta} - e^{-i\theta}}{2\,i}\cdot\frac{e^{i\theta} + e^{-i\theta}}{2} = 2\sin\theta\cos\theta.$$

Example 2. To prove that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Solution.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{e^{\frac{i\pi}{2} - i\theta} + e^{-\frac{i\pi}{2} + i\theta}}{2} = \frac{ie^{-i\theta} + \frac{1}{i}e^{i\theta}}{2} = \frac{e^{i\theta} - e^{-i\theta}}{2\,i} = \sin\theta.$$

Example 3. To prove that $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$.

Solution.

$$\begin{aligned} 2\sin\theta\cos\phi &= 2\,\frac{e^{i\theta} - e^{-i\theta}}{2\,i}\cdot\frac{e^{i\phi} + e^{-i\phi}}{2} \\ &= \frac{e^{i(\theta+\phi)} - e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}}{2\,i} \\ &= \sin(\theta + \phi) + \sin(\theta - \phi). \end{aligned}$$

Exercise 69

By means of the relations (3) and (4), Article 183, prove the following:

1. $\cos^2\theta + \sin^2\theta = 1$.
2. $\cos^2\theta - \sin^2\theta = \cos 2\,\theta$.
3. $\cos 0 = 1$, $\sin 0 = 0$.
4. $e^{\theta + 2n i\pi} = e^{\theta}$, where n is any integer.

5. $\cos(2n\pi + \theta) = \cos\theta.$ 6. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta.$

7. $\cos(\pi - \theta) = -\cos\theta.$ 8. $2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi).$

9. $\sqrt{\frac{1 + \cos 2\theta}{2}} = \cos\theta,\quad \sqrt{\frac{1 - \cos 2\theta}{2}} = \sin\theta.$

10. By actually multiplying the series for e^u and e^v show that

$$e^u \cdot e^v = e^{u+v}.$$

11. Show that $e^{\frac{i\pi}{3}} + e^{\frac{-i\pi}{3}} = 1.$

12. Show that $\cos i\theta$ and $i \sin i\theta$ are real.

13. Prove Euler's theorem,

$$e^{iz} = \cos z + i \sin z,$$

where z is any complex number.

14. Prove DeMoivre's theorem

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz,$$

where z is any complex number, assuming the result of Problem 13.

184. Hyperbolic Functions Defined. Euler's theorem,

$$\cos\theta + i \sin\theta = e^{i\theta}, \tag{1}$$

is not limited to real values of θ, but holds true when θ is imaginary or complex, for if we put for $\cos\theta$, $\sin\theta$ and $e^{i\theta}$ the series which define these functions for complex values of θ, the two sides of (1) become identically equal. We may therefore replace θ by $i\theta$ and again by $-i\theta$ and obtain

$$\cos i\theta + i \sin i\theta = e^{-\theta},$$
$$\cos i\theta - i \sin i\theta = e^{+\theta}.$$

Half the sum and difference give respectively

$$\cos i\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \qquad -i \sin i\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$$

For reasons which will appear presently these expressions are known as the *hyperbolic cosine* of θ and the *hyperbolic sine* of θ respectively. The *hyperbolic tangent* is defined as the ratio of the hyperbolic sine to the hyperbolic cosine. The reciprocals of the hyperbolic cosine, sine and tangent are called the hyperbolic secant, cosecant and cotangent respectively. Their most common abbreviations are cosh,[*] sinh, tanh, sech, csch, coth.

We have accordingly

$$\begin{aligned} \cosh\theta &= \frac{e^\theta+e^{-\theta}}{2} = \cos i\theta, \\ \sinh\theta &= \frac{e^\theta-e^{-\theta}}{2} = -i\sin i\theta, \\ \tanh\theta &= \frac{e^\theta-e^{-\theta}}{e^\theta+e^{-\theta}} = -i\tan i\theta, \end{aligned} \tag{2}$$

and the reciprocal expressions for sech θ, csch θ and coth θ. Notice that so long as θ is real, e^θ and $e^{-\theta}$ are real and each of the hyperbolic functions is real. Their graphs have already been drawn. (Exercise 61, Problem 8, and Fig. 172.)

185. Duality Between the Circular and Hyperbolic Functions. To every property of or relation between the trigonometric or circular functions there exists a corresponding property of or relation between hyperbolic functions.[†] The formulas expressing these properties and relations could all be derived from the expressions of the hyperbolic functions in terms of exponentials (Article 184, (2)), but an easier way to discover them is to substitute the values

$$\cos i\theta = \cosh\theta,\quad \sin i\theta = i\sinh\theta,\quad \tan i\theta = i\tanh\theta, \tag{1}$$

obtained from Article 184, (2), in the corresponding formulas for the circular functions. We shall illustrate both methods in the examples which follow.

EXAMPLE 1. What relation between hyperbolic functions corresponds to the relation $\cos^2\theta + \sin^2\theta = 1$?

Solution. Put for θ, $i\theta$, then $\cos^2 i\theta + \sin^2 i\theta = 1$. Substituting for $\cos i\theta$ and $\sin i\theta$ their values from (1) we have

$$\cosh^2\theta + i^2\sinh^2\theta = 1,$$

or

$$\cosh^2\theta - \sinh^2\theta = 1,$$

which is the required relation.

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[*] Pronounced cosh, shin, than, shec, etc., in academic slang. Some writers use the abbreviations h-cos or hycos, h-sin or hysin, etc.

[†] Both circular and hyperbolic functions are in fact members of a more general class, the *modo-cyclic* functions, defined by the equations

$$\sin_k\theta = \frac{k}{2}(e^{\theta/k} - e^{-\theta/k}),\quad \cos_k\theta = \tfrac{1}{2}(e^{\theta/k} + e^{-\theta/k}),\quad \tan_k\theta = \sin_k\theta/\cos_k\theta, \text{ etc.}$$

If we had been asked to prove the relation $\cosh^2\theta - \sinh^2\theta = 1$, we might have proceeded as follows:

By Article 184, (2), $$\cosh^2\theta = \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 = \frac{e^{2\theta} + 2 + e^{-2\theta}}{4}$$

$$\sinh^2\theta = \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2 = \frac{e^{2\theta} - 2 + e^{-2\theta}}{4}.$$

Subtracting,

$$\cosh^2\theta - \sinh^2\theta = 1.$$

EXAMPLE 2. What property of hyperbolic functions corresponds to the property $\sin(\theta \pm 2n\pi) = \sin\theta$?

Solution. Replace θ by $i\theta$, then

$$\sin(i\theta \pm 2n\pi) = \sin[i(\theta \mp 2ni\pi)] = \sin i\theta.$$

Introducing the relations (1)

$$i\sinh(\theta \mp 2ni\pi) = i\sinh\theta,$$

that is, $$\sinh(\theta \mp 2ni\pi) = \sinh\theta.$$

This shows that the hyperbolic sine has the imaginary period $2i\pi$ just as the circular sine has the real period 2π. Similarly it may be shown that each of the hyperbolic functions has the period $2i\pi$. The hyperbolic tangent and cotangent have the smaller period $i\pi$.

If we had been asked to verify the relation $\sinh(\theta \mp 2ni\pi) = \sinh\theta$, we might have proceeded thus:

$$\sinh(\theta \mp 2ni\pi) = \frac{e^{\theta \mp 2ni\pi} - e^{-\theta \pm 2ni\pi}}{2} = \frac{e^\theta - e^{-\theta}}{2} = \sinh\theta,$$

for according to Article 183, (4), $e^{\pm 2ni\pi} = 1$.

EXAMPLE 3. What formula for hyperbolic functions corresponds to the formula

$$\sin\theta + \sin\phi = 2\sin\frac{\theta + \phi}{2}\cos\frac{\theta - \phi}{2}?$$

Solution. Replace θ by $i\theta$, and ϕ by $i\phi$, we have

$$\sin i\theta + \sin i\phi = 2\sin\frac{i\theta + i\phi}{2}\cos\frac{i\theta - i\phi}{2},$$

whence by (1)

$$i \sinh \theta + i \sinh \phi = 2\, i \sinh \frac{\theta + \phi}{2} \cosh \frac{\theta - \phi}{2}.$$

Dividing by i

$$\sinh \theta + \sinh \phi = 2 \sinh \frac{\theta + \phi}{2} \cosh \frac{\theta - \phi}{2}.$$

EXAMPLE 4. Prove the relation $\dfrac{2 \tanh \theta}{1 + \tanh^2 \theta}$.

Solution. By Article 184, (2),

$$\tanh \theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \qquad 1 + \tanh^2 \theta = 1 + \left(\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}\right)^2 = \frac{2(e^{2\theta} + e^{-2\theta})}{(e^{\theta} + e^{-\theta})^2},$$

hence

$$\begin{aligned}\frac{2 \tanh \theta}{1 + \tanh^2 \theta} &= \frac{2(e^{\theta} - e^{-\theta})}{e^{\theta} + e^{-\theta}} \cdot \frac{(e^{\theta} + e^{-\theta})^2}{2(e^{2\theta} + e^{-2\theta})} \\ &= \frac{(e^{\theta} - e^{-\theta})(e^{\theta} + e^{-\theta})}{e^{2\theta} + e^{-2\theta}} = \frac{e^{2\theta} - e^{-2\theta}}{e^{2\theta} + e^{-2\theta}} \\ &= \tanh 2\theta.\end{aligned}$$

Below we give for purposes of reference and comparison a table of the principal formulas for the circular functions and in parallel column the corresponding formulas for the hyperbolic functions.

186. Table of Formulas.

	Circular Functions.	*Hyperbolic Functions.*
1.	$\sin \theta \cdot \csc \theta = 1.$	$\sinh u \cdot \operatorname{csch} u = 1.$
2.	$\cos \theta \cdot \sec \theta = 1.$	$\cosh u \cdot \operatorname{sech} u = 1.$
3.	$\tan \theta \cdot \cot \theta = 1.$	$\tanh u \cdot \coth u = 1.$
4.	$\cos^2 \theta + \sin^2 \theta = 1.$	$\cosh^2 u - \sinh^2 u = 1.$
5.	$1 + \tan^2 \theta = \sec^2 \theta.$	$1 - \tanh^2 u = \operatorname{sech}^2 u.$
6.	$\cot^2 \theta + 1 = \csc^2 \theta.$	$\coth^2 u - 1 = \operatorname{csch}^2 u.$
7.	$\sin 0 = 0. \quad \sin \frac{\pi}{2} = 1.$	$\sinh 0 = 0. \quad \sinh(\pm \infty) = \infty.$
8.	$\cos 0 = 1. \quad \cos \frac{\pi}{2} = 0.$	$\cosh 0 = 1. \quad \cosh(\pm \infty) = \pm \infty.$
9.	$\tan 0 = 0. \quad \tan \frac{\pi}{2} = \infty.$	$\tanh 0 = 0. \quad \tanh(\pm \infty) = \pm 1.$

10. $\sin\left(\frac{\pi}{2}\pm\theta\right)=\cos\theta.$ $\sinh\left(\frac{\pi i}{2}\pm u\right)=i\cosh u.$

11. $\cos\left(\frac{\pi}{2}\pm\theta\right)=\mp\sin\theta.$ $\cosh\left(\frac{\pi i}{2}\pm u\right)=\pm i\sinh u.$

12. $\tan\left(\frac{\pi}{2}\pm\theta\right)=\pm\cot\theta.$ $\tanh\left(\frac{\pi i}{2}\pm u\right)=\pm\coth u.$

13. $\sin(\pi\pm\theta)=\mp\sin\theta.$ $\sinh(i\pi\pm u)=\mp\sinh u.$
14. $\cos(\pi\pm\theta)=-\cos\theta.$ $\cosh(i\pi\pm u)=-\cosh u.$
15. $\tan(\pi\pm\theta)=\pm\tan\theta.$ $\tanh(i\pi\pm u)=\pm\tanh u.$

16. $\sin(2\pi+\theta)=\sin\theta.$ $\sinh(2i\pi+u)=\sinh u.$
17. $\cos(2\pi+\theta)=\cos\theta.$ $\cosh(2i\pi+u)=\cosh u.$
18. $\tan(2\pi+0)=\tan\theta.$ $\tanh(2i\pi+u)=\tanh u.$

19. $\sin(-\theta)=-\sin\theta.$ $\sinh(-u)=-\sinh u.$
20. $\cos(-\theta)=\cos\theta.$ $\cosh(-u)=\cosh u.$
21. $\tan(-\theta)=-\tan\theta.$ $\tanh(-u)=-\tanh u.$

22. $\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi.$ $\sinh(u\pm v)=\sinh u\cosh v\pm\cosh u\sinh v.$

23. $\cos(\theta\pm\phi)=\cos\theta\cos\phi\mp\sin\theta\sin\phi.$ $\cosh(u\pm v)=\cosh u\cosh v\pm\sinh u\sinh v.$

24. $\tan(\theta\pm\phi)=\frac{\tan\theta\pm\tan\phi}{1\mp\tan\theta\tan\phi}.$ $\tanh(u\pm v)=\frac{\tanh u\pm\tanh v}{1\pm\tanh u\tanh v}.$

25. $\sin 2\theta=2\sin\theta\cos\theta.$ $\sinh 2u=2\sinh u\cosh u.$

26. $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta.$ $\cosh 2u=\cosh^2 u+\sinh^2 u=2\cosh^2 u-1=1+2\sinh^2 u.$

27. $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$ $\tanh 2u=\frac{2\tanh u}{1+\tanh^2 u}.$

28. $\sin 3\theta=3\sin\theta-4\sin^3\theta.$ $\sinh 3u=3\sinh u+4\sinh^3 u.$
29. $\cos 3\theta=4\cos^3\theta-3\cos\theta.$ $\cosh 3u=4\cosh^3 u-3\cosh u.$
30. $\sin\theta+\sin\phi=2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}.$ $\sinh u+\sinh v=2\sinh\frac{u+v}{2}\cosh\frac{u-v}{2}.$

31.	$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$	$\sinh u - \sinh v = 2\cosh\frac{u+v}{2}\sinh\frac{u-v}{2}.$
32.	$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}.$	$\cosh u + \cosh v = 2\cosh\frac{u+v}{2}\cosh\frac{u-v}{2}.$
33.	$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$	$\cosh u - \cosh v = 2\sinh\frac{u+v}{2}\sinh\frac{u-v}{2}.$
34.	$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$	$\cosh\frac{u}{2} = \sqrt{\frac{\cosh u + 1}{2}}.$
35.	$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}.$	$\sinh\frac{u}{2} = \sqrt{\frac{\cosh u - 1}{2}}.$
36.	$\lim\limits_{\theta \doteq 0}\frac{\sin\theta}{\theta} = 1.$	$\lim\limits_{u \doteq 0}\frac{\sinh u}{u} = 1.$
37.	$\lim\limits_{\theta \doteq 0}\frac{\tan\theta}{\theta} = 1.$	$\lim\limits_{u \doteq 0}\frac{\tanh u}{u} = 1.$
38.	$\lim\limits_{n \doteq \infty}\left(\frac{\sin(\theta/n)}{\theta/n}\right)^n = 1.$	$\lim\limits_{n \doteq \infty}\left(\frac{\sinh(u/n)}{u/n}\right)^n = 1.$
39.	$\lim\limits_{n \doteq \infty}\left(\frac{\tan(\theta/n)}{\theta/n}\right)^n = 1.$	$\lim\limits_{n \doteq \infty}\left(\frac{\tanh(u/n)}{u/n}\right)^n = 1.$
40.	$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$	$\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots$
41.	$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$	$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots$
42.	$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta.$	$(\cosh u \pm \sinh u)^n = \cosh nu \pm \sinh nu.$
43.	$\cos\theta \pm i\sin\theta = e^{\pm i\theta}.$	$\cosh u \pm \sinh u = e^{\pm u}.$
44.	$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$	$\sinh u = \frac{e^u - e^{-u}}{2}.$
45.	$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$	$\cosh u = \frac{e^u + e^{-u}}{2}.$

46. $i \sin \theta = \sinh i\theta.$ $\quad i \sinh u = \sin iu.$

47. $\cos \theta = \cosh i\theta.$ $\quad \cosh u = \cos iu.$

48. $i \tan \theta = \tanh i\theta.$ $\quad i \tanh u = \tan iu.$

EXERCISE 70

1. Prove the formulas (7), (8) and (9) for hyperbolic functions.
2. Verify formula (10) for hyperbolic functions.
3. Verify formula (14) for hyperbolic functions.
4. Verify formula (18) for hyperbolic functions.
5. Verify formula (21) for hyperbolic functions.
6. Verify formula (23) for hyperbolic functions.
7. Derive formula (26) for hyperbolic functions.
8. Derive formula (36) for hyperbolic functions.
9. Derive formula (42) for hyperbolic functions.
10. If $y = \cosh x$, show that $x = \log (y \pm \sqrt{y^2 - 1})$.
11. If $y = \tanh x$, show that $x = \frac{1}{2} \log \left(\frac{1+y}{1-y}\right)$.

187. The Inverse Hyperbolic Functions.

Let
$$y = \cosh x = \frac{e^x + e^{-x}}{2}, \tag{1}$$

then $x = \cosh^{-1} y$ is called the inverse hyperbolic cosine y.

Put $e^x = z$, then $e^{-x} = \frac{1}{z}$, and we have from (1)

$$z + \frac{1}{z} = 2y, \text{ or } z^2 - 2yz + 1 = 0,$$

from which, on solving for z,

$$e^x = z = y \pm \sqrt{y^2 - 1},$$

and taking the logarithm of both sides of this equation,

$$x = \cosh^{-1} y = \log (y \pm \sqrt{y^2 - 1}). \tag{2}$$

In like manner from $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$, (3)

$$x = \sinh^{-1} y, \text{ the inverse hyperbolic sine } y.$$

Putting in (3) $e^x = z$, we obtain

$$z - \frac{1}{z} = 2y, \text{ or } z^2 - 2yz - 1 = 0.$$

Solving for z, $$e^x = z = y \pm \sqrt{y^2 + 1}.$$

The minus sign cannot be used, for e^x is positive for every value of x, while $y - \sqrt{y^2 + 1}$ is negative, $\sqrt{y^2 + 1}$ being greater than y. Hence, on taking the logarithm of both sides of the above equation,

$$x = \sinh^{-1} y = \log(y + \sqrt{y^2 + 1}). \qquad (4)$$

If $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, then $x = \tanh^{-1} y$, the inverse hyperbolic tangent y.

Again, we put $e^x = z$ and solve the resulting equation for z,

$$e^x = z = \sqrt{\frac{1 + y}{1 - y}},$$

from which $$x = \tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}, \qquad (5)$$

In the same manner we obtain for the remaining inverse hyperbolic functions

$$x = \coth^{-1} y = \frac{1}{2} \log \frac{y + 1}{y - 1}. \qquad (6)$$

$$x = \operatorname{sech}^{-1} y = \log \frac{1 \pm \sqrt{1 - y^2}}{y}. \qquad (7)$$

$$x = \operatorname{csch}^{-1} y = \log \frac{1 + \sqrt{1 + y^2}}{y}. \qquad (8)$$

188. Geometrical Representation of Hyperbolic Functions. We will now show that the hyperbolic functions may be expressed as ratios of certain lines connected with the equilateral hyperbola, just as the circular functions are expressed as ratios of certain lines connected with the circle. These relations will be best understood by developing the corresponding results for the circle and hyperbola in parallel columns.

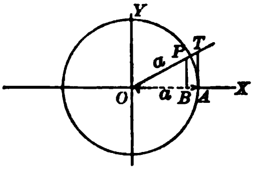

Fig. 180.

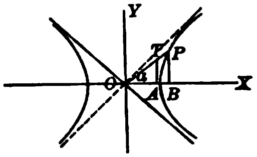

Fig. 181.

Let $x = a \cos \theta$, $y = a \sin \theta$,

then since $\cos^2 \theta + \sin^2 \theta = 1$, $x^2 + y^2 = a^2 (\cos^2 \theta + \sin^2 \theta) = a^2$, that is, x and y are the coördinates of a point P on a circle whose radius is a.

$$\cos \theta = \frac{x}{a} = \frac{OB}{OA},$$

$$\sin \theta = \frac{y}{a} = \frac{BP}{OA},$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{BP}{OB} = \frac{AT}{OA}.$$

If OA is taken for the unit of measure,

OB represents $\cos \theta$,
BP represents $\sin \theta$,
AT represents $\tan \theta$.

Let $x = a \cosh u$, $y = a \sinh u$,

then since $\cosh^2 u - \sinh^2 u = 1$, $x^2 - y^2 = a^2 (\cosh^2 u - \sinh^2 u) = a^2$, that is, x and y are the coördinates of a point P on a hyperbola whose semi-major axis is a.

$$\cosh u = \frac{x}{a} = \frac{OB}{OA},$$

$$\sinh u = \frac{y}{a} = \frac{BP}{OA},$$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{BP}{OB} = \frac{AT}{OA}.$$

If OA is taken for the unit of measure,

OB represents $\cosh u$,
BP represents $\sinh u$,
AT represents $\tanh u$.

It is plain, now, that the hyperbolic functions are related to the hyperbola in the same way that the circular functions are related to the circle. For this reason the functions are named "hyperbolic functions."

189. The Area of a Hyperbolic Sector. In Article 90*a* it was shown that the area of a circular sector is equal to $\frac{a^2\theta}{2}$. We will now derive this result in another way, and show in parallel column that in like manner the area of a hyperbolic sector is equal to $\frac{a^2u}{2}$.

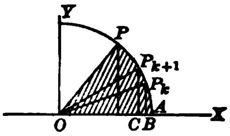

Fig. 182.

Let P be any point on the circle corresponding to θ and

$$P_1, P_2, \cdots P_k, P_{k+1}, \cdots$$

points on the circle corresponding to

$$\frac{\theta}{n}, \frac{2\theta}{n}, \cdots \frac{k\theta}{n}, \frac{(k+1)\theta}{n}, \cdots$$

respectively, n being any integer arbitrarily chosen.

Let the coördinates of P_1, P_2, etc., be x_1, y_1; x_2, y_2; etc., respectively.

The area of triangle OP_kP_{k+1}

$$= \text{area of triangle } OCP_{k+1}$$
$$+ \text{ area of trapezoid } CP_{k+1}P_kB$$
$$- \text{ area of triangle } OBP_k,$$

$$= \frac{x_{k+1}y_{k+1}}{2}$$

$$+ \frac{(x_k - x_{k+1})(y_k + y_{k+1})}{2}$$

$$- \frac{x_ky_k}{2}.$$

Simplifying,

area of triangle OP_kP_{k+1}

$$= \frac{(y_{k+1}x_k - x_{k+1}y_k)}{2}$$

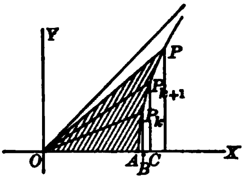

Fig. 183.

Let P be any point on the hyperbola corresponding to u and

$$P_1, P_2, \cdots P_k, P_{k+1}, \cdots$$

points on the hyperbola corresponding to

$$\frac{u}{n}, \frac{2u}{n}, \cdots \frac{ku}{n}, \frac{(k+1)u}{n}, \cdots$$

respectively, n being any integer arbitrarily chosen.

Let the coordinates of P_1, P_2, etc., be x_1, y_1; x_2, y_2; etc., respectively.

The area of triangle OP_kP_{k+1}

$$= \text{area of triangle } OCP_{k+1}$$
$$- \text{ area of trapezoid } CP_{k+1}P_kB$$
$$- \text{ area of triangle } OBP_k,$$

$$= \frac{x_{k+1}y_{k+1}}{2}$$

$$- \frac{(x_{k+1} - x_k)(y_k + y_{k+1})}{2}$$

$$- \frac{x_ky_k}{2}.$$

Simplifying,

area of triangle OP_kP_{k+1}

$$= \frac{(y_{k+1}x_k - x_{k+1}y_k)}{2}$$

$$= \frac{a^2}{2}\left[\sin\frac{(k+1)\theta}{n}\cos\frac{k\theta}{n} - \cos\frac{(k+1)\theta}{n}\sin\frac{k\theta}{n}\right],$$

$$= \frac{a^2}{2}\sin\left[\frac{(k+1)\theta}{n} - \frac{k\theta}{n}\right],$$

$$= \frac{a^2}{2}\sin\frac{\theta}{n}. \qquad (1)$$

Since (1) is the area of any one of the triangles which make up the polygon

$$OAP_1P_2\ .\ .\ .\ P_kP_{k+1}\ .\ .\ .\ P,$$

the entire area of the polygon

$$= n\frac{a^2}{2}\cdot\sin\frac{\theta}{n}$$

$$= \frac{a^2\theta}{2}\cdot\frac{\sin(\theta/n)}{\theta/n}.$$

Now let n be indefinitely increased, then the area of the polygon approaches as its limit the area of the circular sector OAP, which we will denote by S_c, that is,

$$S_c = \lim_{n \doteq \infty}\left[\frac{a^2\theta}{2}\cdot\frac{\sin(\theta/n)}{\theta/n}\right]$$

$$= \frac{a^2\theta}{2}\cdot\lim_{n \doteq \infty}\frac{\sin(\theta/n)}{\theta/n};$$

but by Article 186, (38)

$$\lim_{n \doteq \infty}\frac{\sin(\theta/n)}{\theta/n} = 1,$$

therefore

$$S_c = \frac{a^2\theta}{2}. \qquad (3)$$

$$= \frac{a^2}{2}\left[\sinh\frac{(k+1)u}{n}\cosh\frac{ku}{n} - \cosh\frac{(k+1)u}{n}\sinh\frac{ku}{n}\right],$$

$$= \frac{a^2}{2}\sinh\left[\frac{(k+1)u}{n} - \frac{ku}{n}\right],$$

$$= \frac{a^2}{2}\sinh\frac{u}{n}. \qquad (1)$$

Since (1) is the area of any one of the triangles which make up the polygon

$$OAP_1P_2\ .\ .\ .\ P_kP_{k+1}\ .\ .\ .\ P,$$

the entire area of the polygon

$$= n\frac{a^2}{2}\cdot\sinh\frac{u}{n}$$

$$= \frac{a^2u}{2}\cdot\frac{\sinh(u/n)}{u/n}. \qquad (2)$$

Now let n be indefinitely increased, then the area of the polygon approaches as its limit the area of the hyperbolic sector OAP, which we will denote by S_h, that is,

$$S_h = \lim_{n \doteq \infty}\left[\frac{a^2u}{2}\cdot\frac{\sinh(u/n)}{u/n}\right]$$

$$= \frac{a^2u}{2}\cdot\lim_{n \doteq \infty}\frac{\sinh(u/n)}{u/n};$$

but by Article 186, (38)

$$\lim_{n \doteq \infty}\frac{\sinh(u/n)}{u/n} = 1,$$

therefore

$$S_h = \frac{a^2u}{2}. \qquad (3)$$

From (3) we have

$$\theta = \frac{2\,S_c}{a^2},$$

From (3) we have

$$u = \frac{2\,S_h}{a^2},$$

and to make the analogy between the circular and hyperbolic functions complete we may write

$$\frac{x}{a} = \cos\left(\frac{2\,S}{a^2}\right),$$

$$\frac{y}{a} = \sin\left(\frac{2\,S}{a^2}\right),$$

$$\frac{y}{x} = \tan\left(\frac{2\,S}{a^2}\right),$$

where S is the area of the circular sector AOP, and x, y the coördinates of the point P.

$$\frac{x}{a} = \cosh\left(\frac{2\,S}{a^2}\right),$$

$$\frac{y}{a} = \sinh\left(\frac{2\,S}{a^2}\right),$$

$$\frac{y}{x} = \tanh\left(\frac{2\,S}{a^2}\right),$$

where S is the area of the hyperbolic sector AOP, and x, y the coördinates of the point P.

190. Use of Hyperbolic Functions. A knowledge of hyperbolic functions is not only of great importance for the pursuit of higher mathematics, but also because of the use which is made of these functions in various arts and sciences. Some idea of their usefulness may be gathered from the following statement by C. D. Walcott, secretary of the Smithsonian Institution.

"Hyperbolic functions are extremely useful in every branch of physics and in the applications of physics, whether to observational and experimental sciences or to technology. Thus, whenever an entity (such as light, velocity, electricity or radioactivity) is subject to gradual extinction or absorption, the decay is represented by some form of Hyperbolic Functions. Mercator's projection is likewise computed by Hyperbolic Functions. Whenever mechanical strains are regarded as great enough to be measured they are most simply expressed in terms of Hyperbolic Functions. Hence geological deformations invariably lead to such expressions."

Because of the great importance of hyperbolic functions, a complete hyperbolic functions table has recently been published by the Smithsonian Institution.

EXERCISE 71

1. Show that $\coth^{-1} y = \frac{1}{2} \log \frac{y+1}{y-1}$.

2. Show that $\operatorname{sech}^{-1} y = \log \frac{1 \pm \sqrt{1-y^2}}{y}$.

3. Show that $\operatorname{csch}^{-1} y = \log \frac{1 + \sqrt{1+y^2}}{y}$.

4. If S_h denote the area of the hyperbolic sector (Fig. 183) and x, y the coördinates of the point P, show that $S_h = \frac{a^2}{2} \log \frac{x+y}{a}$.

5. Hence find the area of the hyperbolic sector when $x = \frac{5}{3}$, $a = 1$. *Ans.* $S_h = \frac{1}{2} \log_e 3 = 0.5493$.

6. Compute sinh 1 and cosh 1.

Ans. $\sinh 1 = 1.1752$, $\cosh 1 = 1.5431$.

7. Compute $\sinh^{-1} \frac{12}{5}$ and $\cosh^{-1} \frac{5}{3}$.

Ans. $\sinh^{-1} \frac{12}{5} = \log_e 5 = 1.6094$, $\cosh^{-1} \frac{5}{3} = \log_e 3$ or $\log_e \frac{1}{3}$
$= 1.0986$ or -1.0986.

191. Review. 1. (*a*) Explain how any equation of the form $y = f(x)$ may be graphically represented by a curve. (*b*) Draw the graph for each of the equations $x + y = 3$, $x^2 + y^2 = 3^2$, $x^2 - y^2 = 3^2$.

2. (*a*) Draw on the same sheet of paper (using the same coördinate axes) the six trigonometric curves. (*b*) Draw on one sheet of paper four curves of the type $y = a \sin (x + c)$, giving to a and c the values, $a = 1$, $c = 0$; $a = 1$, $c = \frac{\pi}{2}$; $a = 3$, $c = 0$; $a = 3$, $c = \frac{\pi}{2}$. These curves if properly connected by straight lines form the plan of a double-threaded screw of which these curves represent the spirals.

3. (*a*) Draw the curve $y = \log_e x$. (*b*) Draw the curve $y = \log_e \left(\frac{1}{x}\right)$.

4. Draw on one sheet the curves $y = e^{kx}$, putting k successively equal to 1, -1, $\frac{1}{2}$ and $-\frac{1}{2}$.

5. (*a*) Construct the curve $y = \sin x + \cos x$.

(*b*) Draw the curve $y = \sin x + 2 \cos \left(\frac{x}{2}\right)$.

6. Draw on one sheet of paper the curves of the six hyperbolic functions.

7. Construct the curve $y = e^{-x} \sin x$.

8. (*a*) If $U + iV = (a + ib) + (a + ib)^2 + (a + ib)^3$, find U and V.

(*b*) Add geometrically $3 + 4i$ and $4 - 3i$.

9. (*a*) Express $3 + 4i$ in the trigonometric form.

(*b*) If $U + iV = \frac{1}{x + iy}$, find U and V.

(*c*) If $x + iy = r(\cos\theta + i\sin\theta)$, express x and y in terms of r and θ.

(*d*) In (*c*) express r and θ in terms of x and y.

10. (*a*) Prove that the modulus of a product of two factors is equal to the product of the moduli of the factors. (*b*) Prove that the argument of the product of two factors is equal to the sum of the arguments of the factors. (*c*) Prove that the sum of the conjugates of two complex numbers is equal to the conjugate of the sum of the numbers.

11. (*a*) State and prove DeMoivre's theorem for positive integral exponents. (*b*) By DeMoivre's theorem find $(1 + i)^{10}$, $(\frac{1}{2} + \frac{1}{2} i \sqrt{3})^8$.

12. (*a*) Find all the roots of the equation $x^8 - 1 = 0$.

(*b*) Find all the roots of the equation $x^8 + 1 = 0$.

(*c*) If ω_1 and ω_2 are the imaginary cube roots of unity, show that $\omega_1^2 = \omega_2$, $\omega_2^2 = \omega_1$, $1 + \omega_1 + \omega_2 = 0$.

13. From the relation $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$, find $\cos 5\theta$ and $\sin 5\theta$ each in powers of $\sin\theta$ and $\cos\theta$.

14. (*a*) What is meant by an infinite series, by a convergent infinite series, by a divergent infinite series, by a non-convergent infinite series, by an absolutely convergent series, by a semiconvergent series? (*b*) Give an example of each.

15. (*a*) What is meant by the ratio of convergence or test-ratio of a series?

(*b*) Prove that if the test ratio ultimately approaches some value less than unity the series is absolutely convergent.

16. (*a*) Write down the first five terms of each of the following series: the exponential series, the sine series, the cosine series, the logarithmic series, the binomial series. (*b*) Show that the first three are absolutely convergent for all values of x. (*c*) Show that the last two are absolutely convergent so long as x is less than 1.

17. (*a*) Show that e is greater than 2.5 and less than 2.75.

(*b*) Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{2!} + \cdots$

(*c*) By means of the exponential series compute $e^{-1} = \frac{1}{e} = 0.3679$.

18. Prove that $\lim\limits_{x \doteq 0} \frac{\sin x}{x} = 1$, $\lim\limits_{x \doteq 0} \frac{\tan x}{x} = 1$.

19. Compute $\log_e 2$, $\log_{10} 2$, $\log_3 2$.

20. (*a*) Show how to compute the sine and cosine of 1′.

(*b*) Given sin 1′ and cos 1′, show how to find in succession the functions of 2′, 3′, 4′, etc.

(*c*) Show how a complete table of functions may be constructed when the functions of the angles from 0° to 30° are known.

21. (*a*) When is a series with complex terms said to be convergent?

(*b*) When is such a series said to be absolutely convergent?

(*c*) Show that the series $\Sigma\, u_n = \Sigma\, [r\,(\cos\theta + i \sin\theta)]^n$ is absolutely convergent.

22. (*a*) State and prove Euler's theorem.

(*b*) Show that $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\,\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

(*c*) Show that $e^{i\pi + 2n i\pi} = -1$, and hence $\log_e - 1 = i\pi + 2\,n i\pi$.

23. (*a*) Define each of the hyperbolic functions.

(*b*) Given $e = 2.71828$, $\frac{1}{e} = 0.36788$, compute $\sinh 2 = 3.6269$, $\cosh 2 = 3.7622$, $\sinh \frac{1}{2} = 0.5211$, $\cosh \frac{1}{2} = 1.1276$.

24. (*a*) Show that

$$\cosh u = \cos iu,\ \sinh u = -\,i \sin iu,\ \tanh u = -\,i \tan iu.$$

(*b*) Prove that

$$\cosh^2 u - \sinh^2 u = 1,\ \cosh^2 u + \sinh^2 u = \cosh 2\,u.$$

25. (*a*) If $y = \tanh x$, show that $x = \tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}$.

(*b*) Show that $\sinh^{-1} y = \log (y + \sqrt{y^2 + 1})$.

26. (*a*) Define the hyperbolic functions geometrically. (*b*) Mention some of the uses of hyperbolic functions.

SPHERICAL TRIGONOMETRY

CHAPTER I

INTRODUCTION

1. Definition of Spherical Trigonometry. If three points on any surface are joined by the shortest lines lying in the surface that it is possible to draw between these points a triangle is formed. Every such triangle has six parts, three sides and three angles. In general the sides are not straight lines but geodesic lines, that is, the shortest lines that can be drawn on the surface connecting the points. Thus every class of surfaces gives rise to a special trigonometry whose object is the investigation of the relations between the parts of the triangle and the study of the functions necessary for the determination of the unknown parts of a triangle from a sufficient number of given parts.

If the surface under consideration is the plane, the geodesics are straight lines and the triangles plane triangles, whose properties and those of the functions necessary for their solution have been considered in plane trigonometry. If the points lie on the surface of a sphere the geodesics are arcs of great circles, the triangles are called spherical triangles, and the corresponding trigonometry, spherical trigonometry. Briefly stated,

Spherical Trigonometry *deals with the relations among the six parts of a spherical triangle and the problems which may be solved by means of these relations.* The most important of these problems consist in the computation of the unknown parts of a spherical triangle from three given parts. It will be found that the solution of spherical triangles requires no functions other than those employed in plane trigonometry.

2. The Uses of Spherical Trigonometry. It is obvious that the triangle formed by three points on the earth's surface is not a plane triangle but a spherical triangle, for the distances between them are measured not along straight lines but along arcs of great circles. It is only when the distances are comparatively small that the sides may be considered straight lines and that the formulas of plane trigonometry give fairly approximate results. Hence geodetic surveying, that is surveying on a large scale, requires a knowledge of spherical trigonometry. The same is true of navigation when the bearings and distances of distant ports are under consideration. Strictly speaking since the earth is not a perfect sphere but a spheroid, such problems require a knowledge of spheroidal trigonometry, a branch of trigonometry whose study demands the introduction of functions other than those considered in plane trigonometry, but for many purposes the laws of spherical trigonometry give sufficiently accurate approximations.

While a knowledge of spherical trigonometry is of great importance to the surveyor and navigator, it is of even greater importance to the astronomer. The positions of all heavenly bodies are referred to the surface of an imaginary sphere, the celestial sphere, which encloses them all. In fact it is the dependance of astronomy upon spherical trigonometry that first led to its study by the ancients, long before plane trigonometry was thought of as a separate branch of science. Spherical trigonometry is, as it were, the elder sister of plane trigonometry.

Besides the uses already mentioned, spherical trigonometry furnishes the best possible review and constitutes one of the most interesting applications of the principles of plane trigonometry. Spherical trigonometry embodies the results of plane trigonometry in much the same measure that solid geometry embodies the results of plane geometry.

Finally, spherical trigonometry is worthy of study for its own sake because of the marvellous relations which it reveals and the simplicity, elegance, and beauty of the formulas in which its results are embodied.

3. Spherical Trigonometry Dependent on Solid Geometry. Just as plane trigonometry presupposes a certain knowledge of plane geometry so spherical trigonometry requires an acquaintance with solid geometry, especially with that portion of it which deals with the sphere. The student should, therefore, have a textbook on solid geometry ready at hand while pursuing this study in order to familiarize himself anew with the theorems and definitions which are presupposed in the discussions which follow. He should also provide himself with a small wooden or plaster of paris sphere and construct his figures on it whenever he has difficulty in visualizing the figures called for in his study.

4. Classification of Spherical Triangles. Like plane triangles, spherical triangles are classified in two ways: first, with reference to the sides and second, with reference to the angles.

A spherical triangle is said to be *equilateral*, *isosceles*, or *scalene*, according as it has three, two, or no equal sides. Since each side of a spherical triangle may have any value less than 180°,* one, two, or all three of the sides may be quadrants. If one side is a quadrant, the triangle is called *quadrantal*, if two, *biquadrantal*, if all three, *triquadrantal*.

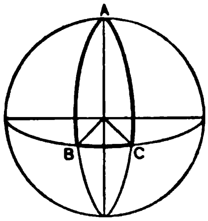

Fig. 1.

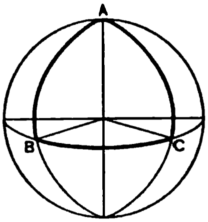

Fig. 2.

A *right spherical triangle* is one which has a right angle; an *oblique spherical triangle* is one which has none of its angles a right angle. Oblique spherical triangles are *obtuse* or *acute* according as they have or have not an obtuse angle. Since the sum of the angles of a spherical

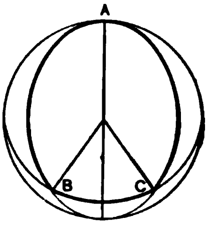

Fig. 3.

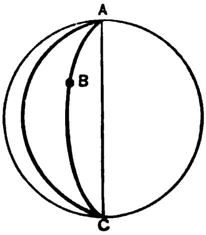

Fig. 4.

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* By the number of degrees in an arc we mean, of course, the number of degrees in the angle which the arc subtends at the center of the sphere. The number of degrees in an arc being given, the length of the arc is at once found from the relation, $s = r\theta$, where r is the radius of the sphere and θ the radian measure of the angle. (See Pl. Trig., Art. 90.)

triangle may have any value between 180° and 540° and no single angle can exceed 180°, a spherical triangle may have two or even three right angles. If it has two right angles it is called *birectangular* (Fig. 1), if three, *trirectangular* (Fig. 2). For the same reason a spherical triangle may have two or even three obtuse angles (Fig. 3).

If two points on a sphere are at the extremities of the same diameter any great circle passing through one of the points will pass also through the other. Two such points, therefore, cannot be the vertices of a spherical triangle, for the great circles connecting these points with any third point will coincide and the resulting figure will not be a triangle but a lune (Fig. 4).

5. Co-lunar Triangles. If the arcs AB, AC (Fig. 5) forming two sides of any spherical triangle be produced, they will meet again in some point A', forming a lune. The third side BC divides this lune into two triangles, the original triangle ABC, and the triangle $A'BC$. The triangle $A'BC$ thus formed is said to be *co-lunar* with the triangle ABC. It is obvious that any given triangle has three co-lunar triangles, one corresponding to each angle of the triangle. Thus the triangle ABC (Fig. 5) has the three co-lunar triangles $A'BC$, $AB'C$, ABC', where A', B', C' are the opposite poles of the vertices A, B, C of the triangle ABC.

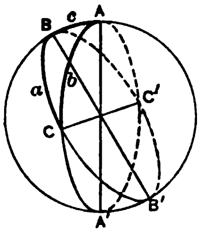

Fig. 5.

Since the angles of a lune are equal, and the sides of the lune semicircles, it follows that the parts of the co-lunar triangles may be immediately expressed in terms of the parts of the original triangle. If we denote the sides of the triangle ABC by a, b, c, and the angles by A, B, C, the corresponding parts of the co-lunar triangles are as follows:

Triangle.	Sides.			Angles.		
ABC	a	b	c	A	B	C
$A'BC$	a	$180° - b$	$180° - c$	A	$180° - B$	$180° - C$
$AB'C$	$180° - a$	b	$180° - c$	$180° - A$	B	$180° - C$
ABC'	$180° - a$	$180° - b$	c	$180° - A$	$180° - B$	C

6. Use of Co-lunar Triangles. Any general formula expressing a relation between the parts of a spherical triangle must continue true when applied to the co-lunar triangles. We may, therefore, substitute in any such formula for any two sides and their opposite angles their supplements, leaving the third side and angle unchanged. This process frequently leads to new relations among the parts of the triangle.

Thus, after it has been shown that for any triangle

$$\cos\frac{a-b}{2}\cos\frac{C}{2}=\sin\frac{A+B}{2}\cos\frac{c}{2},$$

we obtain, by applying this formula to the co-lunar triangle $A'BC$,

$$\cos\frac{a-(180^\circ-b)}{2}\cos\frac{180^\circ-C}{2}=\sin\frac{A+(180^\circ-B)}{2}\cos\frac{180^\circ-c}{2},$$

which reduces to the new formula

$$\sin\frac{a+b}{2}\sin\frac{C}{2}=\cos\frac{A-B}{2}\sin\frac{c}{2}.$$

EXERCISE 1

1. Show that every birectangular spherical triangle is also biquadrantal, and every trirectangular triangle is also triquadrantal.

2. Prove the converse of the proposition in Problem 1.

3. The co-lunar triangles of any right spherical triangle are right spherical triangles, and the co-lunar triangles of any quadrantal triangle are quadrantal.

4. The co-lunar triangles of an equilateral spherical triangle are isosceles.

5. It will be shown later that for any spherical triangle

$$\cos\frac{a+b}{2}\sin\frac{C}{2}=\cos\frac{A+B}{2}\cos\frac{c}{2}.$$

By applying this formula to the co-lunar triangle $A'BC$ show that

$$\sin\frac{a-b}{2}\cos\frac{C}{2}=\sin\frac{A-B}{2}\sin\frac{c}{2}.$$

6. It will be shown later that for any spherical triangle

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}},$$

where $$s=\frac{a+b+c}{2}.$$

By applying this formula to the co-lunar triangle ABC' show that

$$\cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$$

7. In Fig. 6, ABC is any right spherical triangle, right-angled at 1. With B as a pole construct a great circle cutting CB produced in 2 and BA produced in 3. With A as a pole construct a great circle cutting AB produced in 4 and CA produced in 5. The resulting figure is a curvilinear pentagon bordered by five spherical triangles. Show that each of these triangles is right-angled and determine all their parts as indicated in the figure. (Remark. The dashes over the letters indicate complements, thus $\bar{A} = 90^\circ - A$, $\bar{a} = 90^\circ - a$, $\bar{c} = 90^\circ - c$, etc.)

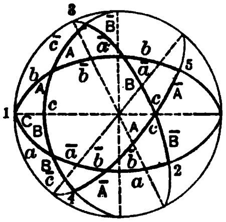

Fig. 6.

7. Polar Triangles. If from the vertices of any spherical triangle ABC as poles, great circles are drawn they will divide the surface of the sphere into eight associated spherical triangles *one of which* is called the *Polar* of the triangle ABC, and is determined as follows:

The great circles whose poles are B and C respectively intersect in two points which lie on opposite sides of the arc BC. Let A' be that one of these two points which lies on the same side of BC as A. The great circles whose poles are C and A respectively intersect in two points which lie on opposite sides of the arc CA. Let B' be that one of the two points which lies on the same side of CA as B. Similarly, let C' be that one of the intersection points of the great circles whose poles are A and B respectively, which lies on the same side of the arc AB as the vertex C. The triangle whose vertices are A', B', C' is the polar of the triangle ABC.

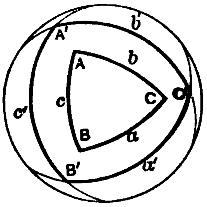

Fig. 7.

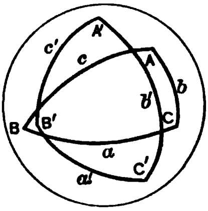

Fig. 8.

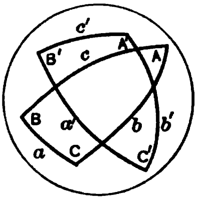

Fig. 9.

Just as in triangle ABC we use A, B, C to denote the angles and a, b, c to denote the sides opposite these angles, so A', B', C' denote the angles and a', b', c' the sides opposite these angles in the polar triangle $A'B'C'$.

It is necessary to recall the two fundamental properties of polar triangles:

I. *The relation of a triangle to its polar is mutual, that is, if $A'B'C'$ is the polar of ABC then ABC is the polar of $A'B'C'$.* Since each of these triangles is the polar of the other, two such triangles are referred to as *polar triangles.*

II. *In two polar triangles each angle is the supplement of the opposite side in the other, and each side the supplement of the opposite angle in the other.* In symbols,

$$\begin{aligned} A + a' &= 180°, \qquad & A' + a &= 180°, \\ B + b' &= 180°, \qquad & B' + b &= 180°, \\ C + c' &= 180°, \qquad & C' + c &= 180°. \end{aligned}$$

8. The Six Cases of Spherical Triangles. It will be shown presently that the six parts of any spherical triangle are so related that when any three are given the remaining three can be found. The three given parts may be:

I. *The three sides.*
II. *The three angles.*
III. *Two sides and the included angle.*
IV. *Two angles and the included side.*
V. *Two sides and the angle opposite one of them.*
VI. *Two angles and the side opposite one of them.*

There are six cases of spherical triangles while there are only three cases of plane triangles. This is because Cases IV and VI above reduce to the same case for plane triangles, since any two angles of a triangle determine the third. Also Case II above is ruled out for plane triangles since the three angles of a plane triangle determine only the shape but not the magnitude of the triangle.

9. Solution of Spherical Triangles. There are two distinct methods of finding the unknown parts of a spherical triangle from three known parts:

I. *The Graphic Method.* This consists of actually constructing the triangle on a material sphere. The unknown parts may then be found by measurement.

II. *The Method of Spherical Trigonometry.* The unknown parts are obtained by computation by means of formulas expressing the relation of the unknown parts to the parts which are given.

The first method is purely geometrical and is subject to all the errors of construction and inaccuracies of measurement pointed out in Pl. Trig., Art. 3. It is valuable as a rough check on the second method rather than as an independent method of solution.

The second method gives the unknown parts to a degree of accuracy limited only by the accuracy of the data and the number of places of the tables employed in the computation. This is the method employed in Geodesy, in Astronomy, and whenever precision is necessary or desirable. The derivation of the formulas employed by the second method and their application to the solution of the six cases of triangles constitutes an important part of Spherical Trigonometry.

10. The Use of the Polar Triangle. By the use of the polar triangle the second, fourth, and sixth case of spherical triangles may be made to depend on the first, third, and fifth respectively. Consider for instance Case II, in which the three angles are given. From the relations of Art. 7 the three sides of the polar triangle are known, this triangle may, therefore, be solved by Case I, and having found the angles of this triangle, the sides of the original triangle are given by the relations of Art. 7. Similarly, Case IV may be solved by Case III, and Case VI by Case V.

Again by means of the polar triangle any known relation between the parts of a triangle may be made to yield another relation, which frequently turns out to be new; for a relation which holds for every triangle must remain true when applied to the polar, that is, it must hold true if we put for each side the supplement of the opposite angle and for each angle the supplement of the opposite side. Thus if in the formula

$$\cos \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}C = \sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c$$

of Art. 6 we put

$$a = 180^\circ - A',\ b = 180^\circ - B',\ C = 180^\circ - c',$$
$$A = 180^\circ - a',\ B = 180^\circ - b',\ c = 180^\circ - C',$$

we obtain

$$\cos\frac{(180^\circ - A') - (180^\circ - B')}{2}\cos\frac{180^\circ - c'}{2} =$$
$$\sin\frac{(180^\circ - a') + (180^\circ - b')}{2}\cos\frac{180^\circ - C'}{2},$$

which on reducing becomes

$$\cos \tfrac{1}{2} (A' - B') \sin \tfrac{1}{2} c' = \sin \tfrac{1}{2} (a' + b') \sin \tfrac{1}{2} C',$$

or dropping accents

$$\cos \tfrac{1}{2} (A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} C.$$

11. Construction of Spherical Triangles.

Case I. *Given the three sides*, a, b, c.

On a sphere lay off an arc BC equal to a.* With B as a pole and an arc equal to c draw a small circle and with C as a pole and an arc equal to b draw another small circle. Either of the intersection points, A, A', of these small circles will be the vertex of a triangle whose other vertices are B and C and whose sides are the three given parts, a, b, c.

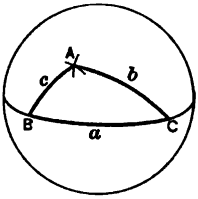

Fig. 10.

Case II. *Given the three angles*, A, B, C.

By Case I construct the polar triangle whose sides are

$$a = 180° - A, \quad b = 180° - B, \quad c = 180° - C.$$

The polar of this triangle will be the required triangle.

Case III. *Given two sides and the included angle*, a, b, C.

On a sphere draw an arc CM of a great circle and on it lay off an arc CB equal to a. Through C draw an arc CN making an angle C with CM.† On CN lay off an arc CA equal to b and join A and B by an arc of a great circle. Then ABC will be the required triangle.

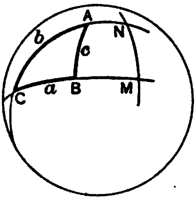

Fig. 11.

Case IV. *Given two angles and the included side*, A, B, c.

By Case III construct the polar triangle whose two sides and included angle are:

$$a = 180° - A, \quad b = 180° - B, \quad C = 180° - c.$$

The polar of this triangle will be the required triangle.

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* To lay off an arc equal to a means to lay off an arc of a great circle containing a degrees. This may be readily done by means of a strip of paper or cardboard equal in length to a semicircumference of the sphere and dividing it into 180 equal divisions. Each division will then represent one degree of angular measure on the sphere.

† This is most easily done as follows: From C as a pole draw the arc of a great circle. Let M be its intersection with CM. On this arc lay off MN equal to C. Join N and C by an arc of a great circle. Then NCM will be the required angle. (Why?)

Case V. *Given two sides and the angle opposite one of these sides,* a, b, A.

We distinguish two cases according as the angle A is acute or obtuse.

I. *A acute.*

On a sphere (Fig. 12) draw two arcs, AM and AN, making an angle A with each other and let A and A' be their points of intersection. On one of these arcs, as AN, lay off AC equal to b. With C as a pole and an arc equal to a describe a small circle.* In general this circle will intersect the arc AM in two points, B and B', either of which, if its angular distance from A is less than 180°, will form the third vertex of a triangle whose other two vertices are A and C and which will contain the three given parts.

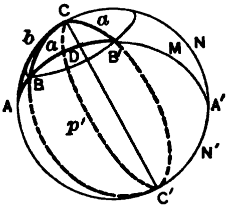

Fig. 12.

Let $p = CD$ be the arc through C which is perpendicular to AM.

(a) If a is less than p, the small circle will not intersect AM and no triangle exists having the given parts. The solution is impossible.

(b) If $a = p$, there is one solution. The resulting triangle has a right angle at D.

(c) If a is greater than p but less than the shorter of the two sides, $AC = b, CA' = 180° - b$, there will be two solutions, ACB and ACB'.

(d) If a is greater than the shorter of the two sides b and $180° - b$ but less than the longer, there will be one solution.

(e) If a is greater than the longer of the two sides b and $180° - b$ there will be no solution.

II. *A obtuse.*

Draw the two arcs AM and AN' (Fig. 12), making the angle A with each other. On one of these arcs, as AN', lay off AC' equal to b. With C' as a pole and an arc equal to a describe a small circle which, in general, will intersect the arc AM in two points, B and B', either of which, if its angular distance from A is less than 180°, will form the third vertex of a triangle whose other two vertices are A and C'.

Let $p' = C'D$ be the arc through C' which is perpendicular to AM. As p is the shortest arc that can be drawn from C to AM, so p' is the longest arc that can be drawn from C' to AM.

---

* This may be done by means of a pair of compasses.

(*a*) If a is greater than p', the small circle will not intersect AM and no triangle exists having the given parts. There is no solution.

(*b*) If $a = p'$, there is one solution. The resulting triangle has a right angle at D.

(*c*) If a is less than p' but greater than the longer of the two sides, $AC' = b$, $C'A' = 180° - b$, there will be two solutions, $AC'B$ and $AC'B'$.

(*d*) If a is less than the longer of the two sides, b and $180° - b$, but greater than the shorter, there will be one solution.

(*e*) If a is less than the shorter of the two sides, b and $180° - b$, there will be no solution.

Case VI. *Given two angles and the side opposite one of them*, A, B, c.

By Case V construct the polar triangle whose parts are $a = 180° - A$, $b = 180° - B$, $A = 180° - a$. The polar of this triangle will be the required triangle. As in Case V, so here there may be either one or two solutions or the solution may be impossible.

12. The General Spherical Triangle. We have defined a spherical triangle as the figure formed by joining three points on a sphere, which lie not in the same great circle, and no two of which are opposite ends of the same diameter, by the *shortest* great arcs. From this it follows that each side of a spherical triangle is less than a semicircumference, and its angular measure less than 180°.

Now the great circle drawn through two points is divided by those points into two arcs either of which may be considered the arc between the two points. If one of these arcs is less than 180° the other will be greater than 180° for their sum is always 360°. Hence if we drop the word *shortest* from the above definition, the resulting definition admits triangles whose sides have any value between 0° and 360°. Such triangles are called *general spherical triangles*. Since the arc between each two vertices may be chosen in two ways there are eight general triangles having the same three vertices. Fig. 13 shows two of these triangles, the triangle $AMBC$ and the triangle $AM'BC$.

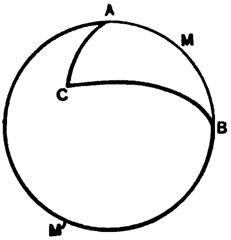

Fig. 13.

The study of general spherical triangles forms the object of Higher Spherical Trigonometry. Their principal applications are found in astronomy where it is frequently necessary to consider triangles whose sides or angles exceed 180°. *We observe that every spherical triangle, one or more of whose parts exceed 180°, may be solved by means of another whose parts are less than 180°, though this is not the simplest way of treating such triangles.* In the present text we shall limit our discussion to triangles which satisfy the first definition, that is, triangles each of whose parts is less than 180°.

EXERCISE 2

1. Prove the two theorems of Art. 7.

2. Prove that the polar of a right spherical triangle is quadrantal, and conversely, that the polar of a quadrantal triangle is a right triangle.

3. Prove that the polar of a birectangular spherical triangle is biquadrantal, and conversely, that the polar of a biquadrantal triangle is birectangular.

4. Prove that a trirectangular triangle is its own polar.

5. If the sides of a triangle are each less than 90° it lies wholly within its polar; if each of its sides is greater than 90° its polar lies wholly within it.

6. In any spherical triangle $a + b + c < 360°$. By applying this relation to the polar show that in every spherical triangle

$$180° < A + B + C < 540°.$$

7. In every spherical triangle the sum of two sides is greater than the third side, that is $a + b > c$. By applying this relation to the polar show that in every spherical triangle the difference between any angle and the sum of the other two is less than 180°, that is, $A + B - C < 180°$.

8. It will be shown later that in any spherical triangle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

By applying this formula to the polar triangle show that also

$$\cos A = - \cos B \cos C + \sin B \sin C \cos a.$$

9. By applying the formulas of Problem 6, Exercise I, to the polar triangle, deduce the two new formulas,

$$\cos\frac{c}{2} = \sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A \sin B}}, \quad \sin\frac{c}{2} = \sqrt{-\frac{\cos S \cos(S-C)}{\sin A \sin B}},$$

where $$S = \frac{A+B+C}{2}.$$

10. Construct the triangle called for in Case IV, Art. 11, without employing the polar triangle.

11. In Case V, Art. 11, write out the conditions under which the construction admits (*a*) one solution, (*b*) two solutions, (*c*) no solution.

CHAPTER II

RIGHT AND QUADRANTAL SPHERICAL TRIANGLES*

13. Formulas for Right Spherical Triangles. Every right triangle has a right angle and five other parts which, beginning with a side including the right angle, are denoted in order by a, B, c, A, b. We shall show that every three of these five parts are so related that when two are given the third may be found. Now the above five parts admit of ten different sets of three, namely:

$$A, a, c;\ \ A, b, c;\ \ A, a, b;\ \ A, B, b;\ \ c, a, b;$$
$$B, b, c;\ \ B, a, c;\ \ B, b, a;\ \ B, A, a;\ \ c, A, B;$$

hence we shall find ten formulas for the right spherical triangle.

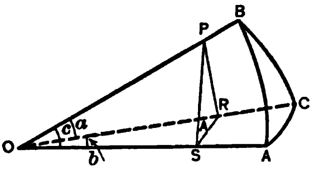

Fig. 14.

Let ABC, Fig. 14, be a right spherical triangle, C the right angle. Let O be the center of the sphere and $O - ABC$ the trihedral angle formed by the planes of the great circles whose arcs are a, b, c, respectively. It is shown in geometry that the face angles BOC, COA, AOB are measured by the arcs a, b, c, respectively, and that the dihedral angles OA, OB, OC are equal to the angles A, B, C, respectively. From any point P in OB draw PR perpendicular to OC, and from R draw RS perpendicular to OA. Join P and S. Then SR is perpendicular to PR (why?), and PS is perpendicular to OA (why?). Hence

triangle ORP has a right angle at R,
triangle OSR has a right angle at S,
triangle OSP has a right angle at S,
triangle PRS has a right angle at R,
and angle PSR equals angle A (why?).†

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* If the class has some knowledge of analytical geometry and the teacher wishes to cover the subject in the least time possible, he may omit the work to Art. 26. The fundamental relations for the oblique triangle as there developed may be specialized for the right triangle by putting $C = 90°$.

† See footnote on page 15.

In triangle PRS

$$\sin A = \frac{RP}{SP} = \frac{RP/OP}{SP/OP} = \frac{\sin ROP}{\sin SOP},$$

or $$\mathbf{\sin \boldsymbol{A} = \sin \boldsymbol{a}/\sin \boldsymbol{c}.} \tag{1}$$

Interchanging letters $$\mathbf{\sin \boldsymbol{B} = \sin \boldsymbol{b}/\sin \boldsymbol{c}.} \tag{2}$$

$$\cos A = \frac{SR}{SP} = \frac{SR/OS}{SP/OS} = \frac{\tan ROS}{\tan POS},$$

or $$\mathbf{\cos \boldsymbol{A} = \tan \boldsymbol{b}/\tan \boldsymbol{c}.} \tag{3}$$

Interchanging letters $$\mathbf{\cos \boldsymbol{B} = \tan \boldsymbol{a}/\tan \boldsymbol{c}.} \tag{4}$$

$$\tan A = \frac{RP}{SR} = \frac{RP/OR}{SR/OR} = \frac{\tan POR}{\sin ROS},$$

or $$\mathbf{\tan \boldsymbol{A} = \tan \boldsymbol{a}/\sin \boldsymbol{b}.} \tag{5}$$

Interchanging letters $$\mathbf{\tan \boldsymbol{B} = \tan \boldsymbol{b}/\sin \boldsymbol{a}.} \tag{6}$$

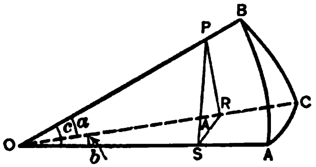

Fig. 14.

$$\begin{aligned}\sin A &= \frac{RP}{SP} = \frac{RP}{OR}\cdot\frac{OR}{OS}\cdot\frac{OS}{SP}\\ &= \tan POR \cdot \sec ROS \cdot \cot POS\\ &= \frac{\tan a}{\tan c}\cdot\frac{1}{\cos b},\end{aligned}$$

whence, substituting the value of $\tan a/\tan c$ from (4), we have

$$\mathbf{\sin \boldsymbol{A} = \cos \boldsymbol{B}/\cos \boldsymbol{b}.} \tag{7}$$

Interchanging letters $$\mathbf{\sin \boldsymbol{B} = \cos \boldsymbol{A}/\cos \boldsymbol{a}.} \tag{8}$$

Once more, $$\cos c = \frac{OS}{OP} = \frac{OR}{OP}\cdot\frac{OS}{OR} = \cos POR \cdot \cos ROS,$$

or $$\mathbf{\cos \boldsymbol{c} = \cos \boldsymbol{a} \cos \boldsymbol{b}.} \tag{9}$$

Finally, substituting in (9) for $\cos a$ and $\cos b$ their values from (7) and (8), we obtain

$$\mathbf{\cos \boldsymbol{c} = \cot \boldsymbol{A} \cot \boldsymbol{B}.} \tag{10}$$

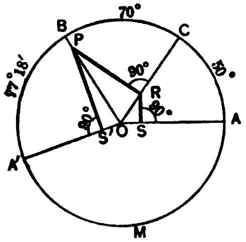

Fig. 15.

† Let the student who has undue difficulty in perceiving these relations construct the trihedral angle and the corresponding spherical triangle as follows: From a piece of cardboard or tin cut out a circle with any radius.

Draw four radii OA, OC, OB, OA', making the angles 50°, 70°, 77° 18′, respectively. Cut the circle along the radii OA and OA', and remove the sector AMA'. Cut the remaining sector partly through along OC and OB and bend the cardboard along these radii

14. Plane and Spherical Right Triangle Formulas Compared. The student will be assisted in remembering the ten formulas of the preceding article if he associates them with the corresponding formulas for the plane right triangle, as shown in the following table:

Plane Right Triangles.		Spherical Right Triangles.	
$\sin A = \frac{a}{c}$	$\sin B = \frac{b}{c}$	$\sin A = \frac{\sin a}{\sin c}$	$\sin B = \frac{\sin b}{\sin c}$
$\cos A = \frac{b}{c}$	$\cos B = \frac{a}{c}$	$\cos A = \frac{\tan b}{\tan c}$	$\cos B = \frac{\tan a}{\tan c}$
$\tan A = \frac{a}{b}$	$\tan B = \frac{b}{a}$	$\tan A = \frac{\tan a}{\sin b}$	$\tan B = \frac{\tan b}{\sin a}$
$\sin A = \cos B$	$\sin B = \cos A$	$\sin A = \frac{\cos B}{\cos b}$	$\sin B = \frac{\cos A}{\cos a}$
$c^2 = a^2 + b^2$		$\cos c = \cos a \cos b$	
$1 = \cot A \cot B$		$\cos c = \cot A \cot B$	

15. Generalization of the Right Triangle Formulas. In Fig. 14 the sides a and b are each less than 90°. It remains to show that the formulas in Art. 13 hold for *all* possible values of a and b.

I. *One side adjacent to the right angle greater than 90° and the other less than 90°.*

In the right triangle ABC (Fig. 16), let a be greater than 90° and b less than 90°. The co-lunar triangle $AB'C$ will have a right angle at C and the adjacent sides b and $a' = 180° - a$, each less than 90°. We may, therefore, apply the formulas of Art. 13 to this triangle. Thus (1) gives

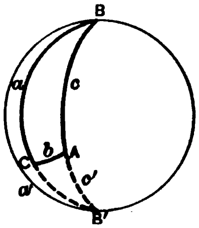

Fig. 16.

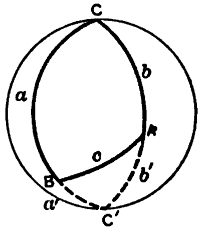

Fig. 17.

$$\sin CAB' = \frac{\sin a'}{\sin c'} = \frac{\sin (180° - a)}{\sin (180° - c)} = \frac{\sin a}{\sin c}, \text{ or } \sin A = \frac{\sin a}{\sin c},$$

that is (1) remains true for the triangle ABC.

---

until OA' meets OA. The figure thus formed will be a right trihedral angle, ABC will form a right spherical triangle, and the lines PR, RS and PS' will form the triangle PRS of Fig. 14.

Similarly each of the other nine formulas will be found true for the triangle ABC.

II. *Each of the sides adjacent to the right angle greater than 90°.*

In the right triangle ABC (Fig. 17), let a and b be each greater than 90°. The co-lunar triangle ABC' will have a right angle at C' and the adjacent sides $a' = 180° - a$ and $b' = 180° - b$, each less than 90°. We may, therefore, apply the formulas of Art. 13 to this triangle. Thus (1) gives

$$\sin BAC' = \frac{\sin a'}{\sin c} = \frac{\sin (180° - a)}{\sin c} = \frac{\sin a}{\sin c}, \text{ or } \sin A = \frac{\sin a}{\sin c},$$

that is (1) holds true for triangle ABC, and similarly each of the other nine formulas will be found true for this case.

This proves that the formulas of Art. 13 may be applied to the solution of every possible right spherical triangle.

16. Napier's Rules of Circular Parts.* Lord Napier, the inventor of logarithms, first succeeded in expressing the ten right triangle formulas by two simple rules. Let us put

$$90° - A = \bar{A}, \quad 90° - c = \bar{c}, \quad 90° - B = \bar{B},$$

then

$$\sin A = \cos \bar{A}, \quad \cos A = \sin \bar{A}, \quad \tan A = \cot \bar{A}, \quad \cot A = \tan \bar{A},$$
$$\sin c = \cos \bar{c}, \text{ etc.}, \qquad \sin B = \cos \bar{B}, \text{ etc.}$$

The ten equations of Art. 13 may then be written as follows, the new formulas being numbered as in Art. 13.

$$\begin{array}{llll}
\sin a = \cos \bar{A} \cos \bar{c} & (1) & \sin \bar{A} = \tan b \tan \bar{c} & (3) \\
\sin b = \cos \bar{B} \cos \bar{c} & (2) & \sin \bar{B} = \tan a \tan \bar{c} & (4) \\
\sin \bar{B} = \cos \bar{A} \cos b & (7) & \sin b = \tan a \tan \bar{A} & (5) \\
\sin \bar{A} = \cos \bar{B} \cos a & (8) & \sin a = \tan b \tan \bar{B} & (6) \\
\sin \bar{c} = \cos a \cos b & (9) & \sin \bar{c} = \tan \bar{A} \tan \bar{B} & (10)
\end{array}$$

Let us now arrange the five parts a, $\bar{B}$, $\bar{c}$, $\bar{A}$, b in their order in a circle as in Fig. 18. Any one of these five parts, as a, being chosen as the *middle* part, the two next to it, as b and $\bar{B}$, are called *adjacent* parts and the remaining two parts,

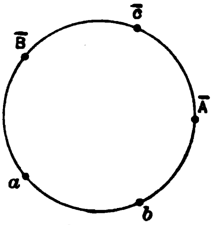

Fig. 18.

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* This and the following article may be omitted by those who prefer to memorize the preceding ten formulas as suggested in Art. 14.

as $\overline{A}$ and $\bar{c}$, are called *opposite* parts. Then each of the five equations on the right are contained in

Rule 1. *The sine of the middle part is equal to the product of the* **tangents** *of the* **adjacent** *parts,*

and the five on the left are contained in

Rule 2. *The sine of the middle part is equal to the product of the* **cosines** *of the* **opposite** *parts.*

These two rules are known as *Napier's Rules of the Circular Parts.*

17. Proof of Napier's Rules of Circular Parts. Napier's rules are commonly looked upon as memory rules which happen to include the ten right triangle formulas. They have been proclaimed the happiest example of artificial memory known to man. Because of their supposed artificial character their value as an instrument in mathematics has been questioned. We shall now show that Napier's rules are not mere mnemotechnic aids but constitute a most remarkable theorem which admits of rigorous proof.

Let ABC_1 be a right spherical triangle, C_1 the right angle. With B as a pole draw a great circle cutting C_1B produced in C_2 and BA produced in C_3. With A as a pole draw a great circle cutting AB produced in C_4 and C_1A produced in C_5. The resulting figure is a spherical pentagon $ABPRS$, bordered by five triangles I, II, III, IV, V.

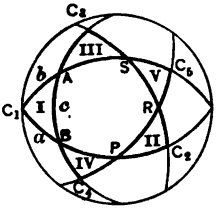


Fig. 19.

Since B is the pole of the arc C_2C_3 the angles at C_2 and C_3 are right angles and since A is the pole of arc C_4C_5 the angles at C_4 and C_5 are right angles. The five triangles are, therefore, right triangles.

Since C_1 and C_2 are right angles, S is the pole of C_1C_2 and consequently SC_1 and SC_2 are quadrants. For like reasons RC_3, RC_4, PC_5, PC_1, BC_2, BC_3, AC_4, AC_5 are quadrants.

With these preliminary observations it is now easy to show that the five triangles I, II, III, IV, V have the same circular parts taken in the same order, while the position of these parts with respect to the right angle is different in each of the triangles.

Let us compare the two triangles ABC_1 and PRC_2 and denote by a_2, B_2, c_2, A_2, b_2 the five parts of II which correspond to a, B, c, A, b of I. Comparing angular measures we have

$$\begin{aligned}
a_2 &= C_2R = C_2C_3 - RC_3 = (180^\circ - B) - 90^\circ = 90^\circ - B = \overline{B},\\
B_2 &= PRC_2 = 180^\circ - PRS = 180^\circ - C_3C_4 = 180^\circ - (C_3B + AC_4 - AB)\\
&= 180^\circ - (90^\circ + 90^\circ - c) = c,\\
c_2 &= PR = C_4R + PC_5 - C_4C_5 = 90^\circ + 90^\circ - C_4AC_5\\
&= 90^\circ + 90^\circ - (180^\circ - A) = A,\\
A_2 &= RPC_2 = 180^\circ - BPR = 180^\circ - C_1C_5 = 180^\circ - (C_1A + AC_5)\\
&= 180^\circ - (b + 90^\circ) = \overline{b},\\
b_2 &= PC_2 = BC_2 - BP = 90^\circ - (C_1P - C_1B) = 90^\circ - (90^\circ - a) = a;
\end{aligned}$$

hence, $a_2 = \overline{B}$, $B_2 = c$, $C_2 = A$, $A_2 = \overline{b}$, $b_2 = a$.

Now the parts of triangle III may be obtained from those of II, the parts of IV from those of III, and the parts of V from those of IV, in exactly the same way that the parts of II were obtained from those of I. Writing corresponding parts under each other, and remembering that to obtain the circular parts we must replace the hypotenuse and angles of each triangle by their complements, we have the following table:

	Actual Parts.	Circular Parts.
Triangle I..................	$a, B, c, A, b,$	$a, \overline{B}, \overline{c}, \overline{A}, b$
Triangle II.................	$\overline{B}, c, A, \overline{b}, a$	$\overline{B}, \overline{c}, \overline{A}, b, a$
Triangle III................	$\overline{c}, A, \overline{b}, \overline{a}, \overline{B}$	$\overline{c}, \overline{A}, b, a, \overline{B}$
Triangle IV.................	$\overline{A}, \overline{b}, \overline{a}, B, \overline{c}$	$\overline{A}, b, a, \overline{B}, \overline{c}$
Triangle V..................	$b, \overline{a}, B, c, \overline{A}$	$b, a, \overline{B}, \overline{c}, \overline{A}$

The column on the right not only shows that each triangle has the same circular parts taken in the same order, but also that the middle part $\overline{c}$ of the first triangle is successively replaced by $\overline{A}$, b, a, $\overline{B}$ in the other four. Now it was shown in Art. 13 (10), (9), that for the triangle ABC_1,

$$\cos c = \cot A \cot B, \quad \text{or} \quad \sin \overline{c} = \tan \overline{A} \tan \overline{B}, \tag{I}$$

$$\cos c = \cos a \cos b, \quad \text{or} \quad \sin \overline{c} = \cos a \cos b, \tag{II}$$

hence formulas (I) and (II) hold when any part other than $\overline{c}$ is taken for the middle part, and thus Napier's rules are shown to be necessarily true.

EXERCISE 3

1. Apply the ten formulas for the right spherical triangle to the polar and obtain ten formulas for the quadrantal spherical triangle.

2. Write out the ten equations for the right spherical triangle by means of Napier's rules.

3. From the relation $\cos c = \cos a \cos b$ show that if a right triangle has only one right angle, the three sides are either all acute, or one is acute and the other two obtuse.

4. From the relation $\cos A = \cos a \sin B$ show that the side a is in the same quadrant as the opposite angle A.

5. If in a right spherical triangle $a = c = 90°$, prove that $\cos b = \cos B$.

6. Also if $a = b$, prove that $\cot B = \cos a$.

Prove the following relations for the right triangle ABC:

7. $\cos^2 A - \sin^2 B = - \sin^2 b \sin^2 A$.
8. $\sin A \sin 2\, b = \sin c \sin 2\, B$.
9. $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b$.
10. $\sin A \cos c = \cos a \cos B$.
11. $\sin b = \cos c \tan a \tan B$.
12. $\sin^2 A \cos^2 b \sin^2 c = \sin^2 c - \sin^2 b$.

18. To Determine the Quadrant of the Unknown Parts in a Right Spherical Triangle. When an unknown part is found from its cosine, tangent, or cotangent, the sign of the function shows whether the part is in the first or second quadrant, that is, whether it is less than 90° or greater than 90°. In the cases where the unknown part is found from the sine, the following theorems enable us to tell, in every case in which the triangle has but one solution, whether the part is greater or less than 90°.

I. *At least one side of every right spherical triangle is in the first quadrant, the remaining two are either both in the first quadrant or both in the second.* For, since the cosine of an angle in the second quadrant is negative, it is plain that the equation

$$\cos c = \cos a \cos b \qquad \text{(Art. 13 (10))}$$

must have either none or two of the angles a, b, c in the second quadrant.

II. *Either of the oblique angles of a right spherical triangle is in the same quadrant as its opposite side.* For since

$$\sin A = \cos B / \cos b \qquad \text{(Art. 13 (7))}$$

and sin A is always positive, it is plain that cos B and cos b must either be both positive or both negative, that is, B and b and similarly A and a, must be in the same quadrant.

19. The Ambiguous Case of Right Spherical Triangles. When the given parts of a right triangle are an angle and the side opposite, the triangle has two solutions. For, the given parts being A and a (Fig. 20), the co-lunar triangle $A'BC$ as well as the triangle ABC has the given parts. Notice that $A'B$ and $A'C$ are the supplements of AB and AC, respectively, and that angle $A'BC$ is the supplement of angle ABC. Both sets of values are given by the formulas, for, A and a being given, c, b, and B are found from their sines (Art. 13, Equations (1), (5) and (8)).

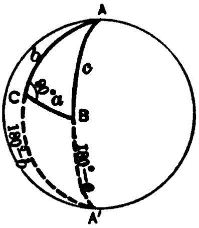


Fig. 20.

20. Solution of Right Spherical Triangles. Napier's rules, or, if it is preferred, the ten formulas in Art. 13, enable us to solve every conceivable right spherical triangle, two parts being given. The procedure in any given case is as follows:

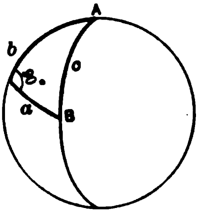


Fig. 21.

I. We consider three parts, two of which are the given parts and the third the part to be found. If these three parts are adjacent we take the middle one for the middle part, if two only are adjacent we take the remaining one for the middle part and by Napier's rules write down the formula relating the three parts.

Thus if A and c are the given parts (Fig. 21), and b is to be found, we take A for the middle part and by Napier's first rule,

$$\sin \bar{A} = \tan b \tan \bar{c}, \text{ that is, } \cos A = \tan b \cot c. \qquad (1)$$

If B is to be found, we take c for the middle part, and again applying Napier's first rule we have

$$\sin \bar{c} = \tan \bar{A} \tan \bar{B}, \text{ that is, } \cos c = \cot A \cot B. \qquad (2)$$

If a is the part required, we take a for the middle part, and applying Napier's second rule, we have

$$\sin a = \cos \bar{A} \cos \bar{c}, \text{ that is, } \sin a = \sin A \sin c. \qquad (3)$$

II. Next we solve the equation for that function which contains the unknown part. Thus to find b, we have from equation (1) above, $\tan b = \cos A \tan c$, to find B we have from (2) $\cot B = \cos c \tan A$, to find a we use equation (3) as it stands.

III. By means of the equations thus obtained and the use of tables we compute the unknown parts, remembering,

(*a*) If the unknown part is found from its cosine, tangent, or cotangent, the algebraic sign of the function determines the quadrant of the angle.

(*b*) If the unknown part is found from its sine, the quadrant of the angle is determined by one of the theorems of Art. 18.

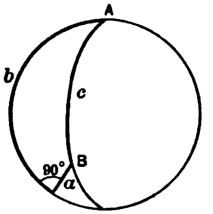
(*c*) If the given parts are an angle and the side opposite, each unknown part has two values which are supplements of each other.

IV. Check. When the unknown parts have been computed, their correctness should be checked by the formula obtained by applying Napier's rules to these parts. Thus in the above example, after b, B, and a have been computed their values must satisfy the formula (a being the middle part)

$$\sin a = \tan \overline{B} \tan b, \text{ that is, } \sin a = \cot B \tan b.$$

EXAMPLE I.

Given
$A = 67^\circ\, 34'\, 40'',$
$c = 137^\circ\, 24'\, 54''.$



Required
$b = 160^\circ\, 40'\, 56'',$
$B = 150^\circ\, 44'\, 00'',$
$a = 35^\circ\, 42'\, 57''.$

Fig. 22.

Solution.

To find b.

$\cos A = \cot c \tan b,$
or, $\tan b = \cos A \tan c.$
$\log \cos A = 9.58141$
$\log \tan c = \underline{9.96334n}$*
$\log \tan b = 9.54475n$
$b = 160^\circ\, 40'\, 56''.$

To find a.

$\sin a = \sin A \sin c.$
$\log \sin A = 9.96586$
$\log \sin c = \underline{9.83038}$
$\log \sin a = 9.79624$
$a = 35^\circ\, 42'\, 57''.$

To find B.

$\cos c = \cot A \cos B,$
or, $\cot B = \cos c \tan A.$
$\log \cos c = 9.86704n$
$\log \tan A = \underline{0.38445}$
$\log \cot B = 0.25149n$
$B = 150^\circ\, 44'\, 00''.$

Check.

$\sin a = \cot B \tan b.$
$\log \cot B = 0.25149n$
$\log \tan b = \underline{9.54475n}$
$\log \sin a = 9.79624$ (check).

---

* *n* written after a logarithm means that the number of which the logarithm is taken (in this case tan *c*) has the negative sign.

In this case, since tan b and cot B are negative, b and B must be taken in the second quadrant, while a is taken in the first quadrant since by Art. 18 it must be in the same quadrant as the opposite angle A.

Example 2.

Given

$B = 25^\circ\, 36'\, 30''$,

$b = 24^\circ\, 20'\, 45''$.

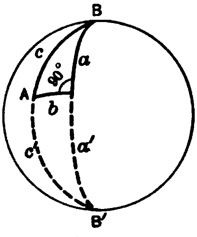


Fig. 23.

Required

$A = 81^\circ\, 48'\, 30''$,

$A' = 98^\circ\, 11'\, 30''$,

$c = 72^\circ\, 30'\, 45''$,

$c' = 107^\circ\, 29'\, 15''$,

$a = 70^\circ\, 44'\, 45''$,

$a' = 109^\circ\, 15'\, 15''$.

Solution.

To find A.

$\cos B = \sin A \cos b$,

or, $\sin A = \cos B/\cos b$.

$\log \cos B = 9.95510$

$\text{colog} \cos b = \underline{0.04045}$

$\log \sin A = 9.99555$

$A = 81^\circ\, 48'\, 30''$.

$A' = 98^\circ\, 11'\, 30''$.

To find c.

$\sin b = \sin B \sin c$,

or, $\sin c = \sin b/\sin B$.

$\log \sin b = 9.61515$

$\text{colog} \sin B = \underline{0.36430}$

$\log \sin c = 9.97945$

$c = 72^\circ\, 30'\, 45''$.

$c' = 107^\circ\, 29'\, 15''$.

To find a.

$\sin a = \cot B \tan b$.

$\log \cot B = 0.31940$

$\log \tan b = \underline{9.65560}$

$\log \sin a = 9.97500$

$a = 70^\circ\, 44'\, 45''$.

$a' = 109^\circ\, 15'\, 15''$.

Check.

$\sin a = \sin c \sin A$.

$\log \sin c = 9.97945$

$\log \sin A = \underline{9.99555}$

$\log \sin a = 9.97500$ (check).

In this case there are two solutions. By Art. 18 a and A must be in the same quadrant, hence the acute values of both a and A belong to one triangle and the obtuse values to another. Again, by Art. 18, the three sides a, b, c are either all in the first quadrant, or two are in the second quadrant, hence c is in the same triangle as a, and c' is in the same triangle as a'.

Exercise 4

When no answer is given the results must be checked. For the number of significant figures to be retained in the answer see Pl. Trig., Art. 44.

Solve the following right spherical triangles when the parts given are:

1. $a = 81^\circ\ 25'$, $b = 101^\circ\ 15'$.
Ans. $A = 81^\circ\ 35'$, $B = 101^\circ\ 08'$, $c = 94^\circ\ 40'$.

2. $c = 86^\circ\ 51'$, $B = 18^\circ\ 04'$.
Ans. $b = 18^\circ\ 02'$, $a = 86^\circ\ 41'$, $A = 88^\circ\ 58'$.

3. $a = 70^\circ\ 28'$, $c = 98^\circ\ 18'$.
Ans. $A = 72^\circ\ 15'$, $B = 114^\circ\ 17'$, $b = 115^\circ\ 35'$.

4. $c = 118^\circ\ 40'$, $A = 128^\circ\ 00'$.
Ans. $a = 136^\circ\ 16'$, $b = 48^\circ\ 24'$, $B = 58^\circ\ 27'$.

5. $A = 81^\circ\ 13'$, $B = 65^\circ\ 24'$.
Ans. $a = 80^\circ\ 20'$, $b = 65^\circ\ 05'$, $c = 85^\circ\ 56'$.

6. $b = 112^\circ\ 49'$, $B = 100^\circ\ 27'$.
Ans. $a = 26^\circ\ 00'$, $A = 27^\circ\ 53'$, $c = 110^\circ\ 24'$;
$a' = 154^\circ\ 00'$, $A' = 152^\circ\ 07'$, $c' = 69^\circ\ 36'$.

7. $c = 81^\circ\ 10'$, $a = 100^\circ\ 47'$.

8. $A = 75^\circ\ 23'$, $B = 75^\circ\ 23'$.

9. $a = 72^\circ\ 15'$, $B = 83^\circ\ 25'$.

10. $b = 148^\circ\ 28'$, $B = 101^\circ\ 04'$.

11. $a = 43^\circ\ 40.5'$, $c = 98^\circ\ 29.1'$.
Ans. $A = 44^\circ\ 17.0'$, $B = 98^\circ\ 11.4'$, $b = 101^\circ\ 46.3'$.

12. $a = 28^\circ\ 47.0'$, $b = 110^\circ\ 27.3'$.
Ans. $A = 30^\circ\ 23.1'$, $B = 100^\circ\ 10.9'$, $c = 107^\circ\ 50.2'$.

13. $b = 74^\circ\ 21.9'$, $A = 38^\circ\ 57.2'$.
Ans. $B = 80^\circ\ 14.7'$, $a = 37^\circ\ 54.1'$, $c = 77^\circ\ 43.3'$.

14. $A = 49^\circ\ 15.0'$, $B = 52^\circ\ 26.0'$.
Ans. $a = 34^\circ\ 33.7'$, $b = 36^\circ\ 24.6'$, $c = 48^\circ\ 29.3'$.

15. $c = 50^\circ\ 20.2'$, $A = 101^\circ\ 29.4'$.
Ans. $a = 131^\circ\ 01.7'$, $b = 166^\circ\ 29.5'$, $B = 162^\circ\ 20.1'$.

16. $a = 32^\circ\ 10.8'$, $A = 42^\circ\ 24.0'$.
Ans. $b = 43^\circ\ 34.8'$, $B = 60^\circ\ 43.2'$, $c = 52^\circ\ 06.0'$;
$b' = 136^\circ\ 25.2'$, $B' = 119^\circ\ 16.8'$, $c' = 127^\circ\ 54.0'$.

17. $c = 95^\circ\, 26.2'$, $b = 12^\circ\, 37.8'$.

18. $a = 119^\circ\, 56.1'$, $b = 151^\circ\, 43.6'$.

19. $A = 70^\circ\, 56.9'$, $B = 39^\circ\, 25.6'$.

20. $b = 112^\circ\, 24.8'$, $B = 94^\circ\, 58.9'$.

21. $a = 41^\circ\, 50'\, 20''$, $b = 50^\circ\, 18'\, 11''$.
Ans. $A = 49^\circ\, 19'\, 29''$, $B = 61^\circ\, 01'\, 33''$, $c = 61^\circ\, 35'\, 05''$.

22. $c = 110^\circ\, 46'\, 20''$, $B = 80^\circ\, 10'\, 30''$.
Ans. $b = 67^\circ\, 06'\, 53''$, $a = 155^\circ\, 46'\, 43''$, $A = 153^\circ\, 58'\, 24''$.

23. $b = 96^\circ\, 49'\, 59''$, $A = 50^\circ\, 12'\, 04''$.
Ans. $a = 50^\circ\, 00'\, 00''$, $B = 95^\circ\, 14'\, 41''$, $c = 94^\circ\, 23'\, 10''$.

24. $A = 46^\circ\, 59'\, 42''$, $B = 57^\circ\, 59'\, 17''$.
Ans. $a = 36^\circ\, 27'\, 00''$, $b = 43^\circ\, 33'\, 30''$, $c = 54^\circ\, 20'\, 03''$.

25. $a = 32^\circ\, 09'\, 17''$, $c = 44^\circ\, 33'\, 17''$.
Ans. $A = 49^\circ\, 20'\, 16''$, $b = 32^\circ\, 41'\, 00''$, $B = 50^\circ\, 19'\, 16''$.

26. $b = 160^\circ\, 00'\, 00''$, $B = 150^\circ\, 00'\, 00''$.
Ans. $a = 140^\circ\, 55'\, 09''$, $A = 112^\circ\, 50'\, 17''$, $c = 43^\circ\, 09'\, 37''$;
$a' = 39^\circ\, 04'\, 51''$, $A' = 67^\circ\, 09'\, 43''$, $c' = 136^\circ\, 50'\, 23''$.

27. $A = 60^\circ\, 45'\, 45''$, $B = 57^\circ\, 56'\, 56''$.

28. $c = 120^\circ\, 23'\, 56''$, $A = 119^\circ\, 34'\, 42''$.

29. $a = 116^\circ\, 52'\, 45''$, $b = 16^\circ\, 06'\, 06''$.

30. $A = 81^\circ\, 58'\, 36''$, $a = 67^\circ\, 20'\, 30''$.

21. Solution of Quadrantal Triangles. The polar of a quadrantal triangle is a right triangle which may be solved by the method of Art. 20 and from it the required parts of the original quadrantal triangle are obtained by means of the relations in Art. 7. Or we may apply the right triangle formulas of Art. 13 to the polar and obtain a new set of formulas for the solution of any quadrantal triangle. Thus formula (1), Art. 13, viz., $\sin A = \sin a/\sin c$, when applied to the polar triangle becomes $\sin(180^\circ - a) = \sin(180^\circ - A)/\sin(180^\circ - C)$ or $\sin a = \sin A/\sin C$. Similarly we obtain each of the following formulas for the solution of quadrantal triangles, C being the angle opposite the quadrant c.

$$\sin a = \sin A/\sin C \tag{1}$$
$$\sin b = \sin B/\sin C \tag{2}$$
$$-\cos a = \tan B/\tan C \tag{3}$$
$$-\cos b = \tan A/\tan C \tag{4}$$
$$\tan a = \tan A/\sin B \tag{5}$$
$$\tan b = \tan B/\sin A \tag{6}$$
$$\sin a = \cos b/\cos B \tag{7}$$
$$\sin b = \cos a/\cos A \tag{8}$$
$$-\cos C = \cos A \cos B \tag{9}$$
$$-\cos C = \cot a \cot b \tag{10}$$

EXAMPLE. Solve the quadrantal triangle in which

$$a = 97^\circ\ 24',\ A = 103^\circ\ 12',\ c = 90^\circ.$$

Solution. The polar triangle has the parts

$$A = 180^\circ - 97^\circ\ 24' = 82^\circ\ 36',\ a = 180^\circ - 103^\circ\ 12' = 76^\circ\ 48',$$
$$C = 180^\circ - 90^\circ = 90^\circ.$$

Solving this right triangle by the method of Art. 20 we find

$$B = 34^\circ\ 20', \qquad b = 33^\circ\ 37', \qquad c = 79^\circ\ 02';$$
$$B' = 145^\circ\ 40', \qquad b' = 146^\circ\ 23', \qquad c' = 100^\circ\ 58'.$$

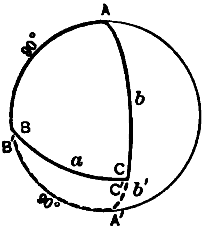


Fig. 24.

The required parts of the quadrantal triangle are, therefore,

$$b = 180^\circ - 34^\circ\ 20' = 145^\circ\ 40',$$
$$b' = 180^\circ - 145^\circ\ 40' = 34^\circ\ 20',$$
$$B = 180^\circ - 33^\circ\ 37' = 146^\circ\ 23',$$
$$B' = 180^\circ - 146^\circ\ 23' = 33^\circ\ 37',$$
$$C = 180^\circ - 79^\circ\ 02' = 100^\circ\ 58',$$
$$C' = 180^\circ - 100^\circ\ 58' = 79^\circ\ 02'.$$

Fig. 24 represents both solutions geometrically.

22. Special Formulas for Angles near 0°, 90° or 180°. An angle near 0° or 180° can not be accurately determined from its cosine, nor an angle near 90° from its sine (see Pl. Trig., Art. 21); in such cases the formulas of Art. 13 are, therefore, no longer adequate. The difficulty may be avoided by employing the following formulas:

A near 0° or 180°, $\tan^2 \frac{1}{2} A = \sin(c - b)/\sin(c + b)$. (1)

B near 0° or 180°, $\tan^2 \frac{1}{2} B = \sin(c - a)/\sin(c + a)$. (2)

a near 0° or 180°, $\tan^2 \frac{1}{2} a = \tan \frac{1}{2}(c + b) \tan \frac{1}{2}(c - b)$. (3)

b near 0° or 180°, $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a)$. (4)

c near 0° or 180°, $\tan^2 \frac{1}{2} c = -\cos(A + B)/\cos(A - B)$ (5)

A near 90°, $\tan^2(45^\circ - \frac{1}{2} A) = \tan \frac{1}{2}(c - a)/\tan \frac{1}{2}(c + a)$ (6)

$= \tan \frac{1}{2}(B - b) \tan \frac{1}{2}(B + b)$. (7)

B near 90°, $\tan^2(45^\circ - \frac{1}{2} B) = \tan \frac{1}{2}(c - b)/\tan \frac{1}{2}(c + b)$ (8)

$= \tan \frac{1}{2}(A - a) \tan \frac{1}{2}(A + a)$. (9)

a near 90°, $\tan^2(45^\circ - \frac{1}{2} a) = \sin(B - b)/\sin(B + b)$. (10)

b near 90°, $\tan^2(45^\circ - \frac{1}{2} b) = \sin(A - a)/\sin(A + a)$. (11)

c near 90°, $\tan^2(45^\circ - \frac{1}{2} c) = \tan \frac{1}{2}(A - a)/\tan \frac{1}{2}(A + a)$ (12)

$= \tan \frac{1}{2}(B - b)/\tan \frac{1}{2}(B + b)$. (13)

To deduce (1) we have

$$\cos A = \tan b/\tan c, \qquad \text{(Art. 13 (3))}$$

$$\frac{1 - \cos A}{1 + \cos A} = \frac{\tan c - \tan b}{\tan c + \tan b}, \qquad \text{(Comp. and div.)}$$

$$\frac{1 - \cos A}{1 + \cos A} = \frac{1 - (1 - 2\sin^2 \frac{1}{2} A)}{1 + (2\cos^2 \frac{1}{2} A - 1)} = \tan^2 \tfrac{1}{2} A, \qquad \text{(Pl. Trig., Art. 111)}$$

and

$$\frac{\tan c - \tan b}{\tan c + \tan b} = \frac{\sin c \cos b - \cos c \sin b}{\sin c \cos b + \cos c \sin b} = \frac{\sin (c - b)}{\sin (c + b)};$$

(Pl. Trig., Art. 109)

hence $\tan^2 \frac{1}{2} A = \sin(c - b)/\sin(c + b)$.

Again, to deduce (13) we proceed as follows:

$$\sin c = \sin b/\sin B, \qquad \text{(Art. 13 (2))}$$

$$\frac{1 - \sin c}{1 + \sin c} = \frac{\sin B - \sin b}{\sin B + \sin b}, \qquad \text{(Comp. and div.)}$$

$$\frac{1 - \sin c}{1 + \sin c} = \frac{1 - 2\sin \frac{1}{2} c \cos \frac{1}{2} c}{1 + 2\sin \frac{1}{2} c \cos \frac{1}{2} c} = \frac{(\cos \frac{1}{2} c - \sin \frac{1}{2} c)^2}{(\cos \frac{1}{2} c + \sin \frac{1}{2} c)^2}$$

(Pl. Trig., Art. 111)

$$= \frac{(1 - \tan \frac{1}{2} c)^2}{(1 + \tan \frac{1}{2} c)^2} = \tan^2 (45^\circ - \tfrac{1}{2} c),$$

(Pl. Trig., Art. 110)

and

$$\frac{\sin B - \sin b}{\sin B + \sin b} = \frac{2\cos \frac{1}{2}(B + b) \sin \frac{1}{2}(B - b)}{2\sin \frac{1}{2}(B + b) \cos \frac{1}{2}(B - b)} = \frac{\tan \frac{1}{2}(B - b)}{\tan \frac{1}{2}(B + b)};$$

(Pl. Trig., Art. 113)

hence $\tan^2 (45^\circ - \frac{1}{2} c) = \tan \frac{1}{2}(B - b)/\tan \frac{1}{2}(B + b)$.

All the other formulas given above may be deduced in a similar manner.

EXERCISE 5

1. Solve the quadrantal triangle given in Art. 21 by using formulas (8), (5), and (1) of that article.

Solve the following quadrantal triangles:

2. $C = 67^\circ\ 12'$, $b = 123^\circ\ 48'$.

Ans. $B = 130^\circ\ 00'$, $A = 52^\circ\ 56'$, $a = 59^\circ\ 56'$.

3. $C = 141^\circ\, 02.8'$, $A = 142^\circ\, 05.9'$.

   Ans. $B = 170^\circ\, 15.0'$, $b = 164^\circ\, 29.3'$, $a = 102^\circ\, 10.5'$.

4. $a = 174^\circ\, 12'\, 49''$, $b = 94^\circ\, 08'\, 20''$.

   Ans. $A = 175^\circ\, 57'\, 10''$, $B = 135^\circ\, 42'\, 50''$, $C = 135^\circ\, 34'\, 07''$.

5. $a = 91^\circ\, 30'$, $b = 92^\circ\, 24'$.

6. $C = 136^\circ\, 14.7'$, $A = 141^\circ\, 45.5'$.

7. $a = 112^\circ\, 56'\, 56''$, $C = 74^\circ\, 45'\, 36''$.

8. In a right spherical triangle one side is $95^\circ\, 52'\, 15''$ and the hypotenuse is $95^\circ\, 44'\, 12''$. Find the angle opposite the given side.

   Ans. $91^\circ\, 15'\, 01''$.

9. Solve the right spherical triangle in which $a = 37^\circ\, 40'\, 12''$, $c = 37^\circ\, 40'\, 20''$.

   Ans. $A = 89^\circ\, 25'\, 32''$, $B = 00^\circ\, 43'\, 32''$, $b = 00^\circ\, 26'\, 36''$.

10. Solve the right spherical triangle in which $a = 34^\circ\, 06'\, 13''$, $A = 34^\circ\, 07'\, 41''$.

    Ans. $b = 87^\circ\, 32'\, 39''$, $B = 88^\circ\, 37'\, 21''$, $c = 87^\circ\, 58'\, 00''$.

11. Prove formulas (2), (5) and (10), Art. 22.

12. Verify formulas (3), (6) and (7), Art. 22.

23. Oblique Spherical Triangles Solved by the Method of Right Triangles. Just as every plane triangle can be solved by considering it the sum or difference of two right triangles formed by drawing a perpendicular from a vertex of the triangle to the opposite side or opposite side produced (Pl. Trig., Art. 52), so likewise every oblique spherical triangle ABC may be solved by considering the triangle as the sum (Fig. 25) or the difference (Fig. 26) of the two right triangles ACD and BCD formed by the perpendicular arc of a great circle drawn from one of the vertices to the opposite side or opposite side produced.

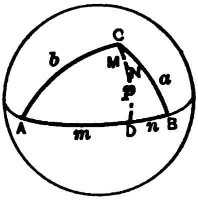


Fig. 25.

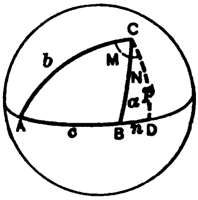


Fig. 26.

We shall denote by m and n the segments AD and DB into which the perpendicular $p = CD$ divides the opposite side c, and by M and N the angles ACD and DCB into which the angle C is divided by the same perpendicular. We then have

$c = m + n$, $C = M + N$ (Fig. 25); $c = m - n$, $C = M - N$ (Fig. 26).

The method of solving oblique spherical triangles by dividing them into right triangles, while exceedingly simple in principle, is not the most convenient method nor the method commonly employed in actual computation. Better methods will be developed in the next chapter and the student is expected to familiarize himself with the methods there presented rather than to depend on the method of the present article.

Case III. Given two sides and the included angle, b, c, A.

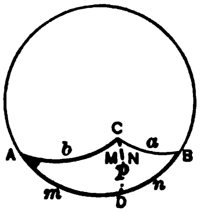


Fig. 27.

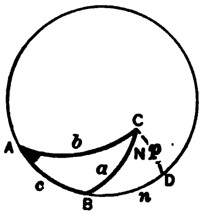


Fig. 28.

Solution. 1. In triangle ACD find p, M and m.
2. $n = c - m$ (Fig. 27), or $n = m - c$ (Fig. 28).
3. In triangle BCD find N, a and B.
4. $C = M + N$ (Fig. 27), or $C = M - N$ (Fig. 28).
5. Check. Repeat the solution drawing the perpendicular from B to the side AC.

Case IV. Given two angles and the included side, B, C, a.

Solution. Solve the polar triangle by Case III and then compute the unknown parts of the original triangle.

Case V. Given two sides and the angle opposite one of them, a, b, A.

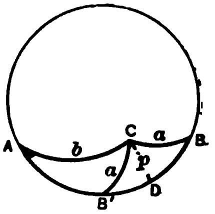


Fig. 29.

Solution. In this case there are two solutions, provided that a is intermediate in value between p and both b and 180° − b (Art. 11).

1. In triangle ACD find p, $M = ACD$, and $m = AD$.

2. In triangle BCD find $N = BCD$, B, and $n = DB$, $AB'C = 180^\circ - B$.

3. $ACB = M + N, ACB' = M - N, AB = m + n, AB' = m - n$.

4. Check. Assume b, c, A as the given parts and find the other parts by Case III.

Case VI. Given two angles and the side opposite one of them, A, B, a.

Solution. Solve the polar triangle by Case V and from it find the unknown parts of the original triangle. As there may be two solutions in Case V so Case VI may have two solutions.

Case I. Given the three sides, a, b, c.

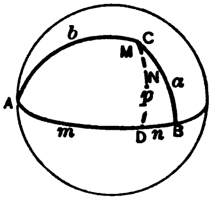


Fig. 30.

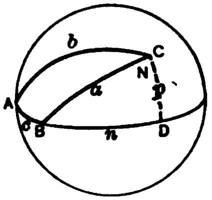


Fig. 31.

Solution. In the triangle ACD we have by Napier's rule

$$\sin \bar{b} = \cos p \cos m, \text{ or } \cos p = \cos b/\cos m.$$

Similarly we have in the triangle BCD

$$\sin \bar{a} = \cos p \cos n, \text{ or } \cos p = \cos a/\cos n.$$

Hence

$$\frac{\cos a}{\cos b} = \frac{\cos m}{\cos n}, \text{ from which } \frac{\cos a - \cos b}{\cos a + \cos b} = \frac{\cos m - \cos n}{\cos m + \cos m}.$$

Now

$$\frac{\cos a - \cos b}{\cos a + \cos b} = \frac{-2 \sin\frac{1}{2}(a+b) \sin\frac{1}{2}(a-b)}{2 \cos\frac{1}{2}(a+b) \cos\frac{1}{2}(a-b)} = -\tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b),$$

so that

$$\tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b) = \tan\tfrac{1}{2}(m+n) \tan\tfrac{1}{2}(m-n),$$

from which

$$\tan\tfrac{1}{2}(m-n) = \tan\tfrac{1}{2}(a+b) \tan\tfrac{1}{2}(a-b) \cot\tfrac{1}{2}c,$$

if
$$m + n = c \text{ (Fig. 30)},$$
or
$$\tan \tfrac{1}{2}(m + n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c,$$
if
$$m - n = c \text{ (Fig. 31)}.$$

We have, therefore, the following steps:

1. Find $\frac{1}{2}(m - n)$ (Fig. 30), or $\frac{1}{2}(m + n)$ (Fig. 31), from the relation
$$\tan \tfrac{1}{2}(m - n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c,$$
$$\tan \tfrac{1}{2}(m + n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c.$$
2. $m = \frac{1}{2}(m + n) + \frac{1}{2}(m - n)$, $n = \frac{1}{2}(m + n) - \frac{1}{2}(m - n)$.
3. In triangle ACD find A and M.
4. In triangle BCD find B and N.
5. $C = M + N$ (Fig. 30), or $C = M - N$ (Fig. 31).
6. Check. Repeat the solution drawing the perpendicular from B on AC or from A on BC.

Case II. Given the three angles, A, B, C.

Solution. Solve the polar triangle by Case I, and from it compute the unknown parts of the original triangle.

EXERCISE 6

1. Show how Case IV may be solved by means of right triangles without using the polar triangle, and outline the steps of the solution.

2. Prove *Bowditch's Rules for Oblique Spherical Triangles* which may be stated as follows: If a spherical triangle is divided into two right triangles by a perpendicular let fall from one of the vertices to the opposite side, and if in the two right triangles the middle parts are so chosen that the perpendicular is an adjacent part in each triangle, then

The sines of the middle parts in the two triangles are proportional to the tangents of the adjacent parts;

but if the perpendicular is an opposite part in each triangle, then

The sines of the middle parts are proportional to the cosines of the opposite parts.

As in the case of Napier's rules, the parts referred to in these rules are the circular parts of the two triangles. By the use of Bowditch's rules the solution of oblique spherical triangles by means of right triangles may be somewhat shortened.

Solve the following triangles by means of right triangles:

3. Given $b = 88^\circ 24'$, $c = 56^\circ 48'$, $A = 128^\circ 16'$;
find $B = 65^\circ 13'$, $C = 49^\circ 28'$, $a = 120^\circ 11'$.

4. Given $a = 103^\circ 44'$, $b = 65^\circ 12'$, $C = 97^\circ 34'$.

5. Given $a = 148^\circ 34.4'$, $b = 142^\circ 11.6'$, $A = 153^\circ 17.6'$;
find $c = 62^\circ 08.6'$, $B = 148^\circ 06.3'$, $C = 130^\circ 21.2'$,
$c' = 7^\circ 18.4'$, $B' = 31^\circ 53.7'$, $C' = 6^\circ 17.6'$.

6. Given $A = 110^\circ$, $B = 62^\circ$, $a = 49^\circ$.

7. Given $A = 80^\circ 20.2'$, $B = 73^\circ 46.7'$, $C = 54^\circ 08.5'$;
find $a = 64^\circ 47.2'$, $b = 61^\circ 47.3'$, $c = 48^\circ 03.4'$.

8. Given $a = 31^\circ 11' 07''$, $b = 32^\circ 19' 18''$, $c = 33^\circ 15' 21''$;
find $A = 59^\circ 29' 42''$, $B = 62^\circ 49' 42''$, $C = 65^\circ 50' 48''$.

9. Given $a = 87^\circ 45' 24''$, $b = 96^\circ 12' 15''$, $c = 100^\circ 08' 56''$.

10. Given $A = 87^\circ 45' 24''$, $B = 96^\circ 12' 15''$, $C = 100^\circ 08' 56''$.

CHAPTER III

PROPERTIES OF OBLIQUE SPHERICAL TRIANGLES

We shall now develop a number of formulas involving the parts of any spherical triangle, from which, if any three parts of the triangle are given, the remaining parts may be derived by computation without first dividing the triangle into right triangles as was done in the last article. Then, in order to facilitate the work of computation, we shall transform these formulas so as to adapt them to the use of logarithms. The actual application of the formulas to the solution of triangles we shall reserve for a separate chapter.

24. The Law of Sines. (*a*) *First Proof.* Let ABC be any spherical triangle, p the perpendicular from one of the vertices C of the triangle to the opposite side AB (Fig. 32) or AB produced (Fig. 33).

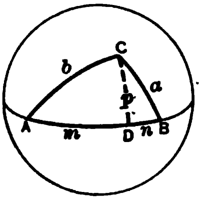


Fig. 32

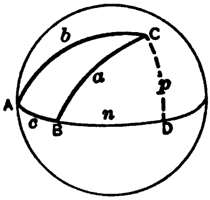


Fig. 33.

By Napier's rules, or the formulas of Art. 13, we have

from triangle ACD $\quad \sin p = \sin b \sin A,$

and from triangle BCD $\quad \sin p = \sin a \sin B \qquad$ (B acute),

or $\quad \sin p = \sin a \sin (180° - B)$

$\qquad = \sin a \sin B \qquad$ (B obtuse).

Hence, whether the perpendicular falls within the triangle or without, we have

$$\sin p = \sin a \sin A = \sin a \sin B.$$

Advancing letters, $\quad \sin c \sin B = \sin b \sin C,$

$$\sin a \sin C = \sin c \sin A$$

These equations may also be written in the form

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}, \tag{1}$$

or in words, ***The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.***

(*b*) *Second Proof.* Let ABC (Fig. 34) be a spherical triangle and O the center of the sphere on which the triangle lies. Draw the radii OA, OB, OC. From C draw CD perpendicular to the plane of AOB and through CD draw planes CDE and CDF perpendicular to OA and OB respectively. Then each of the triangles, OEC, CDE, CDF, OFC, is right-angled, the middle letter being in each case at the right angle. Also since CF and DF are perpendicular to OB, angle CFD is equal to the angle B, and similarly angle CED is equal to the angle A.

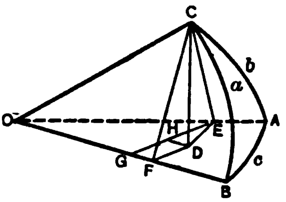


Fig. 34.

Now $CD = CE \sin CED = CE \sin A$,

and $CD = CF \sin CFD = CF \sin B$,

$CE = OC \sin COE = OC \sin b$,

$CF = OC \sin COF = OC \sin a$.

Therefore, substituting in the first two equations for CE and CF their values from the last two, we have

$$OC \sin b \sin A = OC \sin a \sin B,$$

from which

$$\sin a / \sin A = \sin b / \sin B.$$

25. The Law of Cosines. (*a*) *First Proof.* In Figs. 32 and 33 let us denote AD and DB by m and n respectively. By applying Napier's rules, or the formulas of Art. 13, we find

from triangle BCD $\cos a = \cos p \cos n$,

and from triangle ACD $\cos b = \cos p \cos m$.

Now $n = c - m$ (B acute), or $n = m - c$ (B obtuse),

and since $\cos(c - m) = \cos(m - c)$, we have in either case on eliminating $\cos p$ and putting for n its value

$$\begin{aligned}\cos a &= \cos b \cos(c - m)/\cos m \\ &= \cos b \frac{\cos c \cos m + \sin c \sin m}{\cos m} \\ &= \cos b \cos c + \cos b \sin c \tan m.\end{aligned}$$

But by Napier's rules $\tan m = \tan b \cos A$, hence substituting this value in the last equation and remembering that $\cos b \tan b = \sin b$, we have

$$\left.\begin{aligned} &\cos a = \cos b \cos c + \sin b \sin c \cos A. \\ \text{Advancing letters,}\quad &\cos b = \cos c \cos a + \sin c \sin a \cos B, \\ &\cos c = \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (2)$$

These formulas embody the *Law of Cosines: The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides plus the continued product of the sines of these two sides and the cosine of the included angle.*

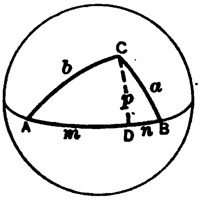


Fig. 32.

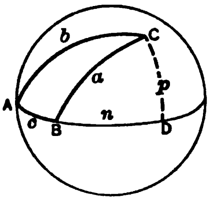


Fig. 33.

(*b*) *Second Proof.* In Fig. 34 draw EG parallel to DF and DH perpendicular to EG, then angle DEH equals angle AOB or c, and we have

$$\frac{HD}{OC} = \frac{HD}{DE} \cdot \frac{DE}{CE} \cdot \frac{CE}{OC} = \sin c \cos A \sin b,$$

$$\frac{HD}{OC} = \frac{OF}{OC} - \frac{OG}{OC} = \frac{OF}{OC} - \frac{OG}{OE} \cdot \frac{OE}{OC} = \cos a - \cos c \cos b.$$

Equating these two values of HD/OC and solving for $\cos a$ we find

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

26. Relation Between Two Angles and Three Sides.

The second of the equations (2) may be written

$$\cos c \cos a + \sin c \sin a \cos B = \cos b,$$

and the first multiplied by $\cos c$ gives

$$\cos c \cos a = \cos b \cos^2 c + \sin b \sin c \cos c \cos A.$$

Subtracting the second of these equations from the first gives

$$\sin c \sin a \cos B = \cos b\,(1 - \cos^2 c) - \sin b \sin c \cos c \cos A.$$

Now $1 - \cos^2 c = \sin^2 c$, hence we may divide the equation by $\sin c$, and obtain

$$\left.\begin{aligned} &\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A. \\ \text{Similarly,}\quad &\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A. \\ &\sin b \cos C = \cos c \sin a - \sin c \cos a \cos B, \\ &\sin b \cos A = \cos a \sin c - \sin a \cos c \cos B, \\ &\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C, \\ &\sin c \cos B = \cos b \sin a - \sin b \cos a \cos C. \end{aligned}\right\} \tag{3}$$

27. Third Proof of the Fundamental Formulas. The three equations (1) Art. 24, (2) Art. 25, and (3) Art. 26, may be derived simultaneously by the method of analytical geometry.* Let ABC be any spherical triangle. Take O, the center of the sphere, for the origin of a system of rectangular coördinates, the plane of BOA for the xy-plane, OB for the direction of the x-axis, and the positive z-axis on the same side of the plane BOA as the vertex C. Join O and C. From C drop the perpendicular CR on BOY, and through CR pass a plane perpendicular to OB cutting OB in S. Then the triangles CRS and CSO have right angles at R and S respectively, and angle RSC equals angle B (why?). Denoting the coördinates of C by x, y, z and the distance OC by r, we have

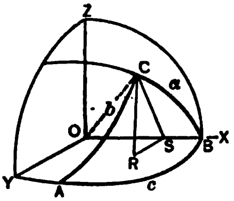


Fig. 35.

$$\begin{aligned} OS &= OC \cos COS, & &\text{or } x = r \cos a, \\ RS &= SC \cos RSC = OC \sin COS \cos RSC, & &\text{or } y = r \sin a \cos B, \\ RC &= SC \sin RSC = OC \sin COS \sin RSC, & &\text{or } z = r \sin a \sin B. \end{aligned}$$

If OA had been taken for the x-axis, the z-axis remaining unchanged, A and a will change places with B and b respectively, and the y coördinates will have opposite signs, so that the new coördinates x', y', z' of C will be

$$x' = r \cos b, \quad y' = - r \sin b \cos A, \quad z' = r \sin b \sin A.$$

But these are the transformed coördinates of a system having the same z-axis while the x- and y-axes are each turned through an angle c, hence the coördinates x, y, z and x', y', z' are related by the transformation formulas,

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* The student without some knowledge of analytical geometry must content himself with the proofs given in the preceding articles and those suggested in the exercises which follow.

$$z = z', \quad x = x' \cos c - y' \sin c, \quad y = x' \sin c + y' \cos c.$$

Substituting in these three formulas the values of x, y, z, x', y', z' in terms of r and the parts of the triangle, we have, after dividing out r,

$$\sin a \sin B = \sin b \sin A, \tag{1}$$
$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{2}$$
$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A. \tag{3}$$

28. Fundamental Relations for the Polar Triangle. If we apply the formulas (1), (2), (3) to the polar triangle, by putting $a = 180° - A'$, $A = 180° - a'$, etc. (Art. 7), and then drop the accents, we find that (1) remains unchanged, while (2) and (3) give rise to the new sets of formulas:

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c, \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} \sin A \cos b &= \cos B \sin C + \sin B \cos C \cos a, \\ \sin A \cos c &= \cos C \sin B + \sin C \cos B \cos a, \\ \sin B \cos c &= \cos C \sin A + \sin C \cos A \cos b, \\ \sin B \cos a &= \cos A \sin C + \sin A \cos C \cos b, \\ \sin C \cos a &= \cos A \sin B + \sin A \cos B \cos c, \\ \sin C \cos b &= \cos B \sin A + \sin B \cos A \cos c. \end{aligned}\right\} \tag{5}$$

29. Arithmetic Solution of Spherical Triangles. The fundamental relations (1), (2), (3) enable us to solve every case of oblique spherical triangles.

Case I. *Given the three sides*, a, b, c.

1. The angle A may be found by the law of cosines.
2. The angles B and C may then be found by the law of sines.

Case III. *Given two sides and the included angle*, a, b, C.

1. The third side may be found by the law of cosines.
2. The angles A and B may then be found by the law of sines.

Case V. *Given two sides and the angle opposite one of them*, a, b, A.

1. The angle B may be found by the law of sines.
2. The third side might be found by the law of cosines but since the law of cosines involves both $\sin c$ and $\cos c$ the formula solved for

either sin c or cos c would involve radical expressions. These may be avoided by using the formula

$$\cos c = \frac{\cos a \cos b - \sin a \sin b \cos A \cos B}{1 - \sin^2 b \sin^2 A},$$

which is obtained by eliminating sin c from the formulas (2) and (3) of Art. 27.

3. The angle C may now be found by the law of sines.

Cases II, IV, VI. These may be referred to Cases I, III, V, respectively, by making use of the polar triangle, or we may apply formulas (1), (4), (5).

While the fundamental relations (1), (2), (3) make it possible to solve each of the six cases of triangles, it is clear that (2) and (3) are not adapted to logarithmic computation. Therefore, in order to facilitate computation, it is desirable to obtain other formulas which enable us to use logarithms. Such formulas will be developed in the following articles.

EXERCISE 7

1. If a', b', c' denote the sides of the polar triangle, show that

$$\sin a : \sin b : \sin c = \sin a' : \sin b' : \sin c'.$$

2. If m is the arc joining the vertex C of a spherical triangle to the middle point of the opposite side, show that

$$\cos a + \cos b = 2 \cos m \cos \tfrac{1}{2} c.$$

3. If the bisector of the angle C meets the opposite side in D, show that

$$\sin a : \sin b = \sin BD : \sin AD.$$

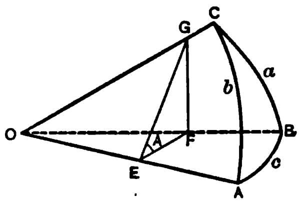


Fig. 36.

4. State in words the laws expressed by formulas (4) and (5), Art. 28.

5. In Fig. 36 let EGF be the triangle in which a plane drawn perpendicular to an edge OA intersects the trihedral angle. Then

$$GF^2 = OF^2 + OG^2 - 2\,OF \cdot OG \cdot \cos a.$$
$$GF^2 = EF^2 + EG^2 - 2\,EF \cdot EG \cdot \cos A.$$

Subtracting and observing that $OF^2 - EF^2 = OE^2, OG^2 - EG^2 = OE^2$, we find

$$2\,OF \cdot OG \cdot \cos a = 2\,OE^2 + 2\,EF \cdot EG \cdot \cos A.$$

which, on dividing by $2\, OF \cdot OG$, leads to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

This constitutes a *fourth proof of the law of cosines.*

6. From the law of cosines

$$\cos A = (\cos a - \cos b \cos c)/\sin b \sin c,$$

show that

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 A}{\sin^2 a} = \frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 a\sin^2 b\sin^2 c}.$$

The expression on the right is symmetrical in a, b, and c, hence

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}, \text{ from which } \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

This constitutes a *fourth proof of the law of sines.*

7. Prove the relation

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Suggestion. Multiply the third of the equations (2), Art. 25, by $\cos b$, substitute in the first equation and divide by $\sin b \sin c$.

8. By interchanging and advancing letters write down five other equations like that in Problem 7.

9. Apply the relations of Problems 7 and 8 to the polar triangle. Do the resulting equations express new relations?

10. Given $b = 135°$, $c = 45°$, $A = 60°$; find the remaining parts to the nearest degree.

Ans. $a = 104°$, $B = 141°$, $C = 39°$.

11. Given $a = 120°$, $b = 60°$, $A = 135°$; find the remaining parts to the nearest minute.

Ans. $B = 45° \, 00'$, $c = 78° \, 28'$, $C = 53° \, 08'$.

12. Given $a = 135°$, $b = 135°$, $c = 45°$; find A, B, C, to the nearest minute.

Ans. $A = B = 114° \, 28'$, $C = 65° \, 32'$.

30. Functions of Half the Angles in Terms of the Sides.

From the law of cosines

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = 1 - 2\sin^2 \tfrac{1}{2} A, \quad \text{(Pl. Trig., Art. 111)}$$

therefore

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

Now

$$\cos(b-c)-\cos a=2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a-b+c)\quad\text{(Pl. Trig., Art. 113)}$$

$$=2\sin(s-c)\sin(s-b),\text{ where } s=\tfrac{1}{2}(a+b+c),$$

therefore

$$2\sin^2\tfrac{1}{2}A=\frac{2\sin(s-b)\sin(s-c)}{\sin \mathrm{b}\sin c},$$

or

$$\left.\begin{aligned}\sin\tfrac{1}{2}A&=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.\\ \text{Similarly,}\quad \sin\tfrac{1}{2}B&=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin \mathrm{a}}},\\ \sin\tfrac{1}{2}C&=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}},\end{aligned}\right\}\qquad(6)$$

$$s=\tfrac{1}{2}(a+b+c).$$

Corresponding formulas for the cosines of half the angles may be obtained by applying the formulas (6) to the co-lunar triangles. Thus by applying the first formula to the co-lunar triangle $AB'C$ whose parts are (Art. 5) $180°-A$, B, $180°-C$, $180°-a$, b, $180°-\mathrm{c}$,

$$\left.\begin{aligned}\text{we obtain}\quad \cos\tfrac{1}{2}A&=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.\\ \text{Similarly,}\quad \cos\tfrac{1}{2}B&=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}},\\ \cos\tfrac{1}{2}C&=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.\end{aligned}\right\}\qquad(7)$$

To find $\tan\frac{1}{2}A$ we divide $\sin\frac{1}{2}A$ by $\cos\frac{1}{2}A$ and obtain

$$\left.\begin{aligned}\tan\tfrac{1}{2}A&=\frac{\tan k}{\sin(s-a)},\\ \tan\tfrac{1}{2}B&=\frac{\tan k}{\sin(s-b)},\\ \tan\tfrac{1}{2}C&=\frac{\tan k}{\sin(s-c)},\\ \text{where}\quad \tan k&=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.\end{aligned}\right\}\qquad(8)$$

k is the arcual radius of the small circle inscribed in the triangle ABC, for if O (Fig. 37) represents the intersection of the arcs bisecting the angles of the triangle, OF, the arc drawn from O perpendicular to one of the sides as AB, will be the arcual radius of the inscribed circle. It follows, just as in the case of plane triangles (Pl. Trig., Art. 68), that $AF = s - a$, hence denoting OF by k and applying Napier's rules to the right triangle AOF, we have

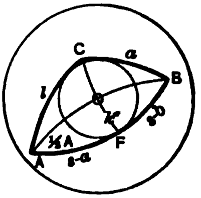


Fig. 37.

$$\sin(s - a) = \cot \tfrac{1}{2} A \tan k,$$

or

$$\tan \tfrac{1}{2} A = \tan k / \sin(s - a).$$

31. Functions of Half the Sides in Terms of the Angles. If we apply the formulas (6) and (7), Art. 30, to the polar triangle (Art. 7), by putting $A = 180° - a'$, $a = 180° - A'$, $B = 180° - b'$, etc., dropping the accents in the final results, we obtain

$$\left.\begin{aligned}\sin \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos(S - A)}{\sin B \sin C}}\\ \sin \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos(S - B)}{\sin C \sin A}}\\ \sin \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos(S - C)}{\sin A \sin B}}\end{aligned}\right\} \qquad (9)$$

$$\left.\begin{aligned}\cos \tfrac{1}{2} a &= \sqrt{\frac{\cos(S - B) \cos(S - C)}{\sin B \sin C}}\\ \cos \tfrac{1}{2} b &= \sqrt{\frac{\cos(S - C) \cos(S - A)}{\sin C \sin A}}\\ \cos \tfrac{1}{2} c &= \sqrt{\frac{\cos(S - A) \cos(S - B)}{\sin A \sin B}}\end{aligned}\right\} \qquad (10)$$

$$S = \tfrac{1}{2}(A + B + C).$$

From (9) and (10) we find

$$\left.\begin{aligned}\tan \tfrac{1}{2} a &= \tan K \cos(S - A),\\ \tan \tfrac{1}{2} b &= \tan K \cos(S - B),\\ \tan \tfrac{1}{2} c &= \tan K \cos(S - C),\\ \text{where}\quad \tan K &= \sqrt{\frac{-\cos S}{\cos(S - A) \cos(S - B) \cos(S - C)}}.\end{aligned}\right\} \qquad (11)$$

K is the arcual radius of the small circle circumscribed about the triangle ABC, for if O (Fig. 38) is the center of this circle, OA, OB, OC the arcs joining the center to the vertices of the triangle, OF the perpendicular arc from O to one of the sides as BC, then $OA = OB = OC = K$, the triangles AOB, BOC, COA are isosceles, and $BF = FC = \frac{1}{2}a$. Furthermore

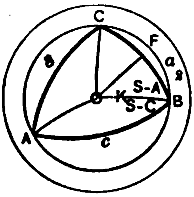


Fig. 38.

$$A = BAO + OAC = ABO + ACO = (B - OBF) + (C - OCF) = B + C - 2\,OBF,$$

hence
$$OBF = \tfrac{1}{2}(B + C - A) = S - A,$$

where
$$S = \tfrac{1}{2}(A + B + C).$$

If now we apply Napier's rules to the right triangle BOF, we find

$$\cos OBF = \cot BO \tan BF$$

or
$$\cos (S - A) = \cot K \tan \tfrac{1}{2} a,$$

from which
$$\tan \tfrac{1}{2} a = \tan K \cos (S - A).$$

EXERCISE 8

1. Prove the formula for sin $\frac{1}{2}C$ (Art. 30) directly by using the relation

$$\cos C = 1 - 2 \sin^2 \tfrac{1}{2} C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

2. Prove the formula for cos $\frac{1}{2}A$ (Art. 30) directly by using the relation

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

and following the method used in deriving the formula for sin $\frac{1}{2}A$.

3. Prove the formula for sin $\frac{1}{2}a$ (Art. 31) directly by using the relation

$$\cos a = 1 - 2 \sin^2 \tfrac{1}{2} a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

4. Prove the formula for cos $\frac{1}{2}a$ (Art. 31) directly by using the relation

$$\cos a = 2 \cos^2 \tfrac{1}{2} a - 1 = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

5. Derive the formula for $\tan \frac{1}{2} a$ (Art. 31) by applying the formula for $\tan \frac{1}{2} A$ (Art. 30) to the polar triangle.

6. Derive the formula for $\cos \frac{1}{2} A$ by applying the formula for $\sin \frac{1}{2} A$ to the co-lunar triangle ABC'.

7. Apply the formula for $\sin \frac{1}{2} A$ to the co-lunar triangle $A'BC$. Does the resulting formula express a new relation?

8. The escribed circles of the triangle ABC are the small circles inscribed in the co-lunar triangles $A'BC$, $AB'C$, ABC'. By applying the formula for $\tan k$ (Art. 30) to these triangles, show that the arcual radii, k_a, k_b, k_c of the escribed circles are given by the formulas

$$\tan k_a = \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}} = \sin s \tan \tfrac{1}{2} A,$$

$$\tan k_b = \sin s \tan \tfrac{1}{2} B, \qquad \tan k_c = \sin s \tan \tfrac{1}{2} C.$$

9. By applying the formula for $\tan K$ (Art. 31) to the co-lunar triangle $A'BC$, show that the arcual radius of the circle circumscribing this triangle is given by the formula

$$\tan K_A = \sqrt{\frac{\cos (S-A)}{-\cos S \cos (S-B) \cos (S-C)}} = -\tan \tfrac{1}{2} a / \cos S,$$

hence also $\tan K_B = -\tan \frac{1}{2} b / \cos S$, $\tan K_C = -\tan \frac{1}{2} c / \cos S$.

10. Show that

$$2 \tan K = \cot k_a + \cot k_b + \cot k_c - \cot k,$$

and

$$2 \cot k = \tan K_A + \tan K_B + \tan K_C - \tan K.$$

32. Delambre's (or Gauss's) Proportions. By Pl. Trig., Art. 106, we have

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

Substituting for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} B$, $\cos \frac{1}{2} A$, $\sin \frac{1}{2} B$, their values from (6) and (7), Art. 30,

$$\begin{aligned}\sin \tfrac{1}{2}(A+B) &= \sqrt{\frac{\sin (s-b) \sin (s-c) \sin s \sin (s-b)}{\sin a \sin b \sin^2 c}} \\ &\quad + \sqrt{\frac{\sin s \sin (s-a) \sin (s-c) \sin (s-a)}{\sin a \sin b \sin^2 c}} \\ &= \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} \cdot \frac{\sin (s-b) + \sin (s-a)}{\sin c} \\ &= \cos \tfrac{1}{2} C \frac{\sin (s-b) + \sin (s-a)}{\sin c}.\end{aligned}$$

Also by Pl. Trig., Art. 111, 113,

$$\sin(s-b)+\sin(s-a) = 2\sin\tfrac{1}{2}(s-b+s-a)\cos\tfrac{1}{2}(s-b-s+a)$$
$$= 2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(a-b),$$

and
$$\sin c = 2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c,$$

hence

$$\frac{\sin(s-b)+\sin(s-a)}{\sin c} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c},$$

and
$$\sin\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)\cos\frac{1}{2}C}{\cos\frac{1}{2}c}.$$

Similarly, we obtain corresponding formulas for $\cos\frac{1}{2}(A+B)$, $\sin\frac{1}{2}(A-B)$ and $\cos\frac{1}{2}(A-B)$. The four formulas, of which the third and fourth may also be obtained by applying the first and second to either one of the co-lunar triangles $A'BC$ or $AB'C$, may be written

$$\left.\begin{aligned}
\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C,\\
\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c &= \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C,\\
\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C,\\
\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c &= \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.
\end{aligned}\right\} \qquad (12)$$

These formulas are known as *Delambre's* or *Gauss's* proportions or equations.

33. Napier's Proportions. If of the equations (12) we divide the first by the second, then the third by the fourth, then the fourth by the second, and finally the third by the first, we obtain the following four new formulas which are known as *Napier's* proportions or analogies.

$$\left.\begin{aligned}
\tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C,\\
\tan\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C,\\
\tan\tfrac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c,\\
\tan\tfrac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.
\end{aligned}\right\} \qquad (13)$$

The second of these formulas may also be obtained by applying the first to either of the co-lunar triangles $A'BC$ or $AB'C$, and the third and fourth by applying the first and second to the polar triangle.

If we divide the first of the equations (13) by the second, or the third by the fourth, we obtain the *law of tangents*

$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

34. Formulas for the Area of a Spherical Triangle. It is shown in Solid Geometry that the area of a spherical triangle is given by the formula

$$T = \frac{\pi R^2 E^\circ}{180^\circ}, \tag{14}$$

where R is the radius of the sphere, and E° the *Spherical Excess* expressed in degrees, that is $E^\circ = A + B + C - 180^\circ$.

If E is the spherical excess expressed in radians, $E = E^\circ \cdot \pi/180$, and (14) becomes

$$T = R^2 E. \tag{15}$$

For a unit sphere ($R = 1$)

$$T = E, \tag{16}$$

hence we have

Theorem I. The area of a spherical triangle on a unit sphere is equal to the spherical excess expressed in radians.

Theorem II. The area of a spherical triangle on any sphere is equal to the area of the corresponding triangle on a unit sphere multiplied by the square of the radius.

The problem of finding various expressions for the area of a spherical triangle resolves itself, therefore, into the problem of finding various expressions for the spherical excess E.

(*a*) *In terms of the angles, A, B, C.*

$$E = 2S - \pi, \quad \text{where} \quad S = \tfrac{1}{2}(A + B + C). \tag{17}$$

(*b*) *In terms of the sides, a, b, c.*

We have

$$\begin{aligned}\sin \tfrac{1}{2} E &= \sin (S - \tfrac{1}{2}\pi) = \sin [\tfrac{1}{2}(A+B) + \tfrac{1}{2}(C - \pi)] \\ &= \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} C.\end{aligned}$$

Substituting for $\sin \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(A+B)$ their values from (12), we have

$$\begin{aligned}\sin \tfrac{1}{2} E &= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c}[\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2}(a+b)] \\ &= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c}(2 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b).\end{aligned}$$

Finally by putting for $\sin \frac{1}{2} C$ and $\cos \frac{1}{2} C$ their values from (6) and (7) we arrive at

Cagnoli's Formula,

$$\sin \tfrac{1}{2} E = \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}, \tag{18}$$

where $n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}$.

Or we may proceed as follows: $\frac{1}{2}(C - E) = \frac{1}{2}\pi - \frac{1}{2}(A + B)$,

and therefore, $\sin \frac{1}{2}(C - E) = \cos \frac{1}{2}(A + B)$.

This value substituted in the second of the equations (12) gives

$$\sin \tfrac{1}{2}(C - E) : \sin \tfrac{1}{2} C = \cos \tfrac{1}{2}(a + b) : \cos \tfrac{1}{2} c.$$

From this proportion we have by division and composition

$$\frac{\sin \frac{1}{2} C - \sin \frac{1}{2}(C - E)}{\sin \frac{1}{2} C + \sin \frac{1}{2}(C - E)} = \frac{\cos \frac{1}{2} c - \cos \frac{1}{2}(a + b)}{\cos \frac{1}{2} c + \cos \frac{1}{2}(a + b)}.$$

On reducing each member of this equation by means of the relations of Art. 113 (Pl. Trig.), we obtain

$$\tan \tfrac{1}{4} E \cot \tfrac{1}{4}(2C - E) = \tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s - c);$$

In like manner, by substituting $\cos \frac{1}{2}(C - E) = \sin \frac{1}{2}(A + B)$ in the first of the equations (12), we find

$$\tan \tfrac{1}{4} E \tan \tfrac{1}{4}(2C - E) = \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b);$$

hence on multiplying these two equations and extracting the square-root we obtain

Lhuilier's Formula,

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}. \tag{19}$$

(*c*) *In terms of two sides and the included angle, a, b, C.*

$$\tan \tfrac{1}{2} E = \frac{\sin (S - \frac{1}{2}\pi)}{\cos (S - \frac{1}{2}\pi)} = \frac{-\cos \frac{1}{2}(A + B + C)}{\sin \frac{1}{2}(A + B + C)}$$

$$= \frac{\sin \frac{1}{2}(A + B) \sin \frac{1}{2} C - \cos \frac{1}{2}(A + B) \cos \frac{1}{2} C}{\sin \frac{1}{2}(A + B) \cos \frac{1}{2} C + \cos \frac{1}{2}(A + B) \sin \frac{1}{2} C}.$$

Substituting for $\sin \frac{1}{2}(A + B)$ and $\cos \frac{1}{2}(A + B)$ their values from (12), we have

$$\tan \tfrac{1}{2} E = \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C [\cos \frac{1}{2}(a - b) - \cos \frac{1}{2}(a + b)]}{\cos \frac{1}{2}(a - b) \cos^2 \frac{1}{2} C + \cos \frac{1}{2}(a + b) \sin^2 \frac{1}{2} C},$$

which readily reduces to

$$\tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C}. \tag{20}$$

35. Plane and Spherical Triangle Formulas Compared. The student will have observed that there is a striking resemblance between the formulas relating to plane triangles and certain of the formulas of the present chapter. In the table below are arranged

Plane Triangles.	Spherical Triangles.
I. *Law of Sines*	I. *Law of Sines*
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$
II. *Law of Cosines*	II. *Law of Cosines*
$a^2 = b^2 + c^2$ $- 2bc \cos A$	$\cos a = \cos b \cos c$ $+ \sin b \sin c \cos A$
III. *Double Formulas*	III. *Delambre's Proportions*
$\sin \frac{1}{2}(A - B) \cdot \frac{1}{2} c$ $= \frac{1}{2}(a - b) \cos \frac{1}{2} C$ $\cos \frac{1}{2}(A - B) \cdot \frac{1}{2} c$ $= \frac{1}{2}(a + b) \sin \frac{1}{2} C$	$\sin \frac{1}{2}(A - B) \sin \frac{1}{2} c$ $= \sin \frac{1}{2}(a - b) \cos \frac{1}{2} C$ $\cos \frac{1}{2}(A - B) \sin \frac{1}{2} c$ $= \sin \frac{1}{2}(a + b) \sin \frac{1}{2} C$
IV. *Law of Tangents*	IV. *Law of Tangents*
$\frac{\frac{1}{2}(a + b)}{\frac{1}{2}(a - b)} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$	$\frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$
V. *Half-angle Formulas*	V. *Half-angle Formulas*
$\sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}$ $\cos \frac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}$ $\tan \frac{1}{2} A = \frac{k}{s - a}$ $k = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$	$\sin \frac{1}{2} A = \sqrt{\frac{\sin(s - b) \sin(s - c)}{\sin b \sin c}}$ $\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s - a)}{\sin b \sin c}}$ $\tan \frac{1}{2} A = \frac{\tan k}{\sin(s - a)}$ $\tan k = \sqrt{\frac{\sin(s - a) \sin(s - b) \sin(s - c)}{\sin s}}$
VI. *Area*	VI. *Area*
$\frac{1}{4} T = \sqrt{\frac{s}{2} \cdot \frac{s - a}{2} \cdot \frac{s - b}{2} \cdot \frac{s - c}{2}}$	$\tan \frac{1}{4} E$ $= \sqrt{\tan \frac{s}{2} \tan \frac{s - a}{2} \tan \frac{s - b}{2} \tan \frac{s - c}{2}}$ $T = r^2 E$

in parallel columns the more important formulas for plane triangles and the corresponding formulas for spherical triangles. The form of some of the formulas for plane triangles has been slightly changed in order to manifest the resemblance in the most striking manner.

36. Derivation of Formulas for Plane Triangles from Those of Spherical Triangles. We will now show that the resemblance between the two sets of formulas is not accidental but is due to a definite relation between plane and spherical triangles. If the vertices of a spherical triangle remain fixed while the radius (r) of the sphere on which the triangle is situated is indefinitely increased, the spherical triangle will approach as a limit the plane triangle having the same vertices. Consequently, for the limit $r = \infty$, the formulas for the spherical triangle must reduce to those for the plane triangle.

Let a', b', c' represent the sides of the spherical triangle expressed in radians, then $a' = a/r$, $b' = b/r$, $c' = c/r$, where a, b, c represent the actual lengths of the sides (Pl. Trig., Art. 90). Also by Pl. Trig., Art. 176, we have

$$\sin a' = \frac{a}{r} - \frac{a^3}{3!\,r^3} + \text{etc.},\quad \cos a' = 1 - \frac{a^2}{2!\,r^2} + \frac{a^4}{4!\,r^4} - \text{etc.},$$

$$\tan a' = \frac{a}{r} + \frac{a^3}{3\,r^3} + \text{etc.}$$

and similar expressions for sin b', sin c', etc.

These expansions involve the radius of the sphere. If now we substitute these expansions in any formula relating to spherical triangles and evaluate the resulting expression for $r = \infty$, the resulting formula will express the corresponding relation between the sides and angles of the plane triangle. We will illustrate the method by some examples.

(*a*) *The Law of Sines.*

$$\frac{\sin A}{\sin B} = \frac{\sin a'}{\sin b'} = \frac{\sin a/r}{\sin b/r} = \frac{\dfrac{a}{r} - \dfrac{a^3}{3!\,r^3} + \text{etc.},}{\dfrac{b}{r} - \dfrac{b^3}{3!\,r^3} + \text{etc.},}.$$

Multiplying both numerator and denominator of the expression on the right by r, and making r infinite, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b},\ \text{the law of sines for plane triangles.}$$

(*b*) *The Law of Cosines.*

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A$$

or
$$\cos\frac{a}{r} = \cos\frac{b}{r}\cos\frac{c}{r} + \sin\frac{b}{r}\sin\frac{c}{r}\cos A,$$

hence

$$1 - \frac{a^2}{2!r^2} + \frac{a^4}{4!r^4} + = \left(1 - \frac{b^2}{2!r^2} + \frac{b^4}{4!r^4} +\right)\left(1 - \frac{c^2}{2!r^2} + \frac{c^4}{4!r^4} +\right)$$
$$+ \left(\frac{b}{r} - \frac{b^3}{3!r^3} +\right)\left(\frac{c}{r} - \frac{c^3}{3!r^3} +\right)\cos A.$$

If we multiply both sides of the equation by $-2r^2$, drop the terms which are common to both sides of the equation, and then make r infinite, we have

$a^2 = b^2 + c^2 - 2bc \cos A$, the law of cosines for plane triangles.

(*c*) *The Law of Tangents.*

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}(a'+b')}{\tan\frac{1}{2}(a'-b')} = \frac{\tan\frac{a+b}{2r}}{\tan\frac{a-b}{2r}} = \frac{\frac{a+b}{2r} + \frac{(a+b)^3}{3(2r)^3} +}{\frac{a-b}{2r} + \frac{(a-b)^3}{3(2r)^3} +}.$$

Multiplying both numerator and denominator on the right by $2r$ and making r infinite, we have

$\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \dfrac{a+b}{a-b}$, the law of tangents for plane triangles.

(*d*) *Area of a Triangle.* As a final example we will deduce *Hero's* formula for the area of a plane triangle from *Lhuillier's* formula for the spherical excess.

Denote $a' + b' + c'$ by $2s'$, then $s' = s/r$, $s' - a' = (s-a)/r$, $s' - b' = (s-b)/r$, etc., and we have from Lhuillier's formula

$$\frac{E}{4} + \frac{E^3}{3\cdot 4^3} + =$$
$$\sqrt{\left(\frac{s}{2r} + \frac{s^3}{3(2r)^3} +\right)\left(\frac{s-a}{2r} + \frac{(s-a)^3}{3(2r)^3} +\right)\left(\frac{s-b}{2r} + \frac{(s-b)^3}{3(2r)^3} +\right)\left(\frac{s-c}{2r} + \frac{(s-c)^3}{3(2r)^3} +\right)}.$$

Multiplying through by $4r^2$ gives

$$r^2E + \frac{r^2E\cdot E^2}{3\cdot 4^2} + =$$
$$\sqrt{\left(s + \frac{s^3}{12r^2} +\right)\left(s-a + \frac{(s-a)^3}{12r^2} +\right)\left(s-b + \frac{(s-b)^3}{12r^2} +\right)\left(s-c + \frac{(s-c)^3}{12r^2} +\right)}.$$

Now as r approaches infinity, E approaches 0, r^2E remains equal to the area of the triangle, hence in the limit

$$r^2E = T = \sqrt{s(s-a)(s-b)(s-c)}$$

which is Hero's formula.

EXERCISE 9

1. Derive the second of the formulas (12).

2. Derive the third and fourth of the formulas (12) by applying the first and second to the co-lunar triangle $AB'C$.

3. Derive the fourth of the formulas (13) by applying the third to the co-lunar triangle $AB'C$.

4. Derive the fourth of the formulas (13) by applying the second to the polar triangle.

5. Show that the area of the co-lunar triangle $A'BC, AB'C, ABC'$ is $r^2(2A - E)$, $r^2(2B - E)$, $r^2(2C - E)$, respectively, where E is the spherical excess of the triangle ABC.

6. Prove that

$$\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s = 4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c.$$

7. If S, S_A, S_B, S_C denote half the sums of the angles of a triangle and its three co-lunars respectively, prove that

$$S + S_A + S_B + S_C = 3\pi.$$

8. If E, E_A, E_B, E_C denote the spherical excesses of a triangle and its three co-lunars respectively, show that $E + E_A + E_B + E_C = 2\pi$, and hence that the sum of the area of these triangles is equal to half the area of the sphere.

9. Deduce the double formula for plane triangles from Delambre's formulas for spherical triangles.

10. Deduce the half-angle formulas for plane triangles from the corresponding formulas for spherical triangles.

11. From the formula $\cos c = \cos a \cos b$ for right spherical triangles deduce the formula $c^2 = a^2 + b^2$ for plane right triangles.

12. If K, K_A, K_B, K_C denote the arcual radii of a triangle and its three co-lunars, show that $\tan K \cot K_A \cot K_B \cot K_C = \cos^2 S$.

CHAPTER IV

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

37. Preliminary Observations. In Art. 23 it was shown that every spherical triangle may be solved by the method of right triangles. Again every spherical triangle may be solved by means of the fundamental relations of Art. 27, as was shown in Art. 29. The purpose of the present chapter is to present the most approved methods, which, though based on apparently more complicated formulas, require, as a rule, the least possible amount of computation, and are, therefore, commonly employed by computers.

The computer will do well to observe the following points:

(*a*) The arrangement of the work should be orderly and methodical. A complete schedule for the tabular work should be made out before the tables are used (Pl. Trig., Art. 70).

(*b*) It will be well to letter the given parts as in the illustrations which follow. Thus if the given parts are two sides and the included angle, call the larger of the two sides a, the other b, and the angle C. This is easier than to rewrite the formulas so as to involve other letters.

(*c*) Remember that a small angle cannot be accurately found from its cosine, nor an angle near 90° from its sine. (Pl. Trig., Art. 21.) Usually there is a choice of formulas which will enable us to avoid any inaccuracies arising from this source.

(*d*) Remember also that the answer cannot be more accurate than the least accurate of the given parts. It is a false show of accuracy to compute the answer to the nearest second when one or more of the given parts have a lesser accuracy. (Pl. Trig., Art. 44, 19.)

(*e*) No result can be relied upon unless it has been checked. When the answer is given, that may be looked upon as a check, in all other cases the computer must provide a check of his own.

38. Case I. Given the Three Sides, *a, b, c*.

Solution.

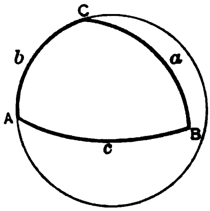
1. To find A, B, C. Use the half-angle formulas (8).
2. Check. Use the law of sines.

Note. If one angle only is required it is better to use (6) or (7).

EXAMPLE.

Given

$a = 123^\circ\, 34'\, 45''$,
$b = 75^\circ\, 56'\, 33''$,
$c = 105^\circ\, 00'\, 18''$.



To find

$A = 121^\circ\, 32'\, 41''$,
$B = 82^\circ\, 52'\, 53''$,
$C = 98^\circ\, 51'\, 55''$.

Fig. 39.

Solution.

1. To find A, B, and C.

$$\tan \tfrac{1}{2} A = \frac{\tan k}{\sin (s-a)}, \quad \tan \tfrac{1}{2} B = \frac{\tan k}{\sin (s-b)}, \quad \tan \tfrac{1}{2} C = \frac{\tan k}{\sin (s-c)},$$

$$\tan k = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}, \quad s = \frac{a+b+c}{2}.$$

$$\begin{aligned}
a &= 123^\circ\, 34'\, 45'' \\
b &= 75^\circ\, 56'\, 33'' \\
c &= \underline{105^\circ\, 00'\, 18''} \\
2s &= 304^\circ\, 31'\, 36'' \\
s &= 152^\circ\, 15'\, 48'' \\
s-a &= 28^\circ\, 41'\, 03'' \\
s-b &= 76^\circ\, 19'\, 15'' \\
s-c &= \underline{47^\circ\, 15'\, 30''} \\
2s &= 152^\circ\, 15'\, 48'' \text{ (check)} \\
\tfrac{1}{2} A &= 60^\circ\, 46'\, 20.7'' \\
\tfrac{1}{2} B &= 41^\circ\, 26'\, 26.4'' \\
\tfrac{1}{2} C &= 49^\circ\, 25'\, 57.7''
\end{aligned}$$

$$\begin{aligned}
\log \sin (s-a) &= 9.68122 \\
\log \sin (s-b) &= 9.98751 \\
\log \sin (s-c) &= 9.86594 \\
\operatorname{colog} \sin s &= \underline{0.33215} \\
\log \tan^2 k &= 9.86682 \\
\log \tan k &= 9.93341 \\
\log \tan \tfrac{1}{2} A &= 0.25219 \\
\log \tan \tfrac{1}{2} B &= 9.94590 \\
\log \tan \tfrac{1}{2} C &= 0.06747 \\
A &= 121^\circ\, 32'\, 41'', \\
B &= 82^\circ\, 52'\, 53'', \\
C &= 98^\circ\, 51'\, 55''.
\end{aligned}$$

2. Check.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\begin{array}{lll}
\log \sin a = 9.92071 & \log \sin b = 9.98680 & \log \sin c = 9.98493 \\
\log \sin A = \underline{9.93056} & \log \sin B = \underline{9.99664} & \log \sin C = \underline{9.99478} \\
 9.99015 & 9.99016 & 9.99015
\end{array}$$

EXERCISE 10

Solve the following oblique triangles:

1. Given $a = 72^\circ\, 16'$, $b = 80^\circ\, 44'$, $c = 41^\circ\, 18'$.
Ans. $A = 73^\circ\, 38'$, $B = 96^\circ\, 12'$, $C = 41^\circ\, 40'$.

2. Given $a = 109^\circ\,45'$, $b = 73^\circ\,56'$, $c = 54^\circ\,32'$.

3. Given $a = 105^\circ\,06.8'$, $b = 93^\circ\,39.9'$, $c = 50^\circ\,20.3'$.
Ans. $A = 106^\circ\,38.0'$, $B = 82^\circ\,04.4'$, $C = 49^\circ\,49.2'$.

4. Given $a = 27^\circ\,43.8'$, $b = 49^\circ\,36.8'$, $c = 55^\circ\,19.7'$.

5. Given $a = 120^\circ\,22'\,40''$ $b = 111^\circ\,34'\,27''$, $c = 96^\circ\,28'\,35''$.
Ans. $A = 126^\circ\,18'\,42''$, $B = 119^\circ\,42'\,08''$, $C = 111^\circ\,51'\,42''$.

6. Given $a = 20^\circ\,45'\,23''$, $b = 55^\circ\,56'\,56''$, $c = 67^\circ\,25'\,54''$.

7. Given $a = 131^\circ\,35'\,04''$, $b = 108^\circ\,30'\,14''$, $c = 84^\circ\,46'\,34''$, $A = 132^\circ\,14'\,21''$. Find B and C.

8. Given $a = 35^\circ\,30'\,24''$, $b = 38^\circ\,57'\,12''$, $c = 56^\circ\,15'\,43''$. Find $B = 47^\circ\,37'\,21''$.

39. Case II. Given the Three Angles, *A, B, C*.

Solution.

1. To find a, b, c. Use the half-angle formula (11).

2. Check. Use the law of sines.

Note. If one side only is required it is better to use (9) or (10).

EXAMPLE.

Given	To find
$A = 121^\circ\,32'\,41''$,	$a = 123^\circ\,34'\,46''$,
$B = 82^\circ\,52'\,53''$,	$b = 75^\circ\,56'\,32''$,
$C = 98^\circ\,51'\,55''$.	$c = 105^\circ\,00'\,18''$.

Solution.

1. To find a, b, c.

$$\tan \tfrac{1}{2} a = \tan K \cos (S - A),\quad \tan \tfrac{1}{2} b = \tan K \cos (S - B),$$

$$\tan \tfrac{1}{2} c = \tan K \cos (S - C),$$

$$\tan K = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}},\quad S = \frac{A + B + C}{2}.$$

$$\begin{array}{rlrl}
A = & 121^\circ\,32'\,41'' & & \\
B = & 82^\circ\,52'\,53'' & S - A = & 30^\circ\,06'\,03.5'' \\
C = & \underline{98^\circ\,51'\,55''} & S - B = & 68^\circ\,45'\,51.5'' \\
2S = & 303^\circ\,17'\,29'' & S - C = & \underline{52^\circ\,46'\,49.5''} \\
S = & 151^\circ\,38'\,44.5'' & S = & 151^\circ\,38'\,44.5'' \text{ (check)}
\end{array}$$

$$\begin{array}{ll}
 & \log(-\cos S) = 9.94450 \\
\log\cos(S-A) = 9.93709 & \operatorname{colog}\cos(S-A) = 0.06291 \\
\log\cos(S-B) = 9.55896 & \operatorname{colog}\cos(S-B) = 0.44104 \\
\log\cos(S-C) = 9.78166 & \operatorname{colog}\cos(S-C) = \underline{0.21834} \\
 & \log\tan^2 K = 0.66679 \\
\log\tan K = \underline{0.33340} & \log\tan K = 0.33340 \\
\log\tan\tfrac{1}{2}a = 0.27049 & \tfrac{1}{2}a = 61^\circ\,47'\,23'' \\
\log\tan\tfrac{1}{2}b = 9.89236 & \tfrac{1}{2}b = 37^\circ\,58'\,16'' \\
\log\tan\tfrac{1}{2}c = 0.11506 & \tfrac{1}{2}c = 52^\circ\,30'\,09''
\end{array}$$

$$a = 123^\circ\,34'\,46'', \quad b = 75^\circ\,56'\,32'', \quad c = 105^\circ\,00'\,18''.$$

2. Check.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$$\begin{array}{lll}
\log\sin A = 9.93056 & \log\sin B = 9.99664 & \log\sin C = 9.99478 \\
\log\sin a = \underline{9.92071} & \log\sin b = \underline{9.98680} & \log\sin c = \underline{9.98493} \\
\qquad\qquad 0.00985 & \qquad\qquad 0.00984 & \qquad\qquad 0.00985
\end{array}$$

Note. Since the sum of the angles of a spherical triangle is always between 180° and 540°, S is necessarily between 90° and 270°, hence, $\cos S$ is always negative and $-\cos S$ positive.

EXERCISE 11

Solve the following triangles:

1. Given $A = 74^\circ\,40'$, $B = 67^\circ\,30'$, $C = 49^\circ\,50'$.
   Ans. $a = 43^\circ\,36'$, $b = 41^\circ\,21'$, $c = 33^\circ\,07'$.
2. Given $A = 125^\circ\,54'$, $B = 55^\circ\,35'$, $C = 45^\circ\,05'$.
3. Given $A = 46^\circ\,59.3'$, $B = 122^\circ\,32.6'$, $C = 139^\circ\,00.3'$.
   Ans. $a = 59^\circ\,27.4'$, $b = 117^\circ\,06.2'$, $c = 123^\circ\,20.0'$.
4. Given $A = 47^\circ\,34.6'$, $B = 74^\circ\,54.7'$, $C = 77^\circ\,24.5'$.
5. Given $A = 59^\circ\,55'\,10''$, $B = 85^\circ\,36'\,50''$, $C = 59^\circ\,55'\,10''$.
   Ans. $a = 51^\circ\,17'\,31''$, $b = 64^\circ\,02'\,47''$, $c = 51^\circ\,17'\,31''$.
6. Given $A = 109^\circ\,35'\,56''$, $B = 111^\circ\,23'\,06''$, $C = 86^\circ\,49'\,19''$.
7. Given $A = 15^\circ\,38'\,06''$, $B = 16^\circ\,06'\,22''$, $C = 159^\circ\,44'\,26''$.
   Find b. *Ans.* $b = 52^\circ\,05'\,54''$.
8. Given $A = 50^\circ\,45'\,23''$, $B = 58^\circ\,01'\,10''$, $C = 87^\circ\,17'\,00''$.
   Find C.

40. Case III. Given Two Sides and the Included Angle, *a, b, C.*

Solution.

1. To find A and B. First find $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ by the first two of Napier's proportions (Art. 33), then

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B), \quad B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

2. To find c. Use either one of Delambre's proportions (Art. 32).
3. Check. Use the law of sines (Art. 24).

EXAMPLE.

Given

$a = 110^\circ\, 30'\, 24''$,
$b = 36^\circ\, 47'\, 36''$,
$C = 135^\circ\, 12'\, 12''$.

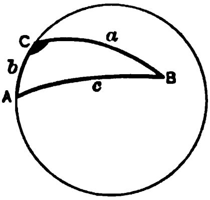


Fig. 40.

To find

$A = 63^\circ\, 57'\, 39''$,
$B = 35^\circ\, 04'\, 03''$,
$c = 132^\circ\, 44'\, 08''$.

Solution.

1. To find A and B.

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

$\frac{1}{2}(a-b) = 36^\circ\, 51'\, 24''$, $\frac{1}{2}(a+b) = 73^\circ\, 39'\, 00''$, $\frac{1}{2}C = 67^\circ\, 36'\, 06''$.

$\log \cos \frac{1}{2}(a-b) = 9.90316$
$\operatorname{colog} \cos \frac{1}{2}(a+b) = 0.55052$
$\log \cot \frac{1}{2} C = \underline{9.61504}$
$\log \tan \frac{1}{2}(A+B) = 0.06872$
$\frac{1}{2}(A+B) = 49^\circ\, 30'\, 51''$
$A = 63^\circ\, 57'\, 39''$.

$\log \sin \frac{1}{2}(a-b) = 9.77802$
$\operatorname{colog} \sin \frac{1}{2}(a+b) = 0.01793$
$\log \cot \frac{1}{2} C = \underline{9.61504}$
$\log \tan \frac{1}{2}(A-B) = 9.41099$
$\frac{1}{2}(A-B) = 14^\circ\, 26'\, 48''$
$B = 35^\circ\, 04'\, 03''$.

2. To find c.

$$\cos \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}(A+B)} \cos \tfrac{1}{2} C.$$

$\log \cos \frac{1}{2}(a-b) = 9.90316$
$\operatorname{colog} \sin \frac{1}{2}(A+B) = 0.11886$
$\log \cos \frac{1}{2} C = \underline{9.58098}$
$\log \cos \frac{1}{2} c = 9.60300$

3. Check.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$\log \sin a = 9.97157$
$\log \sin A = \underline{9.95352}$
0.01805

$$\begin{array}{rr}
 & \log \sin b = 9.77738 \\
\tfrac{1}{2} c = 66^\circ\, 22'\, 04'' & \log \sin B = \underline{9.75932} \\
c = 132^\circ\, 44'\, 08''. & 0.01806 \\
 & \log \sin c = 9.86598 \\
 & \log \sin C = \underline{9.84794} \\
 & 0.01804
\end{array}$$

We might have found c from the third or fourth of Napier's proportions but this would have required us to look up one more logarithm.

EXERCISE 12

Solve the following oblique triangles:

1. Given $a = 140^\circ\, 38'$, $b = 130^\circ\, 28'$, $C = 150^\circ\, 34'$.
   Ans. $A = 161^\circ\, 47'$, $B = 157^\circ\, 58'$, $c = 85^\circ\, 20'$.
2. Given $a = 103^\circ\, 44'$, $b = 64^\circ\, 12'$, $C = 98^\circ\, 33'$.
3. Given $a = 156^\circ\, 12.2'$, $b = 112^\circ\, 48.6'$, $C = 76^\circ\, 32.4'$.
   Ans. $A = 154^\circ\, 04.1'$, $B = 87^\circ\, 27.1'$, $c = 63^\circ\, 48.8'$.
4. Given $a = 27^\circ\, 45.5'$, $b = 22^\circ\, 56.7'$, $C = 156^\circ\, 15.9'$.
5. Given $a = 88^\circ\, 12'\, 20''$, $b = 124^\circ\, 07'\, 17''$, $C = 50^\circ\, 02'\, 02''$.
   Ans. $A = 63^\circ\, 15'\, 10''$, $B = 132^\circ\, 17'\, 59''$, $c = 59^\circ\, 04'\, 25''$.
6. Given $a = 111^\circ\, 11'\, 12''$, $b = 137^\circ\, 56'\, 56''$, $C = 23^\circ\, 15'\, 48''$.
7. Given $b = 68^\circ\, 12'\, 58''$, $c = 80^\circ\, 14'\, 41''$, $A = 17^\circ\, 20'\, 54''$.
   Ans. $B = 52^\circ\, 05'\, 54''$, $C = 123^\circ\, 07'\, 37''$, $a = 20^\circ\, 32'\, 33''$.
8. Given $a = 56^\circ\, 56'\, 56''$, $c = 156^\circ\, 56'\, 56''$, $B = 94^\circ\, 45'\, 45''$.

41. Case IV. Given Two Angles and the Included Side, A, B, c.

Solution.

1. To find a and b. First find $\tfrac{1}{2}(a + b)$ and $\tfrac{1}{2}(a - b)$ by the last two of Napier's proportions (Art. 33), then

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b), \quad b = \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a - b).$$

2. To find c. Use either one of Delambre's proportions (Art. 32).
3. Check. Use the law of sines.

EXAMPLE.

Given	To find
$A = 63^\circ\, 57'\, 39''$,	$a = 110^\circ\, 30'\, 23''$,
$B = 35^\circ\, 04'\, 03''$,	$b = 36^\circ\, 47'\, 37''$,
$c = 132^\circ\, 44'\, 08''$.	$C = 135^\circ\, 12'\, 15''$.

Solution.

1. To find a and b.

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

$\frac{1}{2}(A-B)=14^\circ\,26'\,48''$, $\frac{1}{2}(A+B)=49^\circ\,30'\,51''$, $\frac{1}{2}c=66^\circ\,22'\,04''$.

$$\begin{array}{lll}
\log\cos\frac{1}{2}(A-B)=9.98605 & \quad & \log\sin\frac{1}{2}(A-B)=9.39702\\
\text{colog}\cos\frac{1}{2}(A+B)=0.18758 & & \text{colog}\sin\frac{1}{2}(A+B)=0.11887\\
\log\tan\frac{1}{2}c=\underline{0.35896} & & \log\tan\frac{1}{2}c=\underline{0.35896}\\
\log\tan\frac{1}{2}(a+b)=0.53259 & & \log\tan\frac{1}{2}(a-b)=9.87485\\
\frac{1}{2}(a+b)=73^\circ\,39'\,00'' & & \frac{1}{2}(a-b)=36^\circ\,51'\,23''\\
a=110^\circ\,30'\,23''. & & b=36^\circ\,47'\,37''.
\end{array}$$

2. To find C.

$$\cos\tfrac{1}{2}C=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(a-b)}\sin\tfrac{1}{2}c.$$

$$\begin{array}{ll}
\log\sin\frac{1}{2}(A-B) & =9.39702\\
\text{colog}\sin\frac{1}{2}(a-b) & =0.22199\\
\log\sin\frac{1}{2}c & =\underline{9.96196}\\
\log\cos\frac{1}{2}C & =9.58097\\
\frac{1}{2}C & =67^\circ\,36'\,07.7''\\
C & =135^\circ\,12'\,15''
\end{array}$$

3. Check.

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}.$$

$$\begin{array}{ll}
\log\sin A & =9.95352\\
\log\sin a & =\underline{9.97157}\\
 & \,9.98195\\
\log\sin B & =9.75932\\
\log\sin b & =\underline{9.77738}\\
 & \,9.98194\\
\log\sin C & =9.84793\\
\log\sin c & =\underline{9.86598}\\
 & \,9.98195
\end{array}$$

EXERCISE 13

Solve the following triangles:

1. Given $A=67^\circ\,30'$, $B=45^\circ\,50'$, $c=74^\circ\,20'$.
   Ans. $a=63^\circ\,15'$, $b=53^\circ\,46'$, $C=52^\circ\,27'$.
2. Given $A=126^\circ\,45'$, $B=49^\circ\,52'$, $c=80^\circ\,01'$.
3. Given $B=140^\circ\,43.2'$, $C=100^\circ\,04.6'$, $a=60^\circ\,43.6'$.
   Ans. $b=145^\circ\,55.2'$, $c=119^\circ\,22.6'$, $A=80^\circ\,14.8'$.
4. Given $C=139^\circ\,25.8'$, $A=13^\circ\,56.9'$, $b=29^\circ\,00.8'$.
5. Given $A=153^\circ\,17'\,06''$, $B=78^\circ\,43'\,32''$, $c=86^\circ\,15'\,15''$.
   Ans. $a=88^\circ\,12'\,19''$, $b=78^\circ\,15'\,41''$, $C=152^\circ\,43'\,52''$.
6. Given $a=50^\circ\,34'\,56''$, $B=124^\circ\,10'\,10''$, $C=83^\circ\,25'\,25''$.

42. Case V. Given Two Sides and the Angle Opposite One of Them, *a, b, A.*

In this case there may be two solutions (see Art. 11).

1. To find B. Use the law of sines, $\sin B = \dfrac{\sin b \sin A}{\sin a}$.

Since B is found from its sine it will in general have two values whose sum is 180°. $\sin B = \dfrac{\sin b \sin A}{\sin a} \lesseqqgtr 1$, according as $\sin b \sin A \lesseqqgtr \sin a$, hence B has two values, one value (90°), or no real value, according as $\sin b \sin A \lesseqqgtr \sin a$.

2. To find C. From the second of Napier's proportions

$$\tan \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A-B).$$

Since C is less than 180°, $\tan \frac{1}{2} C$ must be positive. Now $a+b$ is always less than 360°, therefore $\sin \frac{1}{2}(a+b)$ is always positive, hence in order that $\tan \frac{1}{2} C$ may be positive $\sin \frac{1}{2}(a-b)$ and $\cot \frac{1}{2}(A-B)$ must have like signs. Now $\frac{1}{2}(a-b)$ and $\frac{1}{2}(A-B)$ are each numerically less than 90°, hence in order that $\sin \frac{1}{2}(a-b)$ and $\cot \frac{1}{2}(A-B)$ may have like signs, $\frac{1}{2}(a-b)$ and $\frac{1}{2}(A-B)$ and consequently $a-b$ and $A-B$ must have like signs. If both values of B satisfy this condition there are two solutions, if only one value of B satisfies this condition there is only one solution, if neither value of B satisfies this condition there is no solution.

3. To find c. From the fourth of Napier's proportions

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b).$$

4. Check. Use the law of sines, or any other formula involving B, C, and c, which has not been previously used.

The foregoing considerations regarding the number of admissible solutions may be summed up into the following:

Rule.

a. If $\sin a < \sin b \sin A$, there is no solution.
b. If $\sin a = \sin b \sin A$, there is one solution, $B = 90°$.
c. If $\sin a > \sin b \sin A$, each of the two values of B which gives $A - B$ and $a - b$ like signs yields a solution.

Example.

Given.

$a = 62^\circ 15'$,
$b = 103^\circ 19'$,
$A = 53^\circ 43'$.

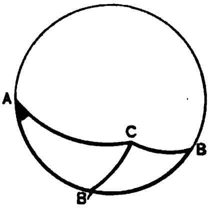


Fig. 41.

To find.

$B = 62^\circ 25'$,
$C = 155^\circ 43'$,
$c = 153^\circ 10'$,
$B' = 117^\circ 35'$,
$C' = 59^\circ 08'$,
$c' = 70^\circ 27'$.

Solution.

1. To find B.

$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

$$\begin{aligned} \log\sin b &= 9.98816 \\ \log\sin A &= 9.90639 \\ \operatorname{colog}\sin a &= \underline{0.05306} \\ \log\sin B &= 9.94761 \end{aligned}$$

$$B = 62^\circ 25' \quad \text{or} \quad B' = 117^\circ 35'.$$

2. To find C.

$$\tan\tfrac{1}{2}C = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}(A-B).$$

3. To find c.

$$\tan\tfrac{1}{2}c = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a-b).$$

$\frac{1}{2}(a+b) = 82^\circ 47'$, $\frac{1}{2}(A+B) = 58^\circ 04'$, $\frac{1}{2}(A+B') = 85^\circ 39'$,
$\frac{1}{2}(a-b) = -20^\circ 32'$, $\frac{1}{2}(A-B) = -4^\circ 21'$, $\frac{1}{2}(A-B') = -31^\circ 56'$.

Since the signs of $a - b$ and $A - B$ are alike for both values of B there are two solutions.

$$\begin{aligned} \log\sin\tfrac{1}{2}(a-b) &= 9.54500n \\ \operatorname{colog}\sin\tfrac{1}{2}(a+b) &= 0.00345 \\ \log\cot\tfrac{1}{2}(A-B) &= 1.11880n \\ \log\cot\tfrac{1}{2}(A-B') &= \underline{0.20534n} \\ \log\tan\tfrac{1}{2}C &= 0.66725 \\ \log\tan\tfrac{1}{2}C' &= 9.75379 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}C &= 77^\circ 51.5'. \\ \tfrac{1}{2}C' &= 29^\circ 33.9'. \\ C &= 155^\circ 43.0'. \\ C' &= 59^\circ 07.8'. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(A+B) &= 9.92874 \\ \operatorname{colog}\sin\tfrac{1}{2}(A-B) &= 1.12005n \\ \log\tan\tfrac{1}{2}(a-b) &= 9.57351n \\ \log\sin\tfrac{1}{2}(A+B') &= 9.99875 \\ \operatorname{colog}\sin\tfrac{1}{2}(A-B') &= \underline{0.27660n} \\ \log\tan\tfrac{1}{2}c &= 0.62230 \\ \log\tan\tfrac{1}{2}c' &= 9.84886 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}c &= 76^\circ 34.8'. \\ \tfrac{1}{2}c' &= 35^\circ 13.5'. \\ c &= 153^\circ 09.6'. \\ c' &= 70^\circ 27.0'. \end{aligned}$$

4. Check.

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{\sin c'}{\sin C'}.$$

$$\begin{aligned} \log\sin b &= 9.98816 \\ \log\sin B &= \underline{9.94761} \\ &\ 0.04055 \end{aligned} \qquad \begin{aligned} \log\sin c &= 9.65466 \\ \log\sin C &= \underline{9.61411} \\ &\ 0.04055 \end{aligned} \qquad \begin{aligned} \log\sin c' &= 9.97421 \\ \log\sin C' &= \underline{9.93366} \\ &\ 0.04055 \end{aligned}$$

EXERCISE 14

Solve the following triangles:

1. $a = 56° 40'$, $b = 30° 50'$, $A = 103° 40'$.
Ans. $B = 36° 36'$, $C = 52° 00'$, $c = 42° 39'$.

2. $b = 44° 45'$, $c = 49° 35'$, $B = 58° 56'$. (Two solutions.)

3. $a = 148° 34.4'$, $b = 142° 11.6'$, $A = 153° 17.6'$.
Ans. $B = 31° 53.7'$, $C = 6° 17.6'$, $c = 7° 18.3'$;
$B' = 148° 06.3'$, $C' = 130° 21.4'$, $c' = 62° 08.8'$.

4. $a = 41° 25.8'$, $b = 19° 57.9'$, $A = 62° 09.5'$. (One solution.)

5. $a = 67° 12' 20''$, $b = 48° 45' 40''$, $B = 42° 20' 30''$.
Ans. $A = 55° 39' 57''$, $C = 116° 34' 18''$, $c = 93° 08' 10''$;
$A' = 124° 20' 03''$, $C' = 24° 32' 15''$, $c' = 27° 37' 20''$.

6. $a = 38° 10' 10''$, $b = 24° 56' 45''$, $B = 65° 25' 00''$. (No solution.)

43. Case VI. Given Two Angles and the Side Opposite One of Them, *A, B, a.*

As in Case V so here there may be two solutions. (See Art. 11.)

1. To find b. Use the law of sines,

$$\sin b = \frac{\sin B \sin a}{\sin A}.$$

2. To find c. From the fourth of Napier's proportions,

$$\cot \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \cot \tfrac{1}{2}(a - b).$$

3. To find C. From the second of Napier's proportions,

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B).$$

4. Check. Use the law of sines, or any other formula involving b, c, and C, which has not been previously used.

To determine the number of solutions we have the following rule which is based upon a process of reasoning exactly analogous to that employed in establishing the corresponding rule in Case V.

Rule.

a. If $\sin A < \sin B \sin a$, there is no solution.

b. If $\sin A = \sin B \sin a$, there is one solution, $b = 90°$.

c. If $\sin A > \sin B \sin a$, each of the two values of b, which gives to $a - b$ and $A - B$ like signs, yields a solution.

Example.

Given	To find
$A = 45^\circ\, 30'$,	$b = 33^\circ\, 38'$,
$B = 37^\circ\, 22'$,	$c = 59^\circ\, 15'$,
$a = 40^\circ\, 36'$.	$C = 109^\circ\, 37'$.

Solution.

1. To find b.

$$\begin{array}{rll} \log\sin B &= 9.78312 & \qquad \tfrac{1}{2}(A+B) = 41^\circ\, 26' \\ \log\sin a &= 9.81343 & \qquad \tfrac{1}{2}(A-B) = 4^\circ\, 04' \\ \operatorname{colog}\sin A &= \underline{0.14676} & \qquad \tfrac{1}{2}(a+b) = 37^\circ\, 06.8' \\ \log\sin b &= 9.74331 & \qquad \tfrac{1}{2}(a-b) = 3^\circ\, 29.2' \\ b &= 33^\circ\, 37.5' & \qquad \tfrac{1}{2}(a+b') = 93^\circ\, 29.2' \\ b' &= 146^\circ\, 22.5' & \qquad \tfrac{1}{2}(a-b') = -52^\circ\, 48.2' \end{array}$$

$A - B$ and $a - b'$ have unlike signs, hence b' does not yield a solution.

2. To find c.

$$\begin{array}{rl} \log\sin\tfrac{1}{2}(A-B) &= 8.85075 \\ \operatorname{colog}\sin\tfrac{1}{2}(A+B) &= 0.17931 \\ \log\cot\tfrac{1}{2}(a-b) &= \underline{1.21507} \\ \log\cot\tfrac{1}{2}c &= 0.24513 \\ \tfrac{1}{2}c &= 29^\circ\, 37.6' \\ c &= 59^\circ\, 15.2' \end{array}$$

3. To find C.

$$\begin{array}{rl} \log\sin\tfrac{1}{2}(a+b) &= 9.78060 \\ \operatorname{colog}\sin\tfrac{1}{2}(a-b) &= 1.21588 \\ \log\tan\tfrac{1}{2}(A-B) &= \underline{8.85185} \\ \log\cot\tfrac{1}{2}C &= 9.84833 \\ \tfrac{1}{2}C &= 54^\circ\, 48.45 \\ C &= 109^\circ\, 36.9' \end{array}$$

4. Check.

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad \begin{array}{rl} \log\sin B &= 9.78312 \\ \log\sin b &= \underline{9.74331} \\ & \ 0.03381 \end{array} \qquad \begin{array}{rl} \log\sin C &= 9.97403 \\ \log\sin c &= \underline{9.93421} \\ & \ 0.03382 \end{array}$$

Exercise 15

Solve the following triangles:

1. $A = 36^\circ\, 20'$, $B = 46^\circ\, 30'$, $a = 42^\circ\, 12'$.
   Ans. $b = 55^\circ\, 19'$, $c = 81^\circ\, 19'$, $C = 119^\circ\, 19'$;
   $b' = 124^\circ\, 51'$, $c' = 162^\circ\, 38'$, $C' = 164^\circ\, 44'$.
2. $A = 60^\circ\, 32'$, $B = 25^\circ\, 56'$, $a = 35^\circ\, 18'$. (One solution.)
3. $A = 73^\circ\, 11.3'$, $B = 61^\circ\, 18.2'$, $a = 46^\circ\, 45.5'$.
   Ans. $b = 41^\circ\, 52.6'$, $c = 41^\circ\, 35.1'$, $C = 60^\circ\, 42.8'$.
4. $A = 103^\circ\, 56.9'$, $B = 79^\circ\, 35.8'$, $a = 127^\circ\, 45.0'$. (Two solutions.)

5. $B = 123^\circ\,40'\,20''$, $C = 159^\circ\,43'\,22''$, $c = 159^\circ\,50'\,05''$.
Ans. $b = 55^\circ\,52'\,30''$, $a = 137^\circ\,21'\,19''$, $A = 137^\circ\,04'\,26''$;
$b' = 124^\circ\,07'\,30''$, $a' = 65^\circ\,39'\,44''$, $A' = 113^\circ\,39'\,16''$.

6. $A = 70^\circ\,45'\,10''$, $B = 119^\circ\,56'\,56''$, $b = 79^\circ\,45'\,02''$. (No solution.)

44. To Find the Area of a Spherical Triangle.

EXAMPLE.

Given $a = 124^\circ\,12'\,31''$, $b = 54^\circ\,18'\,16''$, $c = 97^\circ\,12'\,25''$. Find the spherical excess, and hence the area of the triangle, the radius of the sphere being 3959 miles.

Solution. By Art. 34 we have

$$\tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)}, \qquad T = \frac{\pi R^2 E^\circ}{180^\circ}.$$

$$\begin{aligned}
\tfrac{1}{2}a &= 62^\circ\,06'\,15.5'' \\
\tfrac{1}{2}b &= 27^\circ\,09'\,08'' \\
\tfrac{1}{2}c &= \underline{48^\circ\,36'\,12.5''} \\
s &= 137^\circ\,51'\,36'' \\
\tfrac{1}{2}s &= 68^\circ\,55'\,48'' \\
\tfrac{1}{2}(s-a) &= 6^\circ\,49'\,32.5'' \\
\tfrac{1}{2}(s-b) &= 41^\circ\,46'\,40'' \\
\tfrac{1}{2}(s-c) &= \underline{20^\circ\,19'\,35.5''} \\
\text{(check)} \quad & \ 68^\circ\,55'\,48''
\end{aligned}
\qquad
\begin{aligned}
\log\tan\tfrac{1}{2}s &= 0.41426 \\
\log\tan\tfrac{1}{2}(s-a) &= 9.07809 \\
\log\tan\tfrac{1}{2}(s-b) &= 9.95105 \\
\log\tan\tfrac{1}{2}(s-c) &= \underline{9.56871} \\
\log\tan^2\tfrac{1}{4}E &= 9.01211 \\
\log\tan\tfrac{1}{4}E &= 9.50605 \\
\tfrac{1}{4}E^\circ &= 17^\circ\,46'\,45'' \\
E^\circ &= 71^\circ\,07'\,00''
\end{aligned}$$

$$\begin{aligned}
\log R &= 3.59759 \\
\log R^2 &= 7.19518 \\
\log \pi &= 0.49715 \\
\log E^\circ &= 1.85197 \\
\operatorname{colog} 180^\circ &= \underline{7.74473} \\
\log T &= 7.28903 \\
T &= 19455 \times 10^3 \text{ square miles.}
\end{aligned}$$

45. Applications to Geometry.

EXERCISE 16

Right Spherical Triangles

1. The hypotenuse of an isosceles right spherical triangle is 60°. Find the length of the equal sides. *Ans.* 45°.

2. Find the relations between each two of the three distinct parts of an isosceles right spherical triangle. *Ans.* $\cos c = \cos^2 a = \cot^2 A$.

3. Show that no isosceles right spherical triangle can have its hypotenuse greater than 90° nor its acute angle less than 45°.

4. Find the altitude and angle of an equilateral spherical triangle whose side is 60°. *Ans.* Altitude = 54° 44′, Angle = 70° 32′.

5. If a is the side, A the angle, and p the altitude of an equilateral spherical triangle, show that $\sin \frac{1}{2} a \sin \frac{1}{2} A = \frac{1}{2}$, $\cos p = \dfrac{\cos a}{\cos \frac{1}{2} a}$.

6. The side of a spherical square (a spherical quadrilateral having four equal sides and four equal angles) is 73° 41′, find the angle and length of a diagonal.

Ans. Angle = 118° 04.5′, Diagonal 106° 16′.

7. The side of a regular spherical polygon (a spherical polygon having n equal sides and n equal angles) is a. Find the angle A of the polygon, the perpendicular p from the center of the polygon to one of the sides, and the distance r from the center to one of the vertices of the polygon.

Ans. $\sin \frac{1}{2} A = \dfrac{\cos (\pi/n)}{\cos \frac{1}{2} a}$, $\sin p = \tan \frac{1}{2} a \cot (\pi/n)$, $\sin r = \dfrac{\sin \frac{1}{2} a}{\sin (\pi/n)}$.

8. Find the perimeter of the polygon (Problem 7) when $p = 90°$.

Ans. 2π.

9. Compute the dihedral angles of a regular tetrahedron. Of a regular dodecahedron. *Ans.* 70° 31′ 44″, 116° 33′ 54″.

Suggestion. With a vertex of the polyhedron as a center describe a sphere. The points in which the three edges proceeding from the vertex intersect the sphere determine an equilateral spherical triangle the sides of which are known.

10. Compute the dihedral angles of a regular octahedron. Of a regular icosahedron. *Ans.* 109° 28′ 16″, 138° 11′ 23″.

EXERCISE 17

Oblique Spherical Triangles

1. The three face angles of a trihedral angle are BOC = 84° 24′, COA = 72° 18′, AOB = 60° 18′. Find the dihedral angles.

Ans. OA = 93° 40′, OB = 72° 48′, OC = 60° 36′.

2. Two planes intersect at an angle of 58° 40′. From a point of their line of intersection two lines are drawn, one in each plane,

making the angles 42° 30′ and 64° 24′ with the line of intersection. Find the angle which the lines thus drawn make with each other. *Ans.* 50° 33′.

3. The great pyramid of Gizeh has a square for its base, and the angle between two edges at the vertex measures 96° 01.2′. Find the angle which each face makes with the horizon. *Ans.* 51° 51′.

4. A ten-sided pavilion is covered by a pyramidal roof. Two consecutive hips of the roof make an angle of 30°. Find the angle between two consecutive faces of the roof. *Ans.* 159° 53′.

5. The opposite faces of an obelisk are inclined at an angle of 16°. Find the face angles at the base of the obelisk and the angle between two adjacent faces. *Ans.* 82° 04.6′, 91° 06.6′.

6. The ridges of two gable roofs meet at right angles. The slope of each roof is 60°. Find the angle between the planes of the two roofs, and the angle the valley makes with each ridge. *Ans.* 104° 26.6′, 63° 26.1′.

7. A mason cuts a stone in the shape of a pyramid with a regular hexagonal base. The edges are inclined at an angle of 30° with the base. Find the angle between two adjacent lateral faces, and the inclination of the faces to the base. *Ans.* 149° 18.6′, 39° 13.9′.

8. If α, β, γ are the arcs joining any point in a trirectangular triangle to the vertices of the triangle, show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

9. An oblique parallelopiped has the three edges $OA = 2.59$, $AB = 3.65$, $OC = 7.21$, and the angles $AOB = 72° 16'$, $BOC = 80° 44'$, $COA = 41° 18'$. Find its volume. *Ans.* 21.30.

46. Application to Geography and Navigation.

EXERCISE 18

1. Find the shortest distance measured along a great circle between New York, lat. 40° 42′ 44″ N., long. 74° 00′ 24″ W., and San Francisco, lat. 37° 47′ 55″ N., long. 122° 24′ 32″ W., the earth being considered a perfect sphere, radius 3959 miles. *Ans.* 2564 miles.

2. Find the area of a spherical triangle on the earth's surface ($r = 3959$ miles) whose spherical excess is 1°. *Ans.* 273,575 square miles.

3. Compare the shortest distances in degrees of San Francisco, lat. 37° 47′ 55″ N., long. 122° 24′ 32″ W., and Seattle, lat. 47° 35′ 54″ N., long. 122° 19′ 59″ W., from Tokio, lat. 35° 39′ 18″ N., long. 139° 44′ 30″ E.

4. Find the distance in degrees and the bearing of Rio Janeiro, lat. 22° 55′ S., long. 43° 09′ W., from Cape of Good Hope, lat. 34° 22′ S., long. 18° 30′ E.

Ans. Distance 54° 29′, Bearing S. 84° 45′ W.

5. Find the first and final courses from San Francisco, lat. 37° 47′ 55″ N., long. 122° 24′ 32″ W., to Yokohama, lat. 35° 26′ 52″ N., long. 139° 38′ 41″ E. *Ans.* N. 56° 51′ W., S. 54° 17′ W.

6. A ship sails on an arc of a great circle a distance of 4150 miles from lat. 17° N., long. 130° W., the initial course being S. 54° 20′ W. Taking 1° = 69⅛ miles, what is the latitude and longitude of its final position. *Ans.* Lat. 19° 41′ S., long. 178° 21′ W.

7. A vessel sails from Boston, lat. 42° 21′ N., long. 71° 03′ W., to Cape Town, lat. 33° 56′ S., long. 18° 28′ E. Find at what longitude the ship crosses the Equator and its course at this point.

Ans. Long. 17° 48′ W., course S. 41° 19′ E.

8. Find the distance at which a vessel sailing from Seattle to Tokio will cross the 180th meridian and its latitude at the time of crossing. (See Problem 3.)

9. Find the latitude and longitude of the place where a ship sailing from Cape of Good Hope to Rio Janeiro crosses the meridian at right angles. (See Problem 4.)

Ans. Lat. 34° 43′ S., long. 9° 15′ E.

10. Find the longitude and latitude of the place where a ship sailing from San Francisco to Yokohama crosses the meridian at right angles. (See Problem 5.)

11. The continent of Asia has nearly the shape of an equilateral triangle, each side being approximately 5500 miles. Find the area of the triangle (*a*) regarded as a plane triangle, (*b*) regarded as a spherical triangle, the radius of the earth being assumed 3960 miles.

Ans. 13,098,500 square miles; 17,228,400 square miles.

47. Applications from Astronomy.

EXERCISE 19

(For definitions of terms consult any dictionary or textbook on astronomy.)

1. How many seconds does it take for a star whose declination is +64° 04′ to cross the field of a telescope, the diameter of the field being 36′? *Ans.* 329 seconds.

2. Find the approximate time of sunrise in Seattle, lat. +47° 39′, on Jan. 15, 1913. Suggestion. Look up the sun's declination. *Ans.* $7^h\ 40.5^m$ A.M. local *apparent* time.

3. Find the length of the longest day at Seattle, lat. +47° 39′. Suggestion. When the sun is at its summer solstice its declination is 23° 27′.

4. The moon's most northerly declination during this Saros occurred on March 19, 1913, and was 28° 44′ 10″. Find approximately how long it was below the horizon at San Francisco, lat. 37° 48′ 24″. *Ans.* $8^h\ 56^m$.

5. The zenith distance of the sun was observed to be 45° 26′ the afternoon of a day when its declination was +20° 32′. If the latitude of the place was +37° 10′, what was the local apparent time?

6. The azimuth of the sun was measured and found to be 10° 14.2′ and its zenith distance 25° 12.1′ at a time when its declination was +21° 39.2′, find the latitude of the place. *Ans.* 46° 34.1′.

7. In Problem 6 find the local apparent time. *Ans.* $0^h\ 30^m\ 13^s$.

8. At $1^h\ 15^m\ 16.1^s$ local apparent time the altitude of the sun was found to be 68° 21′ 46″ at a time when its declination was +22° 41′ 30″. Find the latitude of the place.

9. In Problem 8 find the azimuth of the sun.

10. The altitude of the sun was measured and found to be 40° 18′ 25″ at a place whose latitude is 47° 39′ 06″ at $2^h\ 10^m\ 17.8^s$ local apparent time. Find the sun's declination. *Ans.* + 6° 25′ 53″.

11. The northeastern end of the canal Phison on Mars is in Martian latitude 0° 03′ N. and longitude 335° 10′ and the southwestern end

is in latitude 40° 08′ S. and longitude 296° 58′. Find the length of Phison, the diameter of Mars being 4200 miles.

Ans. 1946.6 miles.

12. The declination of Algol is +40° 37′; find the azimuth of the star when setting at Ann Arbor, lat. +42° 17′.

13. The declination of Aldebaran is +16° 20.1′; find the azimuth of the star when setting at Seattle, lat. 47° 39.1′.

Ans. 114° 40.8′.

14. The declination of Procyon is +5° 26′ 55″; find the azimuth of the star when setting at Chicago, lat. 41° 50′ 01″.

15. The declination of 43H Cephei is now (1913) 85° 47′. Find its azimuth at Washington, D. C., lat. 38° 54′, $3^h\ 10^m$ after its meridian passage.

16. The declination of Polaris is now (1913) 88° 50′ 38″. Find its azimuth at Seattle, lat. 47° 39′ 06″, $5^h\ 01^m\ 20^s$ after its meridian passage. *Ans.* 178° 19′ 50″.

17. The right ascension and declination of Regulus are $\alpha = 10^h\ 03^m\ 44.4^s$, $\delta = +\ 12° 23′ 34″$. On May 13, 1913, the moon's right ascension and declination were $\alpha = 9^h\ 58^m\ 37.3^s$, $\delta = +15° 32′ 44″$. Find the angular distance between the moon's center and Regulus. *Ans.* 3° 23′ 20″.

18. The obliquity of the ecliptic is now (1913) 23° 27′ 02″. Find the celestial latitude and longitude of a star for which $\alpha = 3^h\ 15^m\ 20^s$, $\delta = +\ 36° 17′ 56″$.

Ans. $\beta = +\ 17° 33′ 19.7″$, $\lambda = 56° 11′ 24.5″$.

19. What is the greatest altitude of a star on the equator in the meridian of Washington, lat. +38° 53′ 39″? *Ans.* 51° 06′ 21″.

TABLE I.*

COMMON LOGARITHMS OF NUMBERS

Giving Characteristics and Mantissas of Logarithms of Numbers from 1 to 100, and Mantissas only of Numbers from 100 to 10000.

LOGARITHMS OF NUMBERS.

N	Log.	N	Log.	N	Log.	N	Log.
1	0.00000	26	1.41497	51	1.70757	76	1.88081
2	0.30103	27	1.43136	52	1.71600	77	1.88649
3	0.47712	28	1.44716	53	1.72428	78	1.89209
4	0.60206	29	1.46240	54	1.73239	79	1.89763
5	0.69897	30	1.47712	55	1.74036	80	1.90309
6	0.77815	31	1.49136	56	1.74819	81	1.90849
7	0.84510	32	1.50515	57	1.75587	82	1.91381
8	0.90309	33	1.51851	58	1.76343	83	1.91908
9	0.95424	34	1.53148	59	1.77085	84	1.92428
10	1.00000	35	1.54407	60	1.77815	85	1.92942
11	1.04139	36	1.55630	61	1.78533	86	1.93450
12	1.07918	37	1.56820	62	1.79239	87	1.93952
13	1.11394	38	1.57978	63	1.79934	88	1.94448
14	1.14613	39	1.59106	64	1.80618	89	1.94939
15	1.17609	40	1.60206	65	1.81291	90	1.95424
16	1.20412	41	1.61278	66	1.81954	91	1.95904
17	1.23045	42	1.62325	67	1.82607	92	1.96379
18	1.25527	43	1.63347	68	1.83251	93	1.96848
19	1.27875	44	1.64345	69	1.83885	94	1.97313
20	1.30103	45	1.65321	70	1.84510	95	1.97772
21	1.32222	46	1.66276	71	1.85126	96	1.98227
22	1.34242	47	1.67210	72	1.85733	97	1.98677
23	1.36173	48	1.68124	73	1.86332	98	1.99123
24	1.38021	49	1.69020	74	1.86923	99	1.99564
25	1.39794	50	1.69897	75	1.87506	100	2.00000

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* Tables I. and II. reprinted by permission from Murray's "Logarithmic and Trigonometric Tables," published by Longmans, Green & Co.

N	0	1	2	3	4	5	6	7	8	9	D
100	00 000	043	087	130	173	217	260	303	346	389	43
101	432	475	518	561	604	647	689	732	775	817	43
102	860	903	945	988	*030	*072	*115	*157	*199	*242	42
103	01 284	326	368	410	452	494	536	578	620	662	42
104	703	745	787	828	870	912	953	995	*036	*078	42
105	02 119	160	202	243	284	325	366	407	449	490	41
106	531	572	612	653	694	735	776	816	857	898	41
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	40
108	03 342	383	423	463	503	543	583	623	663	703	40
109	743	782	822	862	902	941	981	*021	*060	*100	40
110	04 139	179	218	258	297	336	376	415	454	493	39
111	532	571	610	650	689	727	766	805	844	883	39
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	39
113	05 308	346	385	423	461	500	538	576	614	652	38
114	690	729	767	805	843	881	918	956	994	*032	38
115	06 070	108	145	183	221	258	296	333	371	408	38
116	446	483	521	558	595	633	670	707	744	781	37
117	819	856	893	930	967	*004	*041	*078	*115	*151	37
118	07 188	225	262	298	335	372	408	445	482	518	37
119	555	591	628	664	700	737	773	809	846	882	36
120	918	954	990	*027	*063	*099	*135	*171	*207	*243	36
121	08 279	314	350	386	422	458	493	529	565	600	36
122	636	672	707	743	778	814	849	884	920	955	35
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	35
124	09 342	377	412	447	482	517	552	587	621	656	35
125	691	726	760	795	830	864	899	934	968	*003	35
126	10 037	072	106	140	175	209	243	278	312	346	34
127	380	415	449	483	517	551	585	619	653	687	34
128	721	755	789	823	857	890	924	958	992	*025	34
129	11 059	093	126	160	193	227	261	294	327	361	34
N	0	1	2	3	4	5	6	7	8	9	D

PP	44	43	42	41	40	39	38	37	36
1	4.4	4.3	4.2	4.1	4.0	3.9	3.8	3.7	3.6
2	8.8	8.6	8.4	8.2	8.0	7.8	7.6	7.4	7.2
3	13.2	12.9	12.6	12.3	12.0	11.7	11.4	11.1	10.8
4	17.6	17.2	16.8	16.4	16.0	15.6	15.2	14.8	14.4
5	22.0	21.5	21.0	20.5	20.0	19.5	19.0	18.5	18.0
6	26.4	25.8	25.2	24.6	24.0	23.4	22.8	22.2	21.6
7	30.8	30.1	29.4	28.7	28.0	27.3	26.6	25.9	25.2
8	35.2	34.4	33.6	32.8	32.0	31.2	30.4	29.6	28.8
9	39.6	38.7	37.8	36.9	36.0	35.1	34.2	33.3	32.4

N	0	1	2	3	4	5	6	7	8	9	D
130	11 394	428	461	494	528	561	594	628	661	694	33
131	727	760	793	826	860	893	926	959	992	*024	33
132	12 057	090	123	156	189	222	254	287	320	352	33
133	385	418	450	483	516	548	581	613	646	678	33
134	710	743	775	808	840	872	905	937	969	*001	32
135	13 033	066	098	130	162	194	226	258	290	322	32
136	354	386	418	450	481	513	545	577	609	640	32
137	672	704	735	767	799	830	862	893	925	956	32
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	31
139	14 301	333	364	395	426	457	489	520	551	582	31
140	613	644	675	706	737	768	799	829	860	891	31
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	31
142	15 229	259	290	320	351	381	412	442	473	503	31
143	534	564	594	625	655	685	715	746	776	806	30
144	836	866	897	927	957	987	*017	*047	*077	*107	30
145	16 137	167	197	227	256	286	316	346	376	406	30
146	435	465	495	524	554	584	613	643	673	702	30
147	732	761	791	820	850	879	909	938	967	997	29
148	17 026	056	085	114	143	173	202	231	260	289	29
149	319	348	377	406	435	464	493	522	551	580	29
150	609	638	667	696	725	754	782	811	840	869	29
151	898	926	955	984	*013	*041	*070	*099	*127	*156	29
152	18 184	213	241	270	298	327	355	384	412	441	29
153	469	498	526	554	583	611	639	667	696	724	28
154	752	780	808	837	865	893	921	949	977	*005	28
155	19 033	061	089	117	145	173	201	229	257	285	28
156	312	340	368	396	424	451	479	507	535	562	28
157	590	618	645	673	700	728	756	783	811	838	28
158	866	893	921	948	976	*003	*030	*058	*085	*112	27
159	20 140	167	194	222	249	276	303	330	358	385	27
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D**

PP	35	34	33	32	31	30	29	28	27
1	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7
2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4
3	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1
4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8
5	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5
6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2
7	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9
8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6
9	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3

N	0	1	2	3	4	5	6	7	8	9	D
160	20 412	439	466	493	520	548	575	602	629	656	27
161	683	710	737	763	790	817	844	871	898	925	27
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192	27
163	21 219	245	272	299	325	352	378	405	431	458	27
164	484	511	537	564	590	617	643	669	696	722	26
165	748	775	801	827	854	880	906	932	958	985	26
166	22 011	037	063	089	115	141	167	194	220	246	26
167	272	298	324	350	376	401	427	453	479	505	26
168	531	557	583	608	634	660	686	712	737	763	26
169	789	814	840	866	891	917	943	968	994	*019	26
170	23 045	070	096	121	147	172	198	223	249	274	25
171	300	325	350	376	401	426	452	477	502	528	25
172	553	578	603	629	654	679	704	729	754	779	25
173	805	830	855	880	905	930	955	980	*005	*030	25
174	24 055	080	105	130	155	180	204	229	254	279	25
175	304	329	353	378	403	428	452	477	502	527	25
176	551	576	601	625	650	674	699	724	748	773	25
177	797	822	846	871	895	920	944	969	993	*018	25
178	25 042	066	091	115	139	164	188	212	237	261	24
179	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	24
181	768	792	816	840	864	888	912	935	959	983	24
182	26 007	031	055	079	102	126	150	174	198	221	24
183	245	269	293	316	340	364	387	411	435	458	24
184	482	505	529	553	576	600	623	647	670	694	24
185	717	741	764	788	811	834	858	881	905	928	23
186	951	975	998	*021	*045	*068	*091	*114	*138	*161	23
187	27 184	207	231	254	277	300	323	346	370	393	23
188	416	439	462	485	508	531	554	577	600	623	23
189	646	669	692	715	738	761	784	807	830	852	23
N	0	1	2	3	4	5	6	7	8	9	D

PP	27	26	25	24	23	22
1	2.7	2.6	2.5	2.4	2.3	2.2
2	5.4	5.2	5.0	4.8	4.6	4.4
3	8.1	7.8	7.5	7.2	6.9	6.6
4	10.8	10.4	10.0	9.6	9.2	8.8
5	13.5	13.0	12.5	12.0	11.5	11.0
6	16.2	15.6	15.0	14.4	13.8	13.2
7	18.9	18.2	17.5	16.8	16.1	15.4
8	21.6	20.8	20.0	19.2	18.4	17.6
9	24.3	23.4	22.5	21.6	20.7	19.8

N	0	1	2	3	4	5	6	7	8	9	D
190	875	896	921	944	967	989	*012	*035	*058	*081	23
191	28 103	126	149	171	194	217	240	262	285	307	23
192	330	353	375	398	421	443	466	488	511	533	23
193	556	578	601	623	646	668	691	713	735	758	22
194	780	803	825	847	870	892	914	937	959	981	22
195	29 003	026	048	070	092	115	137	159	181	203	22
196	226	248	270	292	314	336	358	380	403	425	22
197	447	469	491	513	535	557	579	601	623	645	22
198	667	688	710	732	754	776	798	820	842	863	22
199	885	907	929	951	973	994	*016	*038	*060	*081	22
200	30 103	125	146	168	190	211	233	255	276	298	22
201	320	341	363	384	406	428	449	471	492	514	22
202	535	557	578	600	621	643	664	685	707	728	21
203	750	771	792	814	835	856	878	899	920	942	21
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154	21
205	31 175	197	218	239	260	281	302	323	345	366	21
206	387	408	429	450	471	492	513	534	555	576	21
207	597	618	639	660	681	702	723	744	765	785	21
208	806	827	848	869	890	911	931	952	973	994	21
209	32 015	035	056	077	098	118	139	160	181	201	21
210	222	243	263	284	305	325	346	366	387	408	21
211	428	449	469	490	510	531	552	572	593	613	20
212	634	654	675	695	715	736	756	777	797	818	20
213	838	858	879	899	919	940	960	980	*001	*021	20
214	33 041	062	082	102	122	143	163	183	203	224	20
215	244	264	284	304	325	345	365	385	405	425	20
216	445	465	486	506	526	546	566	586	606	626	20
217	646	666	686	706	726	746	766	786	806	826	20
218	846	866	885	905	925	945	965	985	*005	*025	20
219	34 044	064	084	104	124	143	163	183	203	223	20
220	242	262	282	301	321	341	361	380	400	420	20
221	439	459	479	498	518	537	557	577	596	616	20
222	635	655	674	694	713	733	753	772	792	811	19
223	830	850	869	889	908	928	947	967	986	*005	19
224	35 025	044	064	083	102	122	141	160	180	199	19
225	218	238	257	276	295	315	334	353	372	392	19
226	411	430	449	468	488	507	526	545	564	583	19
227	603	622	641	660	679	698	717	736	755	774	19
228	793	813	832	851	870	889	908	927	946	965	19
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154	19
N	0	1	2	3	4	5	6	7	8	9	D

N	0	1	2	3	4	5	6	7	8	9	D
230	36 173	192	211	229	248	267	286	305	324	342	19
231	361	380	399	418	436	455	474	493	511	530	19
232	549	568	586	605	624	642	661	680	698	717	19
233	736	754	773	791	810	829	847	866	884	903	19
234	922	940	959	977	996	*014	*033	*051	*070	*088	18
235	37 107	125	144	162	181	199	218	236	254	273	18
236	291	310	328	346	365	383	401	420	438	457	18
237	475	493	511	530	548	566	585	603	621	639	18
238	658	676	694	712	731	749	767	785	803	822	18
239	840	858	876	894	912	931	949	967	985	*003	18
240	38 021	039	057	075	093	112	130	148	166	184	18
241	202	220	238	256	274	292	310	328	346	364	18
242	382	399	417	435	453	471	489	507	525	543	18
243	561	578	596	614	632	650	668	686	703	721	18
244	739	757	775	792	810	828	846	863	881	899	18
245	917	934	952	970	987	*005	*023	*041	*058	*076	18
246	39 094	111	129	146	164	182	199	217	235	252	18
247	270	287	305	322	340	358	375	393	410	428	18
248	445	463	480	498	515	533	550	568	585	602	18
249	620	637	655	672	690	707	724	742	759	777	17
250	794	811	829	846	863	881	898	915	933	950	17
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	17
252	40 140	157	175	192	209	226	243	261	278	295	17
253	312	329	346	364	381	398	415	432	449	466	17
254	483	500	518	535	552	569	586	603	620	637	17
255	654	671	688	705	722	739	756	773	790	807	17
256	824	841	858	875	892	909	926	943	960	976	17
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	17
258	41 162	179	196	212	229	246	263	280	296	313	17
259	330	347	363	380	397	414	430	447	464	481	17
260	497	514	531	547	564	581	597	614	631	647	17
261	664	681	697	714	731	747	764	780	797	814	17
262	830	847	863	880	896	913	929	946	963	979	16
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	16
264	42 160	177	193	210	226	243	259	275	292	308	16
265	325	341	357	374	390	406	423	439	455	472	16
266	488	504	521	537	553	570	586	602	619	635	16
267	651	667	684	700	716	732	749	765	781	797	16
268	813	830	846	862	878	894	911	927	943	959	16
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	16
N	0	1	2	3	4	5	6	7	8	9	D

N	0	1	2	3	4	5	6	7	8	9	D
270	43 136	152	169	185	201	217	233	249	265	281	16
271	297	318	329	345	361	377	393	409	425	441	16
272	457	473	489	505	521	537	553	569	584	600	16
273	616	632	648	664	680	696	712	727	743	759	16
274	775	791	807	823	838	854	870	886	902	917	16
275	933	949	965	981	996	*012	*028	*044	*059	*075	16
276	44 091	107	122	138	154	170	185	201	217	232	16
277	248	264	279	295	311	326	342	358	373	389	16
278	404	420	436	451	467	483	498	514	529	545	16
279	560	576	592	607	623	638	654	669	685	700	16
280	716	731	747	762	778	793	809	824	840	855	15
281	871	886	902	917	932	948	963	979	994	*010	15
282	45 025	040	056	071	086	102	117	133	148	163	15
283	179	194	209	225	240	255	271	286	301	317	15
284	332	347	362	378	393	408	423	439	454	469	15
285	484	500	515	530	545	561	576	591	606	621	15
286	637	652	667	682	697	712	728	743	758	773	15
287	788	803	818	834	849	864	879	894	909	924	15
288	939	954	969	984	*000	*015	*030	*045	*060	*075	15
289	46 090	105	120	135	150	165	180	195	210	225	15
290	240	255	270	285	300	315	330	345	359	374	15
291	389	404	419	434	449	464	479	494	509	523	15
292	538	553	568	583	598	613	627	642	657	672	15
293	687	702	716	731	746	761	776	790	805	820	15
294	835	850	864	879	894	909	923	938	953	967	15
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	15
296	47 129	144	159	173	188	202	217	232	246	261	15
297	276	290	305	319	334	349	363	378	392	407	15
298	422	436	451	465	480	494	509	524	538	553	15
299	567	582	596	611	625	640	654	669	683	698	15
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307	714	728	742	756	770	785	799	813	827	841	14
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312	415	429	443	457	471	485	499	513	527	541	14
313	554	568	582	596	610	624	638	651	665	679	14
314	693	707	721	734	748	762	776	790	803	817	14
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318	243	256	270	284	297	311	325	338	352	365	14
319	379	393	406	420	433	447	461	474	488	501	14
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326	322	335	348	362	375	388	402	415	428	441	13
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329	720	733	746	759	772	786	799	812	825	838	13
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331	983	996	*009	*022	*035	*048	*061	*075	*088	*101	13
332	52 114	127	140	153	166	179	192	205	218	231	13
333	244	257	270	284	297	310	323	336	349	362	13
334	375	388	401	414	427	440	453	466	479	492	13
335	504	517	530	543	556	569	582	595	608	621	13
336	634	647	660	673	686	699	711	724	737	750	13
337	763	776	789	802	815	827	840	853	866	879	13
338	892	905	917	930	943	956	969	982	994	*007	13
339	53 020	033	046	058	071	084	097	110	122	135	13
340	148	161	173	186	199	212	224	237	250	263	13
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342	403	415	428	441	453	466	479	491	504	517	13
343	529	542	555	567	580	593	605	618	631	643	13
344	656	668	681	694	706	719	732	744	757	769	13
345	782	794	807	820	832	845	857	870	882	895	13
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347	54 033	045	058	070	083	095	108	120	133	145	13
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356	145	157	169	182	194	206	218	230	242	255	12
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366	348	360	372	384	396	407	419	431	443	455	12
367	467	478	490	502	514	526	538	549	561	573	12
368	585	597	608	620	632	644	656	667	679	691	12
369	703	714	726	738	750	761	773	785	797	808	12
370	820	832	844	855	867	879	891	902	914	926	12
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374	287	299	310	322	334	345	357	368	380	392	12
375	403	415	426	438	449	461	473	484	496	507	12
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377	634	646	657	669	680	692	703	715	726	738	11
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380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	11
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383	320	331	343	354	365	377	388	399	410	422	11
384	433	444	456	467	478	490	501	512	524	535	11
385	546	557	569	580	591	602	614	625	636	647	11
386	659	670	681	692	704	715	726	737	749	760	11
387	771	782	794	805	816	827	838	850	861	872	11
388	883	894	906	917	928	939	950	961	973	984	11
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392	329	340	351	362	373	384	395	406	417	428	11
393	439	450	461	472	483	494	506	517	528	539	11
394	550	561	572	583	594	605	616	627	638	649	11
395	660	671	682	693	704	715	726	737	748	759	11
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406	853	863	874	855	895	906	917	927	938	949	11
407	959	970	981	991	*002	*013	*023	*034	*045	*055	11
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409	172	183	194	204	215	225	236	247	257	268	11
410	278	289	300	310	321	331	342	352	363	374	11
411	384	395	405	416	426	437	448	458	469	479	11
412	490	500	511	521	532	542	553	563	574	584	11
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432	548	558	568	579	589	599	609	619	629	639	10
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449	225	234	244	254	263	273	283	292	302	312	10
450	321	331	341	350	360	369	379	389	398	408	10
451	418	427	437	447	456	466	475	485	495	504	10
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454	706	715	725	734	744	753	763	772	782	792	9
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487	753	762	771	780	789	797	806	815	824	833	9
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491	108	117	126	135	144	152	161	170	179	188	9
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493	285	294	302	311	320	329	338	346	355	364	9
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506	415	424	432	441	449	458	467	475	484	492	9
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517	349	357	366	374	383	391	399	408	416	425	8
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529	346	354	362	370	378	387	395	403	411	419	8
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534	754	762	770	779	787	795	803	811	819	827	8
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553	273	280	288	296	304	312	320	327	335	343	8
554	351	359	367	374	382	390	398	406	414	421	8
555	429	437	445	453	461	468	476	484	492	500	8
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557	586	593	601	609	617	624	632	640	648	656	8
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667	413	419	426	432	439	445	452	458	465̄	471	7
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717	552	558	564	570	576	582	588	594	600	606	6
718	612	618	625	631	637	643	649	655	661	667	6
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726	094	100	106	112	118	124	130	136	141	147	6
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756	852	858	864	869	875	881	887	892	898	904	6
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843	583	588	593	598	603	609	614	619	624	629	5
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849	891	896	901	906	911	916	921	927	932	937	5
850	942	947	952	957	962	967	973	978	983	988	5
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	5
852	93 044	049	054	059	064	069	075	080	085	090	5
853	095	100	105	110	115	120	125	131	136	141	5
854	146	151	156	161	166	171	176	181	186	192	5
855	197	202	207	212	217	222	227	232	237	242	5
856	247	252	258	263	268	273	278	283	288	293	5
857	298	303	308	313	318	323	328	334	339	344	5
858	349	354	359	364	369	374	379	384	389	394	5
859	399	404	409	414	420	425	430	435	440	445	5
860	450	455	460	465	470	475	480	485	490	495	5
861	500	505	510	515	520	526	531	536	541	546	5
862	551	556	561	566	571	576	581	586	591	596	5
863	601	606	611	616	621	626	631	636	641	646	5
864	651	656	661	666	671	676	682	687	692	697	5
865	702	707	712	717	722	727	732	737	742	747	5
866	752	757	762	767	772	777	782	787	792	797	5
867	802	807	812	817	822	827	832	837	842	847	5
868	852	857	862	867	872	877	882	887	892	897	5
869	902	907	912	917	922	927	932	937	942	947	5
N	0	1	2	3	4	5	6	7	8	9	D

N	0	1	2	3	4	5	6	7	8	9	D
870	952	957	962	967	972	977	982	987	992	997	5
871	94 002	007	012	017	022	027	032	037	042	047	5
872	052	057	062	067	072	077	082	086	091	096	5
873	101	106	111	116	121	126	131	136	141	146	5
874	151	156	161	166	171	176	181	186	191	196	5
875	201	206	211	216	221	226	231	236	240	245	5
876	250	255	260	265	270	275	280	285	290	295	5
877	300	305	310	315	320	325	330	335	340	345	5
878	349	354	359	364	369	374	379	384	389	394	5
879	399	404	409	414	419	424	429	433	438	443	5
880	448	453	458	463	468	473	478	483	488	493	5
881	498	503	507	512	517	522	527	532	537	542	5
882	547	552	557	562	567	571	576	581	586	591	5
883	596	601	606	611	616	621	626	630	635	640	5
884	645	650	655	660	665	670	675	680	685	689	5
885	694	699	704	709	714	719	724	729	734	738	5
886	743	748	753	758	763	768	773	778	783	787	5
887	792	797	802	807	812	817	822	827	832	836	5
888	841	846	851	856	861	866	871	876	880	885	5
889	890	895	900	905	910	915	919	924	929	934	5
890	939	944	949	954	959	963	968	973	978	983	5
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	5
892	95 036	041	046	051	056	061	066	071	075	080	5
893	085	090	095	100	105	109	114	119	124	129	5
894	134	139	143	148	153	158	163	168	173	177	5
895	182	187	192	197	202	207	211	216	221	226	5
896	231	236	240	245	250	255	260	265	270	274	5
897	279	284	289	294	299	303	308	313	318	323	5
898	328	332	337	342	347	352	357	361	366	371	5
899	376	381	386	390	395	400	405	410	415	419	5
900	424	429	434	439	444	448	453	458	463	468	5
901	472	477	482	487	492	497	501	506	511	516	5
902	521	525	530	535	540	545	550	554	559	564	5
903	569	574	578	583	588	593	598	602	607	612	5
904	617	622	626	631	636	641	646	650	655	660	5
905	665	670	674	679	684	689	694	698	703	708	5
906	713	718	722	727	732	737	742	746	751	756	5
907	761	766	770	775	780	785	789	794	799	804	5
908	809	813	818	823	828	832	837	842	847	852	5
909	856	861	866	871	875	880	885	890	895	899	5
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D**

N	0	1	2	3	4	5	6	7	8	9	D
910	904	909	914	918	923	928	933	938	942	947	5
911	952	957	961	966	971	976	980	985	990	995	5
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	5
913	96 047	052	057	061	066	071	076	080	085	090	5
914	095	099	104	109	114	118	123	128	133	137	5
915	142	147	152	156	161	166	171	175	180	185	5
916	190	194	199	204	209	213	218	223	227	232	5
917	237	242	246	251	256	261	265	270	275	280	5
918	284	289	294	298	303	308	313	317	322	327	5
919	332	336	341	346	350	355	360	365	369	374	5
920	379	384	388	393	398	402	407	412	417	421	5
921	426	431	435	440	445	450	454	459	464	468	5
922	473	478	483	487	492	497	501	506	511	515	5
923	520	525	530	534	539	544	548	553	558	562	5
924	567	572	577	581	586	591	595	600	605	609	5
925	614	619	624	628	633	638	642	647	652	656	5
926	661	666	670	675	680	685	689	694	699	703	5
927	708	713	717	722	727	731	736	741	745	750	5
628	755	759	764	769	774	778	783	788	792	797	5
929	802	806	811	816	820	825	830	834	839	844	5
930	848	853	858	862	867	872	876	881	886	890	5
931	895	900	904	909	914	918	923	928	932	937	5
932	942	946	951	956	960	965	970	974	979	984	5
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	5
934	97 035	039	044	049	053	058	063	067	072	077	5
935	081	086	090	095	100	104	109	114	118	123	5
936	128	132	137	142	146	151	155	160	165	169	5
937	174	179	183	188	192	197	202	206	211	216	5
938	220	225	230	234	239	243	248	253	257	262	5
939	267	271	276	280	285	290	294	299	304	308	5
940	313	317	322	327	331	336	340	345	350	354	5
941	359	364	368	373	377	382	387	391	396	400	5
942	405	410	414	419	424	428	433	437	442	447	5
943	451	456	460	465	470	474	479	483	488	493	5
944	497	502	506	511	516	520	525	529	534	539	5
945	543	548	552	557	562	566	571	575	580	585	5
946	589	594	598	603	607	612	617	621	626	630	5
947	635	640	644	649	653	658	663	667	672	676	5
948	681	685	690	695	699	704	708	713	717	722	5
949	727	731	736	740	745	749	754	759	763	768	5
N	0	1	2	3	4	5	6	7	8	9	D

N	0	1	2	3	4	5	6	7	8	9	D
950	772	777	782	786	791	795	800	804	809	813	5
951	818	823	827	832	836	841	845	850	855	859	5
952	864	868	873	877	882	886	891	896	900	905	5
953	909	914	918	923	928	932	937	941	946	950	5
954	955	959	964	968	973	978	982	987	991	996	5
955	98 000	005	009	014	019	023	028	032	037	041	5
956	046	050	055	059	064	068	073	078	082	087	5
957	091	096	100	105	109	114	118	123	127	132	5
958	137	141	146	150	155	159	164	168	173	177	5
959	182	186	191	195	200	204	209	214	218	223	5
960	227	232	236	241	245	250	254	259	263	268	5
961	272	277	281	286	290	295	299	304	308	313	5
962	318	322	327	331	336	340	345	349	354	358	5
963	363	367	372	376	381	385	390	394	399	403	5
964	408	412	417	421	426	430	435	439	444	448	5
965	453	457	462	466	471	475	480	484	489	493	4
966	498	502	507	511	516	520	525	529	534	538	4
967	543	547	552	556	561	565	570	574	579	583	4
968	588	592	597	601	605	610	614	619	623	628	4
969	632	637	641	646	650	655	659	664	668	673	4
970	677	682	686	691	695	700	704	709	713	717	4
971	722	726	731	735	740	744	749	753	758	762	4
972	767	771	776	780	784	789	793	798	802	807	4
973	811	816	820	825	829	834	838	843	847	851	4
974	856	860	865	869	874	878	883	887	892	896	4
975	900	905	909	914	918	923	927	932	936	941	4
976	945	949	954	958	963	967	972	976	981	985	4
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	4
978	99 034	038	043	047	052	056	061	065	069	074	4
979	078	083	087	092	096	100	105	109	114	118	4
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	4
982	211	216	220	224	229	233	238	242	247	251	4
983	255	260	264	269	273	277	282	286	291	295	4
984	300	304	308	313	317	322	326	330	335	339	4
985	344	348	352	357	361	366	370	374	379	383	4
986	388	392	396	401	405	410	414	419	423	427	4
987	432	436	441	445	449	454	458	463	467	471	4
988	476	480	484	489	493	498	502	506	511	515	4
989	520	524	528	533	537	542	546	550	555	559	4
N	0	1	2	3	4	5	6	7	8	9	D

N	0	1	2	3	4	5	6	7	8	9	D
990	564	568	572	577	581	585	590	594	599	603	4
991	607	612	616	621	625	629	634	638	642	647	4
992	651	656	660	664	669	673	677	682	686	691	4
993	695	699	704	708	712	717	721	726	730	734	4
994	739	743	747	752	756	760	765	769	774	778	4
995	782	787	791	795	800	804	808	813	817	822	4
996	826	830	835	839	843	848	852	856	861	865	4
997	870	874	878	883	887	891	896	900	904	909	4
998	913	917	922	926	930	935	939	944	948	952	4
999	957	961	965	970	974	978	983	987	991	996	4
N	0	1	2	3	4	5	6	7	8	9	D

TABLE II.

FIVE-PLACE LOGARITHMS

OF THE

SINE, COSINE, TANGENT, AND COTANGENT

FOR

EACH MINUTE FROM 0° TO 90°.

0°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	∞	∞	∞	0.00 000	**60**
1	6.46 373	6.46 373	3.53 627	0.00 000	59
2	6.76 476	6.76 476	3.23 524	0.00 000	58
3	6.94 085̄	6.94 085̄	3.05 915	0.00 000	57
4	7.06 579	7.06 579	2.93 421	0.00 000	56
5	7.16 270	7.16 270	2.83 730	0.00 000	55
6	7.24 188	7.24 188	2.75 812	0.00 000	54
7	7.30 882	7.30 882	2.69 118	0.00 000	53
8	7.36 682	7.36 682	2.63 318	0.00 000	52
9	7.41 797	7.41 797	2.58 203	0.00 000	51
10	7.46 373	7.46 373	2.53 627	0.00 000	**50**
11	7.50 512	7.50 512	2.49 488	0.00 000	49
12	7.54 291	7.54 291	2.45 709	0.00 000	48
13	7.57 767	7.57 767	2.42 233	0.00 000	47
14	7.60 985	7.60 986	2.39 014	0.00 000	46
15	7.63 982	7.63 982	2.36 018	0.00 000	45
16	7.66 784	7.66 785̄	2.33 215	0.00 000	44
17	7.69 417	7.69 418	2.30 582	9.99 999	43
18	7.71 900	7.71 900	2.28 100	9.99 999	42
19	7.74 248	7.74 248	2.25 752	9.99 999	41
20	7.76 475	7.76 476	2.23 524	9.99 999	**40**
21	7.78 594	7.78 595	2.21 405̄	9.99 999	39
22	7.80 615̄	7.80 615	2.19 385̄	9.99 999	38
23	7.82 545	7.82 546	2.17 454	9.99 999	37
24	7.84 393	7.84 394	2.15 606	9.99 999	36
25	7.86 166	7.86 167	2.13 833	9.99 999	35
26	7.87 870	7.87 871	2.12 129	9.99 999	34
27	7.89 509	7.89 510	2.10 490	9.99 999	33
28	7.91 088	7.91 089	2.08 911	9.99 999	32
29	7.92 612	7.92 613	2.07 387	9.99 998	31
30	7.94 084	7.94 086	2.05 914	9.99 998	**30**
31	7.95 508	7.95 510	2.04 490	9.99 998	29
32	7.96 887	7.96 889	2.03 111	9.99 998	28
33	7.98 223	7.98 225	2.01 775̄	9.99 998	27
34	7.99 520	7.99 522	2.00 478	9.99 998	26
35	8.00 779	8.00 781	1.99 219	9.99 998	25
36	8.02 002	8.02 004	1.97 996	9.99 998	24
37	8.03 192	8.03 194	1.96 806	9.99 997	23
38	8.04 350	8.04 353	1.95 647	9.99 997	22
39	8.05 478	8.05 481	1.94 519	9.99 997	21
40	8.06 578	8.06 581	1.93 419	9.99 997	**20**
41	8.07 65̄0	8.07 653	1.92 347	9.99 997	19
42	8.08 696	8.08 700	1.91 300	9.99 997	18
43	8.09 718	8.09 722	1.90 278	9.99 997	17
44	8.10 717	8.10 720	1.89 280	9.99 996	16
45	8.11 693	8.11 696	1.88 304	9.99 996	15
46	8.12 647	8.12 651	1.87 349	9.99 996	14
47	8.13 581	8.13 585	1.86 415̄	9.99 996	13
48	8.14 495	8.14 5̄00	1.85 500	9.99 996	12
49	8.15 391	8.15 395	1.84 605̄	9.99 996	11
50	8.16 268	8.16 273	1.83 727	9.99 995	**10**
51	8.17 128	8.17 133	1.82 867	9.99 995	9
52	8.17 971	8.17 976	1.82 024	9.99 995	8
53	8.18 798	8.18 804	1.81 196	9.99 995̄	7
54	8.19 610	8.19 616	1.80 384	9.99 995̄	6
55	8.20 407	8.20 413	1.79 587	9.99 994	5
56	8.21 189	8.21 195	1.78 805̄	9.99 994	4
57	8.21 958	8.21 964	1.78 036	9.99 994	3
58	8.22 713	8.22 720	1.77 280	9.99 994	2
59	8.23 456	8.23 462	1.76 538	9.99 994	1
60	8.24 186	8.24 192	1.75 808	9.99 993	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

89°

1°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.24 186	8.24 192	1.75 808	9.99 993	**60**
1	8.24 903	8.24 910	1.75 090	9.99 993	59
2	8.25 609	8.25 616	1.74 384	9.99 993	58
3	8.26 304	8.26 312	1.73 688	9.99 993	57
4	8.26 988	8.26 996	1.73 004	9.99 992	56
5	8.27 661	8.27 669	1.72 331	9.99 992	55
6	8.28 324	8.28 332	1.71 668	9.99 992	54
7	8.28 977	8.28 986	1.71 014	9.99 992	53
8	8.29 621	8.29 629	1.70 371	9.99 992	52
9	8.30 255	8.30 263	1.69 737	9.99 991	51
10	8.30 879	8.30 888	1.69 112	9.99 991	**50**
11	8.31 495	8.31 505	1.68 495	9.99 991	49
12	8.32 103	8.32 112	1.67 888	9.99 990	48
13	8.32 702	8.32 711	1.67 289	9.99 990	47
14	8.33 292	8.33 302	1.66 698	9.99 990	46
15	8.33 875	8.33 886	1.66 114	9.99 990	45
16	8.34 450	8.34 461	1.65 539	9.99 989	44
17	8.35 018	8.35 029	1.64 971	9.99 989	43
18	8.35 578	8.35 590	1.64 410	9.99 989	42
19	8.36 131	8.36 143	1.63 857	9.99 989	41
20	8.36 678	8.36 689	1.63 311	9.99 988	**40**
21	8.37 217	8.37 229	1.62 771	9.99 988	39
22	8.37 750	8.37 762	1.62 238	9.99 988	38
23	8.38 276	8.38 289	1.61 711	9.99 987	37
24	8.38 796	8.38 809	1.61 191	9.99 987	36
25	8.39 310	8.39 323	1.60 677	9.99 987	35
26	8.39 818	8.39 832	1.60 168	9.99 986	34
27	8.40 320	8.40 334	1.59 666	9.99 986	33
28	8.40 816	8.40 830	1.59 170	9.99 986	32
29	8.41 307	8.41 321	1.58 679	9.99 985	31
30	8.41 792	8.41 807	1.58 193	9.99 985	**30**
31	8.42 272	8.42 287	1.57 713	9.99 985	29
32	8.42 746	8.42 762	1.57 238	9.99 984	28
33	8.43 216	8.43 232	1.56 768	9.99 984	27
34	8.43 680	8.43 696	1.56 304	9.99 984	26
35	8.44 139	8.44 156	1.55 844	9.99 983	25
36	8.44 594	8.44 611	1.55 389	9.99 983	24
37	8.45 044	8.45 061	1.54 939	9.99 983	23
38	8.45 489	8.45 507	1.54 493	9.99 982	22
39	8.45 930	8.45 948	1.54 052	9.99 982	21
40	8.46 366	8.46 385	1.53 615	9.99 982	**20**
41	8.46 799	8.46 817	1.53 183	9.99 981	19
42	8.47 226	8.47 245	1.52 755	9.99 981	18
43	8.47 650	8.47 669	1.52 331	9.99 981	17
44	8.48 069	8.48 089	1.51 911	9.99 980	16
45	8.48 485	8.48 505	1.51 495	9.99 980	15
46	8.48 896	8.48 917	1.51 083	9.99 979	14
47	8.49 304	8.49 325	1.50 675	9.99 979	13
48	8.49 708	8.49 729	1.50 271	9.99 979	12
49	8.50 108	8.50 130	1.49 870	9.99 978	11
50	8.50 504	8.50 527	1.49 473	9.99 978	**10**
51	8.50 897	8.50 920	1.49 080	9.99 977	9
52	8.51 287	8.51 310	1.48 690	9.99 977	8
53	8.51 673	8.51 696	1.48 304	9.99 977	7
54	8.52 055	8.52 079	1.47 921	9.99 976	6
55	8.52 434	8.52 459	1.47 541	9.99 976	5
56	8.52 810	8.52 835	1.47 165	9.99 975	4
57	8.53 183	8.53 208	1.46 792	9.99 975	3
58	8.53 552	8.53 578	1.46 422	9.99 974	2
59	8.53 919	8.53 945	1.46 055	9.99 974	1
60	8.54 282	8.54 308	1.45 692	9.99 974	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

88°

2°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.54 282	8.54 308	1.45 692	9.99 974	**60**
1	8.54 642	8.54 669	1.45 331	9.99 973	59
2	8.54 999	8.55 027	1.44 973	9.99 973	58
3	8.55 354	8.55 382	1.44 618	9.99 972	57
4	8.55 705	8.55 734	1.44 266	9.99 972	56
5	8.56 054	8.56 083	1.43 917	9.99 971	55
6	8.56 400	8.56 429	1.43 571	9.99 971	54
7	8.56 743	8.56 773	1.43 227	9.99 970	53
8	8.57 084	8.57 114	1.42 886	9.99 970	52
9	8.57 421	8.57 452	1.42 548	9.99 969	51
10	8.57 757	8.57 788	1.42 212	9.99 969	**50**
11	8.58 089	8.58 121	1.41 879	9.99 968	49
12	8.58 419	8.58 451	1.41 549	9.99 968	48
13	8.58 747	8.58 779	1.41 221	9.99 967	47
14	8.59 072	8.59 105	1.40 89$\bar{5}$	9.99 967	46
15	8.59 39$\bar{5}$	8.59 428	1.40 572	9.99 967	45
16	8.59 715	8.59 749	1.40 251	9.99 966	44
17	8.60 033	8.60 068	1.39 932	9.99 966	43
18	8.60 349	8.60 384	1.39 616	9.99 96$\bar{5}$	42
19	8.60 662	8.60 698	1.39 302	9.99 964	41
20	8.60 973	8.61 009	1.38 991	9.99 964	**40**
21	8.61 282	8.61 319	1.38 681	9.99 963	39
22	8.61 589	8.61 626	1.38 374	9.99 963	38
23	8.61 894	8.61 931	1.38 069	9.99 962	37
24	8.62 196	8.62 234	1.37 766	9.99 962	36
25	8.62 497	8.62 535	1.37 46$\bar{5}$	9.99 961	35
26	8.62 79$\bar{5}$	8.62 834	1.37 166	9.99 961	34
27	8.63 091	8.63 131	1.36 869	9.99 960	33
28	8.63 385	8.63 426	1.36 574	9.99 960	32
29	8.63 678	8.63 718	1.36 282	9.99 959	31
30	8.63 968	8.64 009	1.35 991	9.99 959	**30**
31	8.64 256	8.64 298	1.35 702	9.99 958	29
32	8.64 543	8.64 585	1.35 41$\bar{5}$	9.99 958	28
33	8.64 827	8.64 870	1.35 130	9.99 957	27
34	8.65 110	8.65 154	1.34 846	9.99 956	26
35	8.65 391	8.65 435	1.34 56$\bar{5}$	9.99 956	25
36	8.65 670	8.65 71$\bar{5}$	1.34 285	9.99 955	24
37	8.65 947	8.65 993	1.34 007	9.99 95$\bar{5}$	23
38	8.66 223	8.66 269	1.33 731	9.99 954	22
39	8.66 497	8.66 543	1.33 457	9.99 954	21
40	8.66 769	8.66 816	1.33 184	9.99 953	**20**
41	8.67 039	8.67 087	1.32 913	9.99 952	19
42	8.67 308	8.67 356	1.32 644	9.99 952	18
43	8.67 575	8.67 624	1.32 376	9.99 951	17
44	8.67 841	8.67 890	1.32 110	9.99 951	16
45	8.68 104	8.68 154	1.31 846	9.99 9$\bar{5}$0	15
46	8.68 367	8.68 417	1.31 583	9.99 949	14
47	8.68 627	8.68 678	1.31 322	9.99 949	13
48	8.68 886	8.68 938	1.31 062	9.99 948	12
49	8.69 144	8.69 196	1.30 804	9.99 948	11
50	8.69 400	8.69 453	1.30 547	9.99 947	**10**
51	8.69 654	8.69 708	1.30 292	9.99 946	9
52	8.69 907	8.69 962	1.30 038	9.99 946	8
53	8.70 159	8.70 214	1.29 786	9.99 94$\bar{5}$	7
54	8.70 409	8.70 465	1.29 535	9.99 944	6
55	8.70 658	8.70 714	1.29 286	9.99 944	5
56	8.70 905	8.70 962	1.29 038	9.99 943	4
57	8.71 151	8.71 208	1.28 792	9.99 942	3
58	8.71 395	8.71 453	1.28 547	9.99 942	2
59	8.71 638	8.71 697	1.28 303	9.99 941	1
60	8.71 880	8.71 940	1.28 060	9.99 940	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

87°

3°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.71 880	8.71 940	1.28 060	9.99 940	**60**
1	8.72 120	8.72 181	1.27 819	9.99 940	59
2	8.72 359	8.72 420	1.27 580	9.99 939	58
3	8.72 597	8.72 659	1.27 341	9.99 938	57
4	8.72 834	8.72 896	1.27 104	9.99 938	56
5	8.73 069	8.73 132	1.26 868	9.99 937	55
6	8.73 303	8.73 366	1.26 634	9.99 936	54
7	8.73 535	8.73 600	1.26 400	9.99 936	53
8	8.73 767	8.73 832	1.26 168	9.99 935	52
9	8.73 997	8.74 063	1.25 937	9.99 934	51
10	8.74 226	8.74 292	1.25 708	9.99 934	**50**
11	8.74 454	8.74 521	1.25 479	9.99 933	49
12	8.74 680	8.74 748	1.25 252	9.99 932	48
13	8.74 906	8.74 974	1.25 026	9.99 932	47
14	8.75 130	8.75 199	1.24 801	9.99 931	46
15	8.75 353	8.75 423	1.24 577	9.99 930	45
16	8.75 575	8.75 645	1.24 355	9.99 929	44
17	8.75 795	8.75 867	1.24 133	9.99 929	43
18	8.76 015	8.76 087	1.23 913	9.99 928	42
19	8.76 234	8.76 306	1.23 694	9.99 927	41
20	8.76 451	8.76 525	1.23 475	9.99 926	**40**
21	8.76 667	8.76 742	1.23 258	9.99 926	39
22	8.76 883	8.76 958	1.23 042	9.99 925	38
23	8.77 097	8.77 173	1.22 827	9.99 924	37
24	8.77 310	8.77 387	1.22 613	9.99 923	36
25	8.77 522	8.77 600	1.22 400	9.99 923	35
26	8.77 733	8.77 811	1.22 189	9.99 922	34
27	8.77 943	8.78 022	1.21 978	9.99 921	33
28	8.78 152	8.78 232	1.21 768	9.99 920	32
29	8.78 360	8.78 441	1.21 559	9.99 920	31
30	8.78 568	8.78 649	1.21 351	9.99 919	**30**
31	8.78 774	8.78 855	1.21 145	9.99 918	29
32	8.78 979	8.79 061	1.20 939	9.99 917	28
33	8.79 183	8.79 266	1.20 734	9.99 917	27
34	8.79 386	8.79 470	1.20 530	9.99 916	26
35	8.79 588	8.79 673	1.20 327	9.99 915	25
36	8.79 789	8.79 875	1.20 125	9.99 914	24
37	8.79 990	8.80 076	1.19 924	9.99 913	23
38	8.80 189	8.80 277	1.19 723	9.99 913	22
39	8.80 388	8.80 476	1.19 524	9.99 912	21
40	8.80 585	8.80 674	1.19 326	9.99 911	**20**
41	8.80 782	8.80 872	1.19 128	9.99 910	19
42	8.80 978	8.81 068	1.18 932	9.99 909	18
43	8.81 173	8.81 264	1.18 736	9.99 909	17
44	8.81 367	8.81 459	1.18 541	9.99 908	16
45	8.81 560	8.81 653	1.18 347	9.99 907	15
46	8.81 752	8.81 846	1.18 154	9.99 906	14
47	8.81 944	8.82 038	1.17 962	9.99 905	13
48	8.82 134	8.82 230	1.17 770	9.99 904	12
49	8.82 324	8.82 420	1.17 580	9.99 904	11
50	8.82 513	8.82 610	1.17 390	9.99 903	**10**
51	8.82 701	8.82 799	1.17 201	9.99 902	9
52	8.82 888	8.82 987	1.17 013	9.99 901	8
53	8.83 075	8.83 175	1.16 825	9.99 900	7
54	8.83 261	8.83 361	1.16 639	9.99 899	6
55	8.83 446	8.83 547	1.16 453	9.99 898	5
56	8.83 630	8.83 732	1.16 268	9.99 898	4
57	8.83 813	8.83 916	1.16 084	9.99 897	3
58	8.83 996	8.84 100	1.15 900	9.99 896	2
59	8.84 177	8.84 282	1.15 718	9.99 895	1
60	8.84 358	8.84 464	1.15 536	9.99 894	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

86°

4°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.84 358	8.84 464	1.15 536	9.99 894	**60**
1	8.84 539	8.84 646	1.15 354	9.99 893	59
2	8.84 718	8.84 826	1.15 174	9.99 892	58
3	8.84 897	8.85 006	1.14 994	9.99 891	57
4	8.85 075	8.85 18$\bar{5}$	1.14 815	9.99 891	56
5	8.85 252	8.85 363	1.14 637	9.99 890	55
6	8.85 429	8.85 540	1.14 460	9.99 889	54
7	8.85 60$\bar{5}$	8.85 717	1.14 283	9.99 888	53
8	8.85 780	8.85 893	1.14 107	9.99 887	52
9	8.85 95$\bar{5}$	8.86 069	1.13 931	9.99 886	51
10	8.86 128	8.86 243	1.13 757	9.99 885	**50**
11	8.86 301	8.86 417	1.13 583	9.99 884	49
12	8.86 474	8.86 591	1.13 409	9.99 883	48
13	8.86 645	8.86 763	1.13 237	9.99 882	47
14	8.86 816	8.86 935	1.13 06$\bar{5}$	9.99 881	46
15	8.86 987	8.87 106	1.12 894	9.99 880	45
16	8.87 156	8.87 277	1.12 723	9.99 879	44
17	8.87 325	8.87 447	1.12 553	9.99 879	43
18	8.87 494	8.87 616	1.12 384	9.99 878	42
19	8.87 661	8.87 78$\bar{5}$	1.12 215	9.99 877	41
20	8.87 829	8.87 953	1.12 047	9.99 876	**40**
21	8.87 99$\bar{5}$	8.88 120	1.11 880	9.99 87$\bar{5}$	39
22	8.88 161	8.88 287	1.11 713	9.99 874	38
23	8.88 326	8.88 453	1.11 547	9.99 873	37
24	8.88 490	8.88 618	1.11 382	9.99 872	36
25	8.88 654	8.88 783	1.11 217	9.99 871	35
26	8.88 817	8.88 948	1.11 052	9.99 870	34
27	8.88 980	8.89 111	1.10 889	9.99 869	33
28	8.89 142	8.89 274	1.10 726	9.99 868	32
29	8.89 304	8.89 437	1.10 563	9.99 867	31
30	8.89 464	8.89 598	1.10 402	9.99 866	**30**
31	8.89 62$\bar{5}$	8.89 760	1.10 240	9.99 86$\bar{5}$	29
32	8.89 784	8.89 920	1.10 080	9.99 864	28
33	8.89 943	8.90 080	1.09 920	9.99 863	27
34	8.90 102	8.90 240	1.09 760	9.99 862	26
35	8.90 260	8.90 399	1.09 601	9.99 861	25
36	8.90 417	8.90 557	1.09 443	9.99 860	24
37	8.90 574	8.90 71$\bar{5}$	1.09 285	9.99 859	23
38	8.90 730	8.90 872	1.09 128	9.99 858	22
39	8.90 885	8.91 029	1.08 971	9.99 857	21
40	8.91 040	8.91 18$\bar{5}$	1.08 815	9.99 856	**20**
41	8.91 19$\bar{5}$	8.91 340	1.08 660	9.99 85$\bar{5}$	19
42	8.91 349	8.91 495	1.08 50$\bar{5}$	9.99 854	18
43	8.91 502	8.91 6$\bar{5}$0	1.08 350	9.99 853	17
44	8.91 655	8.91 803	1.08 197	9.99 852	16
45	8.91 807	8.91 957	1.08 043	9.99 851	15
46	8.91 959	8.92 110	1.07 890	9.99 8$\bar{5}$0	14
47	8.92 110	8.92 262	1.07 738	9.99 848	13
48	8.92 261	8.92 414	1.07 586	9.99 847	12
49	8.92 411	8.92 56$\bar{5}$	1.07 435	9.99 846	11
50	8.92 561	8.92 716	1.07 284	9.99 845	**10**
51	8.92 710	8.92 866	1.07 134	9.99 844	9
52	8.92 859	8.93 016	1.06 984	9.99 843	8
53	8.93 007	8.93 16$\bar{5}$	1.06 835	9.99 842	7
54	8.93 154	8.93 313	1.06 687	9.99 841	6
55	8.93 301	8.93 462	1.06 538	9.99 840	5
56	8.93 448	8.93 609	1.06 391	9.99 839	4
57	8.93 594	8.93 756	1.06 244	9.99 838	3
58	8.93 740	8.93 903	1.06 097	9.99 837	2
59	8.93 88$\bar{5}$	8.94 049	1.05 951	9.99 836	1
60	8.94 030	8.94 195	1.05 80$\bar{5}$	9.99 834	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

85°

5°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.94 030	8.94 195	1.05 805	9.99 834	60
1	8.94 174	8.94 340	1.05 660	9.99 833	59
2	8.94 317	8.94 485	1.05 515	9.99 832	58
3	8.94 461	8.94 630	1.05 370	9.99 831	57
4	8.94 603	8.94 773	1.05 227	9.99 830	56
5	8.94 746	8.94 917	1.05 083	9.99 829	55
6	8.94 887	8.95 060	1.04 940	9.99 828	54
7	8.95 029	8.95 202	1.04 798	9.99 827	53
8	8.95 170	8.95 344	1.04 656	9.99 825	52
9	8.95 310	8.95 486	1.04 514	9.99 824	51
10	8.95 450	8.95 627	1.04 373	9.99 823	50
11	8.95 589	8.95 767	1.04 233	9.99 822	49
12	8.95 728	8.95 908	1.04 092	9.99 821	48
13	8.95 867	8.96 047	1.03 953	9.99 820	47
14	8.96 005	8.96 187	1.03 813	9.99 819	46
15	8.96 143	8.96 325	1.03 675	9.99 817	45
16	8.96 280	8.96 464	1.03 536	9.99 816	44
17	8.96 417	8.96 602	1.03 398	9.99 815	43
18	8.96 553	8.96 739	1.03 261	9.99 814	42
19	8.96 689	8.96 877	1.03 123	9.99 813	41
20	8.96 825	8.97 013	1.02 987	9.99 812	40
21	8.96 960	8.97 150	1.02 850	9.99 810	39
22	8.97 095	8.97 285	1.02 715	9.99 809	38
23	8.97 229	8.97 421	1.02 579	9.99 808	37
24	8.97 363	8.97 556	1.02 444	9.99 807	36
25	8.97 496	8.97 691	1.02 309	9.99 806	35
26	8.97 629	8.97 825	1.02 175	9.99 804	34
27	8.97 762	8.97 959	1.02 041	9.99 803	33
28	8.97 894	8.98 092	1.01 908	9.99 802	32
29	8.98 026	8.98 225	1.01 775	9.99 801	31
30	8.98 157	8.98 358	1.01 642	9.99 800	30
31	8.98 288	8.98 490	1.01 510	9.99 798	29
32	8.98 419	8.98 622	1.01 378	9.99 797	28
33	8.98 549	8.98 753	1.01 247	9.99 796	27
34	8.98 679	8.98 884	1.01 116	9.99 795	26
35	8.98 808	8.99 015	1.00 985	9.99 793	25
36	8.98 937	8.99 145	1.00 855	9.99 792	24
37	8.99 066	8.99 275	1.00 725	9.99 791	23
38	8.99 194	8.99 405	1.00 595	9.99 790	22
39	8.99 322	8.99 534	1.00 466	9.99 788	21
40	8.99 450	8.99 662	1.00 338	9.99 787	20
41	8.99 577	8.99 791	1.00 209	9.99 786	19
42	8.99 704	8.99 919	1.00 081	9.99 785	18
43	8.99 830	9.00 046	0.99 954	9.99 783	17
44	8.99 956	9.00 174	0.99 826	9.99 782	16
45	9.00 082	9.00 301	0.99 699	9.99 781	15
46	9.00 207	9.00 427	0.99 573	9.99 780	14
47	9.00 332	9.00 553	0.99 447	9.99 778	13
48	9.00 456	9.00 679	0.99 321	9.99 777	12
49	9.00 581	9.00 805	0.99 195	9.99 776	11
50	9.00 704	9.00 930	0.99 070	9.99 775	10
51	9.00 828	9.01 055	0.98 945	9.99 773	9
52	9.00 951	9.01 179	0.98 821	9.99 772	8
53	9.01 074	9.01 303	0.98 697	9.99 771	7
54	9.01 196	9.01 427	0.98 573	9.99 769	6
55	9.01 318	9.01 550	0.98 450	9.99 768	5
56	9.01 440	9.01 673	0.98 327	9.99 767	4
57	9.01 561	9.01 796	0.98 204	9.99 765	3
58	9.01 682	9.01 918	0.98 082	9.99 764	2
59	9.01 803	9.02 040	0.97 960	9.99 763	1
60	9.01 923	9.02 162	0.97 838	9.99 761	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

84°

6°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.01 923	9.02 162	0.97 838	9.99 761	**60**
1	9.02 043	9.02 283	0.97 717	9.99 760	59
2	9.02 163	9.02 404	0.97 596	9.99 759	58
3	9.02 283	9.02 525	0.97 47$\bar{5}$	9.99 757	57
4	9.02 402	9.02 645	0.97 35$\bar{5}$	9.99 756	56
5	9.02 520	9.02 766	0.97 234	9.99 75$\bar{5}$	55
6	9.02 639	9.02 885	0.97 11$\bar{5}$	9.99 753	54
7	9.02 757	9.03 00$\bar{5}$	0.96 995	9.99 752	53
8	9.02 874	9.03 124	0.96 876	9.99 751	52
9	9.02 992	9.03 242	0.96 758	9.99 749	51
10	9.03 109	9.03 361	0.96 639	9.99 748	**50**
11	9.03 226	9.03 479	0.96 521	9.99 747	49
12	9.03 342	9.03 597	0.96 403	9.99 745	48
13	9.03 458	9.03 714	0.96 286	9.99 744	47
14	9.03 574	9.03 832	0.96 168	9.99 742	46
15	9.03 690	9.03 948	0.96 052	9.99 741	45
16	9.03 80$\bar{5}$	9.04 065	0.95 93$\bar{5}$	9.99 740	44
17	9.03 920	9.04 181	0.95 819	9.99 738	43
18	9.04 034	9.04 297	0.95 703	9.99 737	42
19	9.04 149	9.04 413	0.95 587	9.99 736	41
20	9.04 262	9.04 528	0.95 472	9.99 734	**40**
21	9.04 376	9.04 643	0.95 357	9.99 733	39
22	9.04 490	9.04 758	0.95 242	9.99 731	38
23	9.04 603	9.04 873	0.95 127	9.99 730	37
24	9.04 715	9.04 987	0.95 013	9.99 728	36
25	9.04 828	9.05 101	0.94 899	9.99 727	35
26	9.04 940	9.05 214	0.94 786	9.99 726	34
27	9.05 052	9.05 328	0.94 672	9.99 724	33
28	9.05 164	9.05 441	0.94 559	9.99 723	32
29	9.05 27$\bar{5}$	9.05 553	0.94 447	9.99 721	31
30	9.05 386	9.05 666	0.94 334	9.99 720	**30**
31	9.05 497	9.05 778	0.94 222	9.99 718	29
32	9.05 607	9.05 890	0.94 110	9.99 717	28
33	9.05 717	9.06 002	0.93 998	9.99 716	27
34	9.05 827	9.06 113	0.93 887	9.99 714	26
35	9.05 937	9.06 224	0.93 776	9.99 713	25
36	9.06 046	9.06 33$\bar{5}$	0.93 665	9.99 711	24
37	9.06 155	9.06 445	0.93 55$\bar{5}$	9.99 710	23
38	9.06 264	9.06 556	0.93 444	9.99 708	22
39	9.06 372	9.06 666	0.93 334	9.99 707	21
40	9.06 481	9.06 775	0.93 225	9.99 705	**20**
41	9.06 589	9.06 88$\bar{5}$	0.93 115	9.99 704	19
42	9.06 696	9.06 994	0.93 006	9.99 702	18
43	9.06 804	9.07 103	0.92 897	9.99 701	17
44	9.06 911	9.07 211	0.92 789	9.99 699	16
45	9.07 018	9.07 320	0.92 680	9.99 698	15
46	9.07 124	9.07 428	0.92 572	9.99 696	14
47	9.07 231	9.07 536	0.92 464	9.99 69$\bar{5}$	13
48	9.07 337	9.07 643	0.92 357	9.99 693	12
49	9.07 442	9.07 751	0.92 249	9.99 692	11
50	9.07 548	9.07 858	0.92 142	9.99 690	**10**
51	9.07 653	9.07 964	0.92 036	9.99 689	9
52	9.07 758	9.08 071	0.91 929	9.99 687	8
53	9.07 863	9.08 177	0.91 823	9.99 686	7
54	9.07 968	9.08 283	0.91 717	9.99 684	6
55	9.08 072	9.08 389	0.91 611	9.99 683	5
56	9.08 176	9.08 49$\bar{5}$	0.91 505	9.99 681	4
57	9.08 280	9.08 600	0.91 400	9.99 680	3
58	9.08 383	9.08 705	0.91 29$\bar{5}$	9.99 678	2
59	9.08 486	9.08 810	0.91 190	9.99 677	1
60	9.08 589	9.08 914	0.91 086	9.99 675	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

83°

7°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.08 589	9.08 914	0.91 086	9.99 675	**60**
1	9.08 692	9.09 019	0.90 981	9.99 674	59
2	9.08 795̄	9.09 123	0.90 877	9.99 672	58
3	9.08 897	9.09 227	0.90 773	9.99 670	57
4	9.08 999	9.09 330	0.90 670	9.99 669	56
5	9.09 101	9.09 434	0.90 566	9.99 667	55
6	9.09 202	9.09 537	0.90 463	9.99 666	54
7	9.09 304	9.09 640	0.90 360	9.99 664	53
8	9.09 405̄	9.09 742	0.90 258	9.99 663	52
9	9.09 506	9.09 845̄	0.90 155	9.99 661	51
10	9.09 606	9.09 947	0.90 053	9.99 659	**50**
11	9.09 707	9.10 049	0.89 951	9.99 658	49
12	9.09 807	9.10 150	0.89 85̄0	9.99 656	48
13	9.09 907	9.10 252	0.89 748	9.99 655̄	47
14	9.10 006	9.10 353	0.89 647	9.99 653	46
15	9.10 106	9.10 454	0.89 546	9.99 651	45
16	9.10 205̄	9.10 555̄	0.89 445	9.99 65̄0	44
17	9.10 304	9.10 656	0.98 344	9.99 648	43
18	9.10 402	9.10 756	0.89 244	9.99 647	42
19	9.10 501	9.10 856	0.89 144	9.99 645̄	41
20	9.10 599	9.10 956	0.89 044	9.99 643	**40**
21	9.10 697	9.11 056	0.88 944	9.99 642	39
22	9.10 795	9.11 155	0.88 845̄	9.99 640	38
23	9.10 893	9.11 254	0.88 746	9.99 638	37
24	9.10 990	9.11 353	0.88 647	9.99 637	36
25	9.11 087	9.11 452	0.88 548	9.99 635	35
26	9.11 184	9.11 551	0.88 449	9.99 633	34
27	9.11 281	9.11 649	0.88 351	9.99 632	33
28	9.11 377	9.11 747	0.88 253	9.99 630	32
29	9.11 474	9.11 845	0.88 155̄	9.99 629	31
30	9.11 570	9.11 943	0.88 057	9.99 627	**30**
31	9.11 666	9.12 040	0.87 960	9.99 625	29
32	9.11 761	9.12 138	0.87 862	9.99 624	28
33	9.11 857	9.12 235̄	0.87 765	9.99 622	27
34	9.11 952	9.12 332	0.87 668	9.99 620	26
35	9.12 047	9.12 428	0.87 572	9.99 618	25
36	9.12 142	9.12 525̄	0.87 475	9.99 617	24
37	9.12 236	9.12 621	0.87 379	9.99 615	23
38	9.12 331	9.12 717	0.87 283	9.99 613	22
39	9.12 425̄	9.12 813	0.87 187	9.99 612	21
40	9.12 519	9.12 909	0.87 091	9.99 610	**20**
41	9.12 612	9.13 004	0.86 996	9.99 608	19
42	9.12 706	9.13 099	0.86 901	9.99 607	18
43	9.12 799	9.13 194	0.86 806	9.99 605̄	17
44	9.12 892	9.13 289	0.86 711	9.99 603	16
45	9.12 985	9.13 384	0.86 616	9.99 601	15
46	9.13 078	9.13 478	0.86 522	9.99 600	14
47	9.13 171	9.13 573	0.86 427	9.99 598	13
48	9.13 263	9.13 667	0.86 333	9.99 596	12
49	9.13 355	9.13 761	0.86 239	9.99 595̄	11
50	9.13 447	9.13 854	0.86 146	9.99 593	**10**
51	9.13 539	9.13 948	0.86 052	9.99 591	9
52	9.13 630	9.14 041	0.85 959	9.99 589	8
53	9.13 722	9.14 134	0.85 866	9.99 588	7
54	9.13 813	9.14 227	0.85 773	9.99 586	6
55	9.13 904	9.14 320	0.85 680	9.99 584	5
56	9.13 994	9.14 412	0.85 588	9.99 582	4
57	9.14 085	9.14 504	0.85 496	9.99 581	3
58	9.14 175	9.14 597	0.85 403	9.99 579	2
59	9.14 266	9.14 688	0.85 312	9.99 577	1
60	9.14 356	9.14 780	0.85 220	9.99 575	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

82°

8°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.14 356	9.14 780	0.85 220	9.99 575	**60**
1	9.14 445	9.14 872	0.85 128	9.99 574	59
2	9.14 535̄	9.14 963	0.85 037	9.99 572	58
3	9.14 624	9.15 054	0.84 946	9.99 570	57
4	9.14 714	9.15 145	0.84 855̄	9.99 568	56
5	9.14 803	9.15 236	0.84 764	9.99 566	55
6	9.14 891	9.15 327	0.84 673	9.99 565̄	54
7	9.14 980	9.15 417	0.84 583	9.99 563	53
8	9.15 069	9.15 508	0.84 492	9.99 561	52
9	9.15 157	9.15 598	0.84 402	9.99 559	51
10	9.15 245	9.15 688	0.84 312	9.99 557	**50**
11	9.15 333	9.15 777	0.84 223	9.99 556	49
12	9.15 421	9.15 867	0.84 133	9.99 554	48
13	9.15 508	9.15 956	0.84 044	9.99 552	47
14	9.15 596	9.16 046	0.83 954	9.99 550	46
15	9.15 683	9.16 135̄	0.83 865	9.99 548	45
16	9.15 770	9.16 224	0.83 776	9.99 546	44
17	9.15 857	9.16 312	0.83 688	9.99 545̄	43
18	9.15 944	9.16 401	0.83 599	9.99 543	42
19	9.16 030	9.16 489	0.83 511	9.99 541	41
20	9.16 116	9.16 577	0.83 423	9.99 539	**40**
21	9.16 203	9.16 665	0.83 335̄	9.99 537	39
22	9.16 289	9.16 753	0.83 247	9.99 535	38
23	9.16 374	9.16 841	0.83 159	9.99 533	37
24	9.16 460	9.16 928	0.83 072	9.99 532	36
25	9.16 545	9.17 016	0.82 984	9.99 530	35
26	9.16 631	9.17 103	0.82 897	9.99 528	34
27	9.16 716	9.17 190	0.82 810	9.99 526	33
28	9.16 801	9.17 277	0.82 723	9.99 524	32
29	9.16 886	9.17 363	0.82 637	9.99 522	31
30	9.16 970	9.17 45̄0	0.82 550	9.99 520	**30**
31	9.17 055̄	9.17 536	0.82 464	9.99 518	29
32	9.17 139	9.17 622	0.82 378	9.99 517	28
33	9.17 223	9.17 708	0.82 292	9.99 515̄	27
34	9.17 307	9.17 794	0.82 206	9.99 513	26
35	9.17 391	9.17 880	0.82 120	9.99 511	25
36	9.17 474	9.17 965	0.82 035̄	9.99 509	24
37	9.17 558	9.18 051	0.81 949	9.99 507	23
38	9.17 641	9.18 136	0.81 864	9.99 505	22
39	9.17 724	9.18 221	0.81 779	9.99 503	21
40	9.17 807	9.18 306	0.81 694	9.99 501	**20**
41	9.17 890	9.18 391	0.81 609	9.99 499	19
42	9.17 973	9.18 475	0.81 525̄	9.99 497	18
43	9.18 055	9.18 560	0.81 440	9.99 495	17
44	9.18 137	9.18 644	0.81 356	9.99 494	16
45	9.18 220	9.18 728	0.81 272	9.99 492	15
46	9.18 302	9.18 812	0.81 188	9.99 490	14
47	9.18 383	9.18 896	0.81 104	9.99 488	13
48	9.18 465	9.18 979	0.81 021	9.99 486	12
49	9.18 547	9.19 063	0.80 937	9.99 484	11
50	9.18 628	9.19 146	0.80 854	9.99 482	**10**
51	9.18 709	9.19 229	0.80 771	9.99 480	9
52	9.18 790	9.19 312	0.80 688	9.99 478	8
53	9.18 871	9.19 395	0.80 605̄	9.99 476	7
54	9.18 952	9.19 478	0.80 522	9.99 474	6
55	9.19 033	9.19 561	0.80 439	9.99 472	5
56	9.19 113	9.19 643	0.80 357	9.99 470	4
57	9.19 193	9.19 725	0.80 275̄	9.99 468	3
58	9.19 273	9.19 807	0.80 193	9.99 466	2
59	9.19 353	9.19 889	0.80 111	9.99 464	1
60	9.19 433	9.19 971	0.80 029	9.99 462	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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9°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.19 433	9.19 971	0.80 029	9.99 462	60
1	9.19 513	9.20 053	0.79 947	9.99 460	59
2	9.19 592	9.20 134	0.79 866	9.99 458	58
3	9.19 672	9.20 216	0.79 784	9.99 456	57
4	9.19 751	9.20 297	0.79 703	9.99 454	56
5	9.19 830	9.20 378	0.79 622	9.99 452	55
6	9.19 909	9.20 459	0.79 541	9.99 450	54
7	9.19 988	9.20 540	0.79 460	9.99 448	53
8	9.20 067	9.20 621	0.79 379	9.99 446	52
9	9.20 145	9.20 701	0.79 299	9.99 444	51
10	9.20 223	9.20 782	0.79 218	9.99 442	50
11	9.20 302	9.20 862	0.79 138	9.99 440	49
12	9.20 380	9.20 942	0.79 058	9.99 438	48
13	9.20 458	9.21 022	0.78 978	9.99 436	47
14	9.20 535	9.21 102	0.78 898	9.99 434	46
15	9.20 613	9.21 182	0.78 818	9.99 432	45
16	9.20 691	9.21 261	0.78 739	9.99 429	44
17	9.20 768	9.21 341	0.78 659	9.99 427	43
18	9.20 845	9.21 420	0.78 580	9.99 425	42
19	9.20 922	9.21 499	0.78 501	9.99 423	41
20	9.20 999	9.21 578	0.78 422	9.99 421	40
21	9.21 076	9.21 657	0.78 343	9.99 419	39
22	9.21 153	9.21 736	0.78 264	9.99 417	38
23	9.21 229	9.21 814	0.78 186	9.99 415	37
24	9.21 306	9.21 893	0.78 107	9.99 413	36
25	9.21 382	9.21 971	0.78 029	9.99 411	35
26	9.21 458	9.22 049	0.77 951	9.99 409	34
27	9.21 534	9.22 127	0.77 873	9.99 407	33
28	9.21 610	9.22 205	0.77 795	9.99 404	32
29	9.21 685	9.22 283	0.77 717	9.99 402	31
30	9.21 761	9.22 361	0.77 639	9.99 400	30
31	9.21 836	9.22 438	0.77 562	9.99 398	29
32	9.21 912	9.22 516	0.77 484	9.99 396	28
33	9.21 987	9.22 593	0.77 407	9.99 394	27
34	9.22 062	9.22 670	0.77 330	9.99 392	26
35	9.22 137	9.22 747	0.77 253	9.99 390	25
36	9.22 211	9.22 824	0.77 176	9.99 388	24
37	9.22 286	9.22 901	0.77 099	9.99 385	23
38	9.22 361	9.22 977	0.77 023	9.99 383	22
39	9.22 435	9.23 054	0.76 946	9.99 381	21
40	9.22 509	9.23 130	0.76 870	9.99 379	20
41	9.22 583	9.23 206	0.76 794	9.99 377	19
42	9.22 657	9.23 283	0.76 717	9.99 375	18
43	9.22 731	9.23 359	0.76 641	9.99 372	17
44	9.22 805	9.23 435	0.76 565	9.99 370	16
45	9.22 878	9.23 510	0.76 490	9.99 368	15
46	9.22 952	9.23 586	0.76 414	9.99 366	14
47	9.23 025	9.23 661	0.76 339	9.99 364	13
48	9.23 098	9.23 737	0.76 263	9.99 362	12
49	9.23 171	9.23 812	0.76 188	9.99 359	11
50	9.23 244	9.23 887	0.76 113	9.99 357	10
51	9.23 317	9.23 962	0.76 038	9.99 355	9
52	9.23 390	9.24 037	0.75 963	9.99 353	8
53	9.23 462	9.24 112	0.75 888	9.99 351	7
54	9.23 535	9.24 186	0.75 814	9.99 348	6
55	9.23 607	9.24 261	0.75 739	9.99 346	5
56	9.23 679	9.24 335	0.75 665	9.99 344	4
57	9.23 752	9.24 410	0.75 590	9.99 342	3
58	9.23 823	9.24 484	0.75 516	9.99 340	2
59	9.23 895	9.24 558	0.75 442	9.99 337	1
60	9.23 967	9.24 632	0.75 368	9.99 335	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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10°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.23 967	9.24 632	0.75 368	9.99 335	60
1	9.24 039	9.24 706	0.75 294	9.99 333	59
2	9.24 110	9.24 779	0.75 221	9.99 331	58
3	9.24 181	9.24 853	0.75 147	9.99 328	57
4	9.24 253	9.24 926	0.75 074	9.99 326	56
5	9.24 324	9.25̄ 000	0.75 000	9.99 324	55
6	9.24 395̄	9.25 073	0.74 927	9.99 322	54
7	9.24 466	9.25 146	0.74 854	9.99 319	53
8	9.24 536	9.25 219	0.74 781	9.99 317	52
9	9.24 607	9.25 292	0.74 708	9.99 315̄	51
10	9.24 677	9.25 365̄	0.74 635	9.99 313	50
11	9.24 748	9.25 437	0.74 563	9.99 310	49
12	9.24 818	9.25 510	0.74 490	9.99 308	48
13	9.24 888	9.25 582	0.74 418	9.99 306	47
14	9.24 958	9.25 655̄	0.74 345	9.99 304	46
15	9.25 028	9.25 727	0.74 273	9.99 301	45
16	9.25 098	9.25 799	0.74 201	9.99 299	44
17	9.25 168	9.25 871	0.74 129	9.99 297	43
18	9.25 237	9.25 943	0.74 057	9.99 294	42
19	9.25 307	9.26 015̄	0.73 985	9.99 292	41
20	9.25 376	9.26 086	0.73 914	9.99 290	40
21	9.25 445	9.26 158	0.73 842	9.99 288	39
22	9.25 514	9.26 229	0.73 771	9.99 285	38
23	9.25 583	9.26 301	0.73 699	9.99 283	37
24	9.25 652	9.26 372	0.73 628	9.99 281	36
25	9.25 721	9.26 443	0.73 557	9.99 278	35
26	9.25 790	9.26 514	0.73 486	9.99 276	34
27	9.25 858	9.26 585̄	0.73 415	9.99 274	33
28	9.25 927	9.26 655	0.73 345̄	9.99 271	32
29	9.25 995	9.26 726	0.73 274	9.99 269	31
30	9.26 063	9.26 797	0.73 203	9.99 267	30
31	9.26 131	9.26 867	0.73 133	9.99 264	29
32	9.26 199	9.26 937	0.73 063	9.99 262	28
33	9.26 267	9.27 008	0.72 992	9.99 260	27
34	9.26 335	9.27 078	0.72 922	9.99 257	26
35	9.26 403	9.27 148	0.72 852	9.99 255̄	25
36	9.26 470	9.27 218	0.72 782	9.99 252	24
37	9.26 538	9.27 288	0.72 712	9.99 250	23
38	9.26 605	9.27 357	0.72 643	9.99 248	22
39	9.26 672	9.27 427	0.72 573	9.99 245	21
40	9.26 739	9.27 496	0.72 504	9.99 243	20
41	9.26 806	9.27 566	0.72 434	9.99 241	19
42	9.26 873	9.27 635	0.72 365̄	9.99 238	18
43	9.26 940	9.27 704	0.72 296	9.99 236	17
44	9.27 007	9.27 773	0.72 227	9.99 233	16
45	9.27 073	9.27 842	0.72 158	9.99 231	15
46	9.27 140	9.27 911	0.72 089	9.99 229	14
47	9.27 206	9.27 980	0.72 020	9.99 226	13
48	9.27 273	9.28 049	0.71 951	9.99 224	12
49	9.27 339	9.28 117	0.71 883	9.99 221	11
50	9.27 405̄	9.28 186	0.71 814	9.99 219	10
51	9.27 471	9.28 254	0.71 746	9.99 217	9
52	9.27 537	9.28 323	0.71 677	9.99 214	8
53	9.27 602	9.28 391	0.71 609	9.99 212	7
54	9.27 668	9.28 459	0.71 541	9.99 209	6
55	9.27 734	9.28 527	0.71 473	9.99 207	5
56	9.27 799	9.28 595̄	0.71 405	9.99 204	4
57	9.27 864	9.28 662	0.71 338	9.99 202	3
58	9.27 930	9.28 730	0.71 270	9.99 200	2
59	9.27 995̄	9.28 798	0.71 202	9.99 197	1
60	9.28 060	9.28 865	0.71 135̄	9.99 195̄	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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11°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.28 060	9.28 865	0.71 135	9.99 195	60
1	9.28 125	9.28 933	0.71 067	9.99 192	59
2	9.28 190	9.29 000	0.71 000	9.99 190	58
3	9.28 254	9.29 067	0.70 933	9.99 187	57
4	9.28 319	9.29 134	0.70 866	9.99 185	56
5	9.28 384	9.29 201	0.70 799	9.99 182	55
6	9.28 448	9.29 268	0.70 732	9.99 180	54
7	9.28 512	9.29 335	0.70 665	9.99 177	53
8	9.28 577	9.29 402	0.70 598	9.99 175	52
9	9.28 641	9.29 468	0.70 532	9.99 172	51
10	9.28 705	9.29 535	0.70 465	9.99 170	50
11	9.28 769	9.29 601	0.70 399	9.99 167	49
12	9.28 833	9.29 668	0.70 332	9.99 165	48
13	9.28 896	9.29 734	0.70 266	9.99 162	47
14	9.28 960	9.29 800	0.70 200	9.99 160	46
15	9.29 024	9.29 866	0.70 134	9.99 157	45
16	9.29 087	9.29 932	0.70 068	9.99 155	44
17	9.29 150	9.29 998	0.70 002	9.99 152	43
18	9.29 214	9.30 064	0.69 936	9.99 150	42
19	9.29 277	9.30 130	0.69 870	9.99 147	41
20	9.29 340	9.30 195	0.69 805	9.99 145	40
21	9.29 403	9.30 261	0.69 739	9.99 142	39
22	9.29 466	9.30 326	0.69 674	9.99 140	38
23	9.29 529	9.30 391	0.69 609	9.99 137	37
24	9.29 591	9.30 457	0.69 543	9.99 135	36
25	9.29 654	9.30 522	0.69 478	9.99 132	35
26	9.29 716	9.30 587	0.69 413	9.99 130	34
27	9.29 779	9.30 652	0.69 348	9.99 127	33
28	9.29 841	9.30 717	0.69 283	9.99 124	32
29	9.29 903	9.30 782	0.69 218	9.99 122	31
30	9.29 966	9.30 846	0.69 154	9.99 119	30
31	9.30 028	9.30 911	0.69 089	9.99 117	29
32	9.30 090	9.30 975	0.69 025	9.99 114	28
33	9.30 151	9.31 040	0.68 960	9.99 112	27
34	9.30 213	9.31 104	0.68 896	9.99 109	26
35	9.30 275	9.31 168	0.68 832	9.99 106	25
36	9.30 336	9.31 233	0.68 767	9.99 104	24
37	9.30 398	9.31 297	0.68 703	9.99 101	23
38	9.30 459	9.31 361	0.68 639	9.99 099	22
39	9.30 521	9.31 425	0.68 575	9.99 096	21
40	9.30 582	9.31 489	0.68 511	9.99 093	20
41	9.30 643	9.31 552	0.68 448	9.99 091	19
42	9.30 704	9.31 616	0.68 384	9.99 088	18
43	9.30 765	9.31 679	0.68 321	9.99 086	17
44	9.30 826	9.31 743	0.68 257	9.99 083	16
45	9.30 887	9.31 806	0.68 194	9.99 080	15
46	9.30 947	9.31 870	0.68 130	9.99 078	14
47	9.31 008	9.31 933	0.68 067	9.99 075	13
48	9.31 068	9.31 996	0.68 004	9.99 072	12
49	9.31 129	9.32 059	0.67 941	9.99 070	11
50	9.31 189	9.32 122	0.67 878	9.99 067	10
51	9.31 250	9.32 185	0.67 815	9.99 064	9
52	9.31 310	9.32 248	0.67 752	9.99 062	8
53	9.31 370	9.32 311	0.67 689	9.99 059	7
54	9.31 430	9.32 373	0.67 627	9.99 056	6
55	9.31 490	9.32 436	0.67 564	9.99 054	5
56	9.31 549	9.32 498	0.67 502	9.99 051	4
57	9.31 609	9.32 561	0.67 439	9.99 048	3
58	9.31 669	9.32 623	0.67 377	9.99 046	2
59	9.31 728	9.32 685	0.67 315	9.99 043	1
60	9.31 788	9.32 747	0.67 253	9.99 040	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

78°

12°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.31 788	9.32 747	0.67 253	9.99 040	60
1	9.31 847	9.32 810	0.67 190	9.99 038	59
2	9.31 907	9.32 872	0.67 128	9.99 035	58
3	9.31 966	9.32 933	0.67 067	9.99 032	57
4	9.32 025̄	9.32 995	0.67 005̄	9.99 030	56
5	9.32 084	9.33 057	0.66 943	9.99 027	55
6	9.32 143	9.33 119	0.66 881	9.99 024	54
7	9.32 202	9.33 180	0.66 820	9.99 022	53
8	9.32 261	9.33 242	0.66 758	9.99 019	52
9	9.32 319	9.33 303	0.66 697	9.99 016	51
10	9.32 378	9.33 365̄	0.66 635	9.99 013	50
11	9.32 437	9.33 426	0.66 574	9.99 011	49
12	9.32 495	9.33 487	0.66 513	9.99 008	48
13	9.32 553	9.33 548	0.66 452	9.99 005	47
14	9.32 612	9.33 609	0.66 391	9.99 002	46
15	9.32 670	9.33 670	0.66 330	9.99 000	45
16	9.32 728	9.33 731	0.66 269	9.98 997	44
17	9.32 786	9.33 792	0.66 208	9.98 994	43
18	9.32 844	9.33 853	0.66 147	9.98 991	42
19	9.32 902	9.33 913	0.66 087	9.98 989	41
20	9.32 960	9.33 974	0.66 026	9.98 986	40
21	9.33 018	9.34 034	0.65 966	9.98 983	39
22	9.33 075	9.34 095̄	0.65 905	9.98 980	38
23	9.33 133	9.34 155	0.65 845̄	9.98 978	37
24	9.33 190	9.34 215	0.65 785̄	9.98 975̄	36
25	9.33 248	9.34 276	0.65 724	9.98 972	35
26	9.33 305	9.34 336	0.65 664	9.98 969	34
27	9.33 362	9.34 396	0.65 604	9.98 967	33
28	9.33 420	9.34 456	0.65 544	9.98 964	32
29	9.33 477	9.34 516	0.65 484	9.98 961	31
30	9.33 534	9.34 576	0.65 424	9.98 958	30
31	9.33 591	9.34 635	0.65 365̄	9.98 955	29
32	9.33 647	9.34 695̄	0.65 305	9.98 953	28
33	9.33 704	9.34 755̄	0.65 245	9.98 950	27
34	9.33 761	9.34 814	0.65 186	9.98 947	26
35	9.33 818	9.34 874	0.65 126	9.98 944	25
36	9.33 874	9.34 933	0.65 067	9.98 941	24
37	9.33 931	9.34 992	0.65 008	9.98 938	23
38	9.33 987	9.35 051	0.64 949	9.98 936	22
39	9.34 043	9.35 111	0.64 889	9.98 933	21
40	9.34 100	9.35 170	0.64 830	9.98 930	20
41	9.34 156	9.35 229	0.64 771	9.98 927	19
42	9.34 212	9.35 288	0.64 712	9.98 924	18
43	9.34 268	9.35 347	0.64 653	9.98 921	17
44	9.34 324	9.35 405	0.64 595̄	9.98 919	16
45	9.34 380	9.35 464	0.64 536	9.98 916	15
46	9.34 436	9.35 523	0.64 477	9.98 913	14
47	9.34 491	9.35 581	0.64 419	9.98 910	13
48	9.34 547	9.35 640	0.64 360	9.98 907	12
49	9.34 602	9.35 698	0.64 302	9.98 904	11
50	9.34 658	9.35 757	0.64 243	9.98 901	10
51	9.34 713	9.35 815̄	0.64 185	9.98 898	9
52	9.34 769	9.35 873	0.64 127	9.98 896	8
53	9.34 824	9.35 931	0.64 069	9.98 893	7
54	9.34 879	9.35 989	0.64 011	9.98 890	6
55	9.34 934	9.36 047	0.63 953	9.98 887	5
56	9.34 989	9.36 105	0.63 895̄	9.98 884	4
57	9.35 044	9.36 163	0.63 837	9.98 881	3
58	9.35 099	9.36 221	0.63 779	9.98 878	2
59	9.35 154	9.36 279	0.63 721	9.98 875	1
60	9.35 209	9.36 336	0.63 664	9.98 872	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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13°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.35 209	9.36 336	0.63 664	9.98 872	60
1	9.35 263	9.36 394	0.63 606	9.98 869	59
2	9.35 318	9.36 452	0.63 548	9.98 867	58
3	9.35 373	9.36 509	0.63 491	9.98 864	57
4	9.35 427	9.36 566	0.63 434	9.98 861	56
5	9.35 481	9.36 624	0.63 376	9.98 858	55
6	9.35 536	9.36 681	0.63 319	9.98 855̄	54
7	9.35 590	9.36 738	0.63 262	9.98 852	53
8	9.35 644	9.36 795	0.63 205̄	9.98 849	52
9	9.35 698	9.36 852	0.63 148	9.98 846	51
10	9.35 752	9.36 909	0.63 091	9.98 843	50
11	9.35 806	9.36 966	0.63 034	9.98 840	49
12	9.35 860	9.37 023	0.62 977	9.98 837	48
13	9.35 914	9.37 080	0.62 920	9.98 834	47
14	9.35 968	9.37 137	0.62 863	9.98 831	46
15	9.36 022	9.37 193	0.62 807	9.98 828	45
16	9.36 075	9.37 25̄0	0.62 750	9.98 825	44
17	9.36 129	9.37 306	0.62 694	9.98 822	43
18	9.36 182	9.37 363	0.62 637	9.98 819	42
19	9.36 236	9.37 419	0.62 581	9.98 816	41
20	9.36 289	9.37 476	0.62 524	9.98 813	40
21	9.36 342	9.37 532	0.62 468	9.98 810	39
22	9.36 395	9.37 588	0.62 412	9.98 807	38
23	9.36 449	9.37 644	0.62 356	9.98 804	37
24	9.36 502	9.37 700	0.62 300	9.98 801	36
25	9.36 555̄	9.37 756	0.62 244	9.98 798	35
26	9.36 608	9.37 812	0.62 188	9.98 795	34
27	9.36 660	9.37 868	0.62 132	9.98 792	33
28	9.36 713	9.37 924	0.62 076	9.98 789	32
29	9.36 766	9.37 980	0.62 020	9.98 786	31
30	9.36 819	9.38 035	0.61 965̄	9.98 783	30
31	9.36 871	9.38 091	0.61 909	9.98 780	29
32	9.36 924	9.38 147	0.61 853	9.98 777	28
33	9.36 976	9.38 202	0.61 798	9.98 774	27
34	9.37 028	9.38 257	0.61 743	9.98 771	26
35	9.37 081	9.38 313	0.61 687	9.98 768	25
36	9.37 133	9.38 368	0.61 632	9.98 765̄	24
37	9.37 185	9.38 423	0.61 577	9.98 762	23
38	9.37 237	9.38 479	0.61 521	9.98 759	22
39	9.37 289	9.38 534	0.61 466	9.98 756	21
40	9.37 341	9.38 589	0.61 411	9.98 753	20
41	9.37 393	9.38 644	0.61 356	9.98 75̄0	19
42	9.37 445	9.38 699	0.61 301	9.98 746	18
43	9.37 497	9.38 754	0.61 246	9.98 743	17
44	9.37 549	9.38 808	0.61 192	9.98 740	16
45	9.37 600	9.38 863	0.61 137	9.98 737	15
46	9.37 652	9.38 918	0.61 082	9.98 734	14
47	9.37 703	9.38 972	0.61 028	9.98 731	13
48	9.37 755̄	9.39 027	0.60 973	9.98 728	12
49	9.37 806	9.39 082	0.60 918	9.98 725̄	11
50	9.37 858	9.39 136	0.60 864	9.98 722	10
51	9.37 909	9.39 190	0.60 810	9.98 719	9
52	9.37 960	9.39 245̄	0.60 755	9.98 715	8
53	9.38 011	9.39 299	0.60 701	9.98 712	7
54	9.38 062	9.39 353	0.60 647	9.98 709	6
55	9.38 113	9.39 407	0.60 593	9.98 706	5
56	9.38 164	9.39 461	0.60 539	9.98 703	4
57	9.38 215	9.39 515	0.60 485̄	9.98 700	3
58	9.38 266	9.39 569	0.60 431	9.98 697	2
59	9.38 317	9.39 623	0.60 377	9.98 694	1
60	9.38 368	9.39 677	0.60 323	9.98 690	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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14°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.38 368	9.39 677	0.60 323	9.98 690	**60**
1	9.38 418	9.39 731	0.60 269	9.98 687	59
2	9.38 469	9.39 785̄	0.60 215	9.98 684	58
3	9.38 519	9.39 838	0.60 162	9.98 681	57
4	9.38 570	9.39 892	0.60 108	9.98 678	56
5	9.38 620	9.39 945	0.60 055̄	9.98 675̄	55
6	9.38 670	9.39 999	0.60 001	9.98 671	54
7	9.38 721	9.40 052	0.59 948	9.98 668	53
8	9.38 771	9.40 106	0.59 894	9.98 665	52
9	9.38 821	9.40 159	0.59 841	9.98 662	51
10	9.38 871	9.40 212	0.59 788	9.98 659	**50**
11	9.38 921	9.40 266	0.59 734	9.98 656	49
12	9.38 971	9.40 319	0.59 681	9.98 652	48
13	9.39 021	9.40 372	0.59 628	9.98 649	47
14	9.39 071	9.40 425̄	0.59 575	9.98 646	46
15	9.39 121	9.40 478	0.59 522	9.98 643	45
16	9.39 170	9.40 531	0.59 469	9.98 640	44
17	9.39 220	9.40 584	0.59 416	9.98 636	43
18	9.39 270	9.40 636	0.59 364	9.98 633	42
19	9.39 319	9.40 689	0.59 311	9.98 630	41
20	9.39 369	9.40 742	0.59 258	9.98 627	**40**
21	9.39 418	9.40 795̄	0.59 205	9.98 623	39
22	9.39 467	9.40 847	0.59 153	9.98 620	38
23	9.39 517	9.40 900	0.59 100	9.98 617	37
24	9.39 566	9.40 952	0.59 048	9.98 614	36
25	9.39 615̄	9.41 005̄	0.58 995	9.98 610	35
26	9.39 664	9.41 057	0.58 943	9.98 607	34
27	9.39 713	9.41 109	0.58 891	9.98 604	33
28	9.39 762	9.41 161	0.58 839	9.98 601	32
29	9.39 811	9.41 214	0.58 786	9.98 597	31
30	9.39 860	9.41 266	0.58 734	9.98 594	**30**
31	9.39 909	9.41 318	0.58 682	9.98 591	29
32	9.39 958	9.41 370	0.58 630	9.98 588	28
33	9.40 006	9.41 422	0.58 578	9.98 584	27
34	9.40 055̄	9.41 474	0.58 526	9.98 581	26
35	9.40 103	9.41 526	0.58 474	9.98 578	25
36	9.40 152	9.41 578	0.58 422	9.98 574	24
37	9.40 200	9.41 629	0.58 371	9.98 571	23
38	9.40 249	9.41 681	0.58 319	9.98 568	22
39	9.40 297	9.41 733	0.58 267	9.98 565̄	21
40	9.40 346	9.41 784	0.58 216	9.98 561	**20**
41	9.40 394	9.41 836	0.58 164	9.98 558	19
42	9.40 442	9.41 887	0.58 113	9.98 555̄	18
43	9.40 490	9.41 939	0.58 061	9.98 551	17
44	9.40 538	9.41 990	0.58 010	9.98 548	16
45	9.40 586	9.42 041	0.57 959	9.98 545̄	15
46	9.40 634	9.42 093	0.57 907	9.98 541	14
47	9.40 682	9.42 144	0.57 856	9.98 538	13
48	9.40 730	9.42 195	0.57 805̄	9.98 535̄	12
49	9.40 778	9.42 246	0.57 754	9.98 531	11
50	9.40 825	9.42 297	0.57 703	9.98 528	**10**
51	9.40 873	9.42 348	0.57 652	9.98 525̄	9
52	9.40 921	9.42 399	0.57 601	9.98 521	8
53	9.40 968	9.42 450	0.57 5̄50	9.98 518	7
54	9.41 016	9.42 501	0.57 499	9.98 515̄	6
55	9.41 063	9.42 552	0.57 448	9.98 511	5
56	9.41 111	9.42 603	0.57 397	9.98 508	4
57	9.41 158	9.42 653	0.57 347	9.98 505̄	3
58	9.41 205	9.42 704	0.57 296	9.98 501	2
59	9.41 252	9.42 755̄	0.57 245	9.98 498	1
60	9.41 300	9.42 805	0.57 195̄	9.98 494	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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15°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.41 300	9.42 805	0.57 195	9.98 494	**60**
1	9.41 347	9.42 856	0.57 144	9.98 491	59
2	9.41 394	9.42 906	0.57 094	9.98 488	58
3	9.41 441	9.42 957	0.57 043	9.98 484	57
4	9.41 488	9.43 007	0.56 993	9.98 481	56
5	9.41 535	9.43 057	0.56 943	9.98 477	55
6	9.41 582	9.43 108	0.56 892	9.98 474	54
7	9.41 628	9.43 158	0.56 842	9.98 471	53
8	9.41 675	9.43 208	0.56 792	9.98 467	52
9	9.41 722	9.43 258	0.56 742	9.98 464	51
10	9.41 768	9.43 308	0.56 692	9.98 460	**50**
11	9.41 815	9.43 358	0.56 642	9.98 457	49
12	9.41 861	9.43 408	0.56 592	9.98 453	48
13	9.41 908	9.43 458	0.56 542	9.98 450	47
14	9.41 954	9.43 508	0.56 492	9.98 447	46
15	9.42 001	9.43 558	0.56 442	9.98 443	45
16	9.42 047	9.43 607	0.56 393	9.98 440	44
17	9.42 093	9.43 657	0.56 343	9.98 436	43
18	9.42 140	9.43 707	0.56 293	9.98 433	42
19	9.42 186	9.43 756	0.56 244	9.98 429	41
20	9.42 232	9.43 806	0.56 194	9.98 426	**40**
21	9.42 278	9.43 855	0.56 145	9.98 422	39
22	9.42 324	9.43 905	0.56 095	9.98 419	38
23	9.42 370	9.43 954	0.56 046	9.98 415	37
24	9.42 416	9.44 004	0.55 996	9.98 412	36
25	9.42 461	9.44 053	0.55 947	9.98 409	35
26	9.42 507	9.44 102	0.55 898	9.98 405	34
27	9.42 553	9.44 151	0.55 849	9.98 402	33
28	9.42 599	9.44 201	0.55 799	9.98 398	32
29	9.42 644	9.44 250	0.55 750	9.98 395	31
30	9.42 690	9.44 299	0.55 701	9.98 391	**30**
31	9.42 735	9.44 348	0.55 652	9.98 388	29
32	9.42 781	9.44 397	0.55 603	9.98 384	28
33	9.42 826	9.44 446	0.55 554	9.98 381	27
34	9.42 872	9.44 495	0.55 505	9.98 377	26
35	9.42 917	9.44 544	0.55 456	9.98 373	25
36	9.42 962	9.44 592	0.55 408	9.98 370	24
37	9.43 008	9.44 641	0.55 359	9.98 366	23
38	9.43 053	9.44 690	0.55 310	9.98 363	22
39	9.43 098	9.44 738	0.55 262	9.98 359	21
40	9.43 143	9.44 787	0.55 213	9.98 356	**20**
41	9.43 188	9.44 836	0.55 164	9.98 352	19
42	9.43 233	9.44 884	0.55 116	9.98 349	18
43	9.43 278	9.44 933	0.55 067	9.98 345	17
44	9.43 323	9.44 981	0.55 019	9.98 342	16
45	9.43 367	9.45 029	0.54 971	9.98 338	15
46	9.43 412	9.45 078	0.54 922	9.98 334	14
47	9.43 457	9.45 126	0.54 874	9.98 331	13
48	9.43 502	9.45 174	0.54 826	9.98 327	12
49	9.43 546	9.45 222	0.54 778	9.98 324	11
50	9.43 591	9.45 271	0.54 729	9.98 320	**10**
51	9.43 635	9.45 319	0.54 681	9.98 317	9
52	9.43 680	9.45 367	0.54 633	9.98 313	8
53	9.43 724	9.45 415	0.54 585	9.98 309	7
54	9.43 769	9.45 463	0.54 537	9.98 306	6
55	9.43 813	9.45 511	0.54 489	9.98 302	5
56	9.43 857	9.45 559	0.54 441	9.98 299	4
57	9.43 901	9.45 606	0.54 394	9.98 295	3
58	9.43 946	9.45 654	0.54 346	9.98 291	2
59	9.43 990	9.45 702	0.54 298	9.98 288	1
60	9.44 034	9.45 750	0.54 250	9.98 284	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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16°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.44 034	9.45 75̄0	0.54 250	9.98 284	**60**
1	9.44 078	9.45 797	0.54 203	9.98 281	59
2	9.44 122	9.45 845̄	0.54 155	9.98 277	58
3	9.44 166	9.45 892	0.54 108	9.98 273	57
4	9.44 210	9.45 940	0.54 060	9.98 270	56
5	9.44 253	9.45 987	0.54 013	9.98 266	55
6	9.44 297	9.46 035̄	0.53 965	9.98 262	54
7	9.44 341	9.46 082	0.53 918	9.98 259	53
8	9.44 385̄	9.46 130	0.53 870	9.98 255	52
9	9.44 428	9.46 177	0.53 823	9.98 251	51
10	9.44 472	9.46 224	0.53 776	9.98 248	**50**
11	9.44 516	9.46 271	0.53 729	9.98 244	49
12	9.44 559	9.46 319	0.53 681	9.98 240	48
13	9.44 602	9.46 366	0.53 634	9.98 237	47
14	9.44 646	9.46 413	0.53 587	9.98 233	46
15	9.44 689	9.46 460	0.53 540	9.98 229	45
16	9.44 733	9.46 507	0.53 493	9.98 226	44
17	9.44 776	9.46 554	0.53 446	9.98 222	43
18	9.44 819	9.46 601	0.53 399	9.98 218	42
19	9.44 862	9.46 648	0.53 352	9.98 215̄	41
20	9.44 905	9.46 694	0.53 306	9.98 211	**40**
21	9.44 948	9.46 741	0.53 259	9.98 207	39
22	9.44 992	9.46 788	0.53 212	9.98 204	38
23	9.45 035̄	9.46 835̄	0.53 165	9.98 200	37
24	9.45 077	9.46 881	0.53 119	9.98 196	36
25	9.45 120	9.46 928	0.53 072	9.98 192	35
26	9.45 163	9.46 975̄	0.53 025	9.98 189	34
27	9.45 206	9.47 021	0.52 979	9.98 185̄	33
28	9.45 249	9.47 068	0.52 932	9.98 181	32
29	9.45 292	9.47 114	0.52 886	9.98 177	31
30	9.45 334	9.47 160	0.52 840	9.98 174	**30**
31	9.45 377	9.47 207	0.52 793	9.98 170	29
32	9.45 419	9.47 253	0.52 747	9.98 166	28
33	9.45 462	9.47 299	0.52 701	9.98 162	27
34	9.45 504	9.47 346	0.52 654	9.98 159	26
35	9.45 547	9.47 392	0.52 608	9.98 155̄	25
36	9.45 589	9.47 438	0.52 562	9.98 151	24
37	9.45 632	9.47 484	0.52 516	9.98 147	23
38	9.45 674	9.47 530	0.52 470	9.98 144	22
39	9.45 716	9.47 576	0.52 424	9.98 140	21
40	9.45 758	9.47 622	0.52 378	9.98 136	**20**
41	9.45 801	9.47 668	0.52 332	9.98 132	19
42	9.45 843	9.47 714	0.52 286	9.98 129	18
43	9.45 885̄	9.47 760	0.52 240	9.98 125̄	17
44	9.45 927	9.47 806	0.52 194	9.98 121	16
45	9.45 969	9.47 852	0.52 148	9.98 117	15
46	9.46 011	9.47 897	0.52 103	9.98 113	14
47	9.46 053	9.47 943	0.52 057	9.98 110	13
48	9.46 095̄	9.47 989	0.52 011	9.98 106	12
49	9.46 136	9.48 035̄	0.51 965	9.98 102	11
50	9.46 178	9.48 080	0.51 920	9.98 098	**10**
51	9.46 220	9.48 126	0.51 374	9.98 094	9
52	9.46 262	9.48 171	0.51 829	9.98 090	8
53	9.46 303	9.48 217	0.51 783	9.98 087	7
54	9.46 345̄	9.48 262	0.51 738	9.98 083	6
55	9.46 386	9.48 307	0.51 693	9.98 079	5
56	9.46 428	9.48 353	0.51 647	9.98 075	4
57	9.46 469	9.48 398	0.51 602	9.98 071	3
58	9.46 511	9.48 443	0.51 557	9.98 067	2
59	9.46 552	9.48 489	0.51 511	9.98 063	1
60	9.46 594	9.48 534	0.51 466	9.98 060	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

73°

17°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.46 594	9.48 534	0.51 466	9.98 060	**60**
1	9.46 63$\bar{5}$	9.48 579	0.51 421	9.98 056	59
2	9.46 676	9.48 624	0.51 376	9.98 052	58
3	9.46 717	9.48 669	0.51 331	9.98 048	57
4	9.46 758	9.48 714	0.51 286	9.98 044	56
5	9.46 800	9.48 759	0.51 241	9.98 040	55
6	9.46 841	9.48 804	0.51 196	9.98 036	54
7	9.46 882	9.48 849	0.51 151	9.98 032	53
8	9.46 923	9.48 894	0.51 106	9.98 029	52
9	9.46 964	9.48 939	0.51 061	9.98 02$\bar{5}$	51
10	9.47 00$\bar{5}$	9.48 984	0.51 016	9.98 021	**50**
11	9.47 045	9.49 029	0.50 971	9.98 017	49
12	9.47 086	9.49 073	0.50 927	9.98 013	48
13	9.47 127	9.49 118	0.50 882	9.98 009	47
14	9.47 168	9.49 163	0.50 837	9.98 005	46
15	9.47 209	9.49 207	0.50 793	9.98 001	45
16	9.47 249	9.49 252	0.50 748	9.97 997	44
17	9.47 290	9.49 296	0.50 704	9.97 993	43
18	9.47 330	9.49 341	0.50 659	9.97 989	42
19	9.47 371	9.49 385	0.50 61$\bar{5}$	9.97 986	41
20	9.47 411	9.49 430	0.50 570	9.97 982	**40**
21	9.47 452	9.49 474	0.50 526	9.97 978	39
22	9.47 492	9.49 519	0.50 481	9.97 974	38
23	9.47 533	9.49 563	0.50 437	9.97 970	37
24	9.47 573	9.49 607	0.50 393	9.97 966	36
25	9.47 613	9.49 652	0.50 348	9.97 962	35
26	9.47 654	9.49 696	0.50 304	9.97 958	34
27	9.47 694	9.49 740	0.50 260	9.97 954	33
28	9.47 734	9.49 784	0.50 216	9.97 9$\bar{5}$0	32
29	9.47 774	9.49 828	0.50 172	9.97 946	31
30	9.47 814	9.49 872	0.50 128	9.97 942	**30**
31	9.47 854	9.49 916	0.50 084	9.97 938	29
32	9.47 894	9.49 960	0.50 040	9.97 934	28
33	9.47 934	9.50 04	0.49 996	9.97 930	27
34	9.47 974	9.50 048	0.49 952	9.97 926	26
35	9.48 014	9.50 092	0.49 908	9.97 922	25
36	9.48 054	9.50 136	0.49 864	9.97 918	24
37	9.48 094	9.50 180	0.49 820	9.97 914	23
38	9.48 133	9.50 223	0.49 777	9.97 910	22
39	9.48 173	9.50 267	0.49 733	9.97 906	21
40	9.48 213	9.50 311	0.49 689	9.97 902	**20**
41	9.48 252	9.50 35$\bar{5}$	0.49 645	9.97 898	19
42	9.48 292	9.50 398	0.49 602	9.97 894	18
43	9.48 332	9.50 442	0.49 558	9.97 890	17
44	9.48 371	9.50 485	0.49 51$\bar{5}$	9.97 886	16
45	9.48 411	9.50 529	0.49 471	9.97 882	15
46	9.48 450	9.50 572	0.49 428	9.97 878	14
47	9.48 490	9.50 616	0.49 384	9.97 874	13
48	9.48 529	9.50 659	0.49 341	9.97 870	12
49	9.48 568	9.50 703	0.49 297	9.97 866	11
50	9.48 607	9.50 746	0.49 254	9.97 861	**10**
51	9.48 647	9.50 789	0.49 211	9.97 857	9
52	9.48 686	9.50 833	0.49 167	9.97 853	8
53	9.48 725	9.50 876	0.49 124	9.97 849	7
54	9.48 764	9.50 919	0.49 081	9.97 845	6
55	9.48 803	9.50 962	0.49 038	9.97 841	5
56	9.48 842	9.51 005	0.48 99$\bar{5}$	9.97 837	4
57	9.48 881	9.51 048	0.48 952	9.97 833	3
58	9.48 920	9.51 092	0.48 908	9.97 829	2
59	9.48 959	9.51 13$\bar{5}$	0.48 865	9.97 82$\bar{5}$	1
60	9.48 998	9.51 178	0.48 822	9.97 821	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

72°

18°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.48 998	9.51 178	0.48 822	9.97 821	**60**
1	9.49 037	9.51 221	0.48 779	9.97 817	59
2	9.49 076	9.51 264	0.48 736	9.97 812	58
3	9.49 115	9.51 306	0.48 694	9.97 808	57
4	9.49 153	9.51 349	0.48 651	9.97 804	56
5	9.49 192	9.51 392	0.48 608	9.97 800	55
6	9.49 231	9.51 435	0.48 565	9.97 796	54
7	9.49 269	9.51 478	0.48 522	9.97 792	53
8	9.49 308	9.51 520	0.48 480	9.97 788	52
9	9.49 347	9.51 563	0.48 437	9.97 784	51
10	9.49 385	9.51 606	0.48 394	9.97 779	**50**
11	9.49 424	9.51 648	0.48 352	9.97 775	49
12	9.49 462	9.51 691	0.48 309	9.97 771	48
13	9.49 500	9.51 734	0.48 266	9.97 767	47
14	9.49 539	9.51 776	0.48 224	9.97 763	46
15	9.49 577	9.51 819	0.48 181	9.97 759	45
16	9.49 615	9.51 861	0.48 139	9.97 754	44
17	9.49 654	9.51 903	0.48 097	9.97 750	43
18	9.49 692	9.51 946	0.48 054	9.97 746	42
19	9.49 730	9.51 988	0.48 012	9.97 742	41
20	9.49 768	9.52 031	0.47 969	9.97 738	**40**
21	9.49 806	9.52 073	0.47 927	9.97 734	39
22	9.49 844	9.52 115	0.47 885	9.97 729	38
23	9.49 882	9.52 157	0.47 843	9.97 725	37
24	9.49 920	9.52 200	0.47 800	9.97 721	36
25	9.49 958	9.52 242	0.47 758	9.97 717	35
26	9.49 996	9.52 284	0.47 716	9.97 713	34
27	9.50 034	9.52 326	0.47 674	9.97 708	33
28	9.50 072	9.52 368	0.47 632	9.97 704	32
29	9.50 110	9.52 410	0.47 590	9.97 700	31
30	9.50 148	9.52 452	0.47 548	9.97 696	**30**
31	9.50 185	9.52 494	0.47 506	9.97 691	29
32	9.50 223	9.52 536	0.47 464	9.97 687	28
33	9.50 261	9.52 578	0.47 422	9.97 683	27
34	9.50 298	9.52 620	0.47 380	9.97 679	26
35	9.50 336	9.52 661	0.47 339	9.97 674	25
36	9.50 374	9.52 703	0.47 297	9.97 670	24
37	9.50 411	9.52 745	0.47 255	9.97 666	23
38	9.50 449	9.52 787	0.47 213	9.97 662	22
39	9.50 486	9.52 829	0.47 171	9.97 657	21
40	9.50 523	9.52 870	0.47 130	9.97 653	**20**
41	9.50 561	9.52 912	0.47 088	9.97 649	19
42	9.50 598	9.52 953	0.47 047	9.97 645	18
43	9.50 635	9.52 995	0.47 005	9.97 640	17
44	9.50 673	9.53 037	0.46 963	9.97 636	16
45	9.50 710	9.53 078	0.46 922	9.97 632	15
46	9.50 747	9.53 120	0.46 880	9.97 628	14
47	9.50 784	9.53 161	0.46 839	9.97 623	13
48	9.50 821	9.53 202	0.46 798	9.97 619	12
49	9.50 858	9.53 244	0.46 756	9.97 615	11
50	9.50 896	9.53 285	0.46 715	9.97 610	**10**
51	9.50 933	9.53 327	0.46 673	9.97 606	9
52	9.50 970	9.53 368	0.46 632	9.97 602	8
53	9.51 007	9.53 409	0.46 591	9.97 597	7
54	9.51 043	9.53 450	0.46 550	9.97 593	6
55	9.51 080	9.53 492	0.46 508	9.97 589	5
56	9.51 117	9.53 533	0.46 467	9.97 584	4
57	9.51 154	9.53 574	0.46 426	9.97 580	3
58	9.51 191	9.53 615	0.46 385	9.97 576	2
59	9.51 227	9.53 656	0.46 344	9.97 571	1
60	9.51 264	9.53 697	0.46 303	9.97 567	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

71°

19°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.51 264	9.53 697	0.46 303	9.97 567	**60**
1	9.51 301	9.53 738	0.46 262	9.97 563	59
2	9.51 338	9.53 779	0.46 221	9.97 558	58
3	9.51 374	9.53 820	0.46 180	9.97 554	57
4	9.51 411	9.53 861	0.46 139	9.97 5̄50	56
5	9.51 447	9.53 902	0.46 098	9.97 545	55
6	9.51 484	9.53 943	0.46 057	9.97 541	54
7	9.51 520	9.53 984	0.46 016	9.97 536	53
8	9.51 557	9.54 025̄	0.45 975	9.97 532	52
9	9.51 593	9.54 065	0.45 935̄	9.97 528	51
10	9.51 629	9.54 106	0.45 894	9.97 523	**50**
11	9.51 666	9.54 147	0.45 853	9.97 519	49
12	9.51 702	9.54 187	0.45 813	9.97 515̄	48
13	9.51 738	9.54 228	0.45 772	9.97 510	47
14	9.51 774	9.54 269	0.45 731	9.97 506	46
15	9.51 811	9.54 309	0.45 691	9.97 501	45
16	9.51 847	9.54 35̄0	0.45 650	9.97 497	44
17	9.51 883	9.54 390	0.45 610	9.97 492	43
18	9.51 919	9.54 431	0.45 569	9.97 488	42
19	9.51 955	9.54 471	0.45 529	9.97 484	41
20	9.51 991	9.54 512	0.45 488	9.97 479	**40**
21	9.52 027	9.54 552	0.45 448	9.97 475̄	39
22	9.52 063	9.54 593	0.45 407	9.97 470	38
23	9.52 099	9.54 633	0.45 367	9.97 466	37
24	9.52 135̄	9.54 673	0.45 327	9.97 461	36
25	9.52 171	9.54 714	0.45 286	9.97 457	35
26	9.52 207	9.54 754	0.45 246	9.97 453	34
27	9.52 242	9.54 794	0.45 206	9.97 448	33
28	9.52 278	9.54 835̄	0.45 165	9.97 444	32
29	9.52 314	9.54 875̄	0.45 125	9.97 439	31
30	9.52 35̄0	9.54 915̄	0.45 085	9.97 435̄	**30**
31	9.52 385	9.54 955	0.45 045̄	9.97 430	29
32	9.52 421	9.54 995	0.45 005̄	9.97 426	28
33	9.52 456	9.55 035	0.44 965̄	9.97 421	27
34	9.52 492	9.55 075	0.44 925̄	9.97 417	26
35	9.52 527	9.55 115	0.44 885̄	9.97 412	25
36	9.52 563	9.55 155	0.44 845̄	9.97 408	24
37	9.52 598	9.55 195	0.44 805̄	9.97 403	23
38	9.52 634	9.55 235	0.44 765̄	9.97 399	22
39	9.52 669	9.55 275	0.44 725̄	9.97 394	21
40	9.52 705̄	9.55 315̄	0.44 685	9.97 390	**20**
41	9.52 740	9.55 355̄	0.44 645	9.97 385	19
42	9.52 775	9.55 395̄	0.44 605	9.97 381	18
43	9.52 811	9.55 434	0.44 566	9.97 376	17
44	9.52 846	9.55 474	0.44 526	9.97 372	16
45	9.52 881	9.55 514	0.44 486	9.97 367	15
46	9.52 916	9.55 554	0.44 446	9.97 363	14
47	9.52 951	9.55 593	0.44 407	9.97 358	13
48	9.52 986	9.55 633	0.44 367	9.97 353	12
49	9.53 021	9.55 673	0.44 327	9.97 349	11
50	9.53 056	9.55 712	0.44 288	9.97 344	**10**
51	9.53 092	9.55 752	0.44 248	9.97 340	9
52	9.53 126	9.55 791	0.44 209	9.97 335	8
53	9.53 161	9.55 831	0.44 169	9.97 331	7
54	9.53 196	9.55 870	0.44 130	9.97 326	6
55	9.53 231	9.55 910	0.44 090	9.97 322	5
56	9.53 266	9.55 949	0.44 051	9.97 317	4
57	9.53 301	9.55 989	0.44 011	9.97 312	3
58	9.53 336	9.56 028	0.43 972	9.97 308	2
59	9.53 370	9.56 067	0.43 933	9.97 303	1
60	9.53 405	9.56 107	0.43 893	9.97 299	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

70°

20°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.53 405	9.56 107	0.43 893	9.97 299	**60**
1	9.53 440	9.56 146	0.43 854	9.97 294	59
2	9.53 475	9.56 185	0.43 815	9.97 289	58
3	9.53 509	9.56 224	0.43 776	9.97 285	57
4	9.53 544	9.56 264	0.43 736	9.97 280	56
5	9.53 578	9.56 303	0.43 697	9.97 276	55
6	9.53 613	9.56 342	0.43 658	9.97 271	54
7	9.53 647	9.56 381	0.43 619	9.97 266	53
8	9.53 682	9.56 420	0.43 580	9.97 262	52
9	9.53 716	9.56 459	0.43 541	9.97 257	51
10	9.53 751	9.56 498	0.43 502	9.97 252	**50**
11	9.53 785	9.56 537	0.43 463	9.97 248	49
12	9.53 819	9.56 576	0.43 424	9.97 243	48
13	9.53 854	9.56 615	0.43 385	9.97 238	47
14	9.53 888	9.56 654	0.43 346	9.97 234	46
15	9.53 922	9.56 693	0.43 307	9.97 229	45
16	9.53 957	9.56 732	0.43 268	9.97 224	44
17	9.53 991	9.56 771	0.43 229	9.97 220	43
18	9.54 025	9.56 810	0.43 190	9.97 215	42
19	9.54 059	9.56 849	0.43 151	9.97 210	41
20	9.54 093	9.56 887	0.43 113	9.97 206	**40**
21	9.54 127	9.56 926	0.43 074	9.97 201	39
22	9.54 161	9.56 965	0.43 035	9.97 196	38
23	9.54 195	9.57 004	0.42 996	9.97 192	37
24	9.54 229	9.57 042	0.42 958	9.97 187	36
25	9.54 263	9.57 081	0.42 919	9.97 182	35
26	9.54 297	9.57 120	0.42 880	9.97 178	34
27	9.54 331	9.57 158	0.42 842	9.97 173	33
28	9.54 365	9.57 197	0.42 803	9.97 168	32
29	9.54 399	9.57 235	0.42 765	9.97 163	31
30	9.54 433	9.57 274	0.42 726	9.97 159	**30**
31	9.54 466	9.57 312	0.42 688	9.97 154	29
32	9.54 500	9.57 351	0.42 649	9.97 149	28
33	9.54 534	9.57 389	0.42 611	9.97 145	27
34	9.54 567	9.57 428	0.42 572	9.97 140	26
35	9.54 601	9.57 466	0.42 534	9.97 135	25
36	9.54 635	9.57 504	0.42 496	9.97 130	24
37	9.54 668	9.57 543	0.42 457	9.97 126	23
38	9.54 702	9.57 581	0.42 419	9.97 121	22
39	9.54 735	9.57 619	0.42 381	9.97 116	21
40	9.54 769	9.57 658	0.42 342	9.97 111	**20**
41	9.54 802	9.57 696	0.42 304	9.97 107	19
42	9.54 836	9.57 734	0.42 266	9.97 102	18
43	9.54 869	9.57 772	0.42 228	9.97 097	17
44	9.54 903	9.57 810	0.42 190	9.97 092	16
45	9.54 936	9.57 849	0.42 151	9.97 087	15
46	9.54 969	9.57 887	0.42 113	9.97 083	14
47	9.55 003	9.57 925	0.42 075	9.97 078	13
48	9.55 036	9.57 963	0.42 037	9.97 073	12
49	9.55 069	9.58 001	0.41 999	9.97 068	11
50	9.55 102	9.58 039	0.41 961	9.97 063	**10**
51	9.55 136	9.58 077	0.41 923	9.97 059	9
52	9.55 169	9.58 115	0.41 885	9.97 054	8
53	9.55 202	9.58 153	0.41 847	9.97 049	7
54	9.55 235	9.58 191	0.41 809	9.97 044	6
55	9.55 268	9.58 229	0.41 771	9.97 039	5
56	9.55 301	9.58 267	0.41 733	9.97 035	4
57	9.55 334	9.58 304	0.41 696	9.97 030	3
58	9.55 367	9.58 342	0.41 658	9.97 025	2
59	9.55 400	9.58 380	0.41 620	9.97 020	1
60	9.55 433	9.58 418	0.41 582	9.97 015	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

69°

21°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.55 433	9.58 418	0.41 582	9.97 015	**60**
1	9.55 466	9.58 455	0.41 545	9.97 010	59
2	9.55 499	9.58 493	0.41 507	9.97 005	58
3	9.55 532	9.58 531	0.41 469	9.97 001	57
4	9.55 564	9.58 569	0.41 431	9.96 996	56
5	9.55 597	9.58 606	0.41 394	9.96 991	55
6	9.55 630	9.58 644	0.41 356	9.96 986	54
7	9.55 663	9.58 681	0.41 319	9.96 981	53
8	9.55 695	9.58 719	0.41 281	9.96 976	52
9	9.55 728	9.58 757	0.41 243	9.96 971	51
10	9.55 761	9.58 794	0.41 206	9.96 966	**50**
11	9.55 793	9.58 832	0.41 168	9.96 962	49
12	9.55 826	9.58 869	0.41 131	9.96 957	48
13	9.55 858	9.58 907	0.41 093	9.96 952	47
14	9.55 891	9.58 944	0.41 056	9.96 947	46
15	9.55 923	9.58 981	0.41 019	9.96 942	45
16	9.55 956	9.59 019	0.40 981	9.96 937	44
17	9.55 988	9.59 056	0.40 944	9.96 932	43
18	9.56 021	9.59 094	0.40 906	9.96 927	42
19	9.56 053	9.59 131	0.40 869	9.96 922	41
20	9.56 085	9.59 168	0.40 832	9.96 917	**40**
21	9.56 118	9.59 205	0.40 795	9.96 912	39
22	9.56 150	9.59 243	0.40 757	9.96 907	38
23	9.56 182	9.59 280	0.40 720	9.96 903	37
24	9.56 215	9.59 317	0.40 683	9.96 898	36
25	9.56 247	9.59 354	0.40 646	9.96 893	35
26	9.56 279	9.59 391	0.40 609	9.96 888	34
27	9.56 311	9.59 429	0.40 571	9.96 883	33
28	9.56 343	9.59 466	0.40 534	9.96 878	32
29	9.56 375	9.59 503	0.40 497	9.96 873	31
30	9.56 408	9.59 540	0.40 460	9.96 868	**30**
31	9.56 440	9.59 577	0.40 423	9.96 863	29
32	9.56 472	9.59 614	0.40 386	9.96 858	28
33	9.56 504	9.59 651	0.40 349	9.96 853	27
34	9.56 536	9.59 688	0.40 312	9.96 848	26
35	9.56 568	9.59 725	0.40 275	9.96 843	25
36	9.56 599	9.59 762	0.40 238	9.96 838	24
37	9.56 631	9.59 799	0.40 201	9.96 833	23
38	9.56 663	9.59 835	0.40 165	9.96 828	22
39	9.56 695	9.59 872	0.40 128	9.96 823	21
40	9.56 727	9.59 909	0.40 091	9.96 818	**20**
41	9.56 759	9.59 946	0.40 054	9.96 813	19
42	9.56 790	9.59 983	0.40 017	9.96 808	18
43	9.56 822	9.60 019	0.39 981	9.96 803	17
44	9.56 854	9.60 056	0.39 944	9.96 798	16
45	9.56 886	9.60 093	0.39 907	9.96 793	15
46	9.56 917	9.60 130	0.39 870	9.96 788	14
47	9.56 949	9.60 166	0.39 834	9.96 783	13
48	9.56 980	9.60 203	0.39 797	9.96 778	12
49	9.57 012	9.60 240	0.39 760	9.96 772	11
50	9.57 044	9.60 276	0.39 724	9.96 767	**10**
51	9.57 075	9.60 313	0.39 687	9.96 762	9
52	9.57 107	9.60 349	0.39 651	9.96 757	8
53	9.57 138	9.60 386	0.39 614	9.96 752	7
54	9.57 169	9.60 422	0.39 578	9.96 747	6
55	9.57 201	9.60 459	0.39 541	9.96 742	5
56	9.57 232	9.60 495	0.39 505	9.96 737	4
57	9.57 264	9.60 532	0.39 468	9.96 732	3
58	9.57 295	9.60 568	0.39 432	9.96 727	2
59	9.57 326	9.60 605	0.39 395	9.96 722	1
60	9.57 358	9.60 641	0.39 359	9.96 717	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

68°

22°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.57 358	9.60 641	0.39 359	9.96 717	**60**
1	9.57 389	9.60 677	0.39 323	9.96 711	59
2	9.57 420	9.60 714	0.39 286	9.96 706	58
3	9.57 451	9.60 75̄0	0.39 250	9.96 701	57
4	9.57 482	9.60 786	0.39 214	9.96 696	56
5	9.57 514	9.60 823	0.39 177	9.96 691	55
6	9.57 545̄	9.60 859	0.39 141	9.96 686	54
7	9.57 576	9.60 895	0.39 105̄	9.96 681	53
8	9.57 607	9.60 931	0.39 069	9.96 676	52
9	9.57 638	9.60 967	0.39 033	9.96 670	51
10	9.57 669	9.61 004	0.38 996	9.96 665	**50**
11	9.57 700	9.61 040	0.38 960	9.96 660	49
12	9.57 731	9.61 076	0.38 924	9.96 655	48
13	9.57 762	9.61 112	0.38 888	9.96 65̄0	47
14	9.57 793	9.61 148	0.38 852	9.96 645̄	46
15	9.57 824	9.61 184	0.38 816	9.96 640	45
16	9.57 855̄	9.61 220	0.38 780	9.96 634	44
17	9.57 885	9.61 256	0.38 744	9.96 629	43
18	9.57 916	9.61 292	0.38 708	9.96 624	42
19	9.57 947	9.61 328	0.38 672	9.96 619	41
20	9.57 978	9.61 364	0.38 636	9.96 614	**40**
21	9.58 008	9.61 400	0.38 600	9.96 608	39
22	9.58 039	9.61 436	0.38 564	9.96 603	38
23	9.58 070	9.61 472	0.38 528	9.96 598	37
24	9.58 101	9.61 508	0.38 492	9.96 593	36
25	9.58 131	9.61 544	0.38 456	9.96 588	35
26	9.58 162	9.61 579	0.38 421	9.96 582	34
27	9.58 192	9.61 615	0.38 385̄	9.96 577	33
28	9.58 223	9.61 651	0.38 349	9.96 572	32
29	9.58 253	9.61 687	0.38 313	9.96 567	31
30	9.58 284	9.61 722	0.38 278	9.96 562	**30**
31	9.58 314	9.61 758	0.38 242	9.96 556	29
32	9.58 345̄	9.61 794	0.38 206	9.96 551	28
33	9.58 375	9.61 830	0.38 170	9.96 546	27
34	9.58 406	9.61 865	0.38 135̄	9.96 541	26
35	9.58 436	9.61 901	0.38 099	9.96 535	25
36	9.58 467	9.61 936	0.38 064	9.96 530	24
37	9.58 497	9.61 972	0.38 028	9.96 525̄	23
38	9.58 527	9.62 008	0.37 992	9.96 520	22
39	9.58 557	9.62 043	0.37 957	9.96 514	21
40	9.58 588	9.62 079	0.37 921	9.96 509	**20**
41	9.58 618	9.62 114	0.37 886	9.96 504	19
42	9.58 648	9.62 15̄0	0.37 850	9.96 498	18
43	9.58 678	9.62 185	0.37 815̄	9.96 493	17
44	9.58 709	9.62 221	0.37 779	9.96 488	16
45	9.58 739	9.62 256	0.37 744	9.96 483	15
46	9.58 769	9.62 292	0.37 708	9.96 477	14
47	9.58 799	9.62 327	0.37 673	9.96 472	13
48	9.58 829	9.62 362	0.37 638	9.96 467	12
49	9.58 859	9.62 398	0.37 602	9.96 461	11
50	9.58 889	9.62 433	0.37 567	9.96 456	**10**
51	9.58 919	9.62 468	0.37 532	9.96 451	9
52	9.58 949	9.62 504	0.37 496	9.96 445	8
53	9.58 979	9.62 539	0.37 461	9.96 440	7
54	9.59 009	9.62 574	0.37 426	9.96 435̄	6
55	9.59 039	9.62 609	0.37 391	9.96 429	5
56	9.59 069	9.62 645̄	0.37 355	9.96 424	4
57	9.59 098	9.62 680	0.37 320	9.96 419	3
58	9.59 128	9.62 715̄	0.37 285	9.96 413	2
59	9.59 158	9.62 750	0.37 25̄0	9.96 408	1
60	9.59 188	9.62 785	0.37 215̄	9.96 403	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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23°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.59 188	9.62 78$\bar{5}$	0.37 21$\bar{5}$	9.96 403	**60**
1	9.59 218	9.62 820	0.37 180	9.96 397	59
2	9.59 247	9.62 855	0.37 14$\bar{5}$	9.96 392	58
3	9.59 277	9.62 890	0.37 110	9.96 387	57
4	9.59 307	9.62 926	0.37 074	9.96 381	56
5	9.59 336	9.62 961	0.37 039	9.96 376	55
6	9.59 366	9.62 996	0.37 004	9.96 370	54
7	9.59 396	9.63 031	0.36 969	9.96 36$\bar{5}$	53
8	9.59 425	9.63 066	0.36 934	9.96 360	52
9	9.59 45$\bar{5}$	9.63 101	0.36 899	9.96 354	51
10	9.59 484	9.63 135	0.36 86$\bar{5}$	9.96 349	**50**
11	9.59 514	9.63 170	0.36 830	9.96 343	49
12	9.59 543	9.63 205	0.36 79$\bar{5}$	9.96 338	48
13	9.59 573	9.63 240	0.36 760	9.96 333	47
14	9.59 602	9.63 275	0.36 72$\bar{5}$	9.96 327	46
15	9.59 632	9.63 310	0.36 690	9.96 322	45
16	9.59 661	9.63 34$\bar{5}$	0.36 655	9.96 316	44
17	9.59 690	9.63 379	0.36 621	9.96 311	43
18	9.59 720	9.63 414	0.36 586	9.96 305	42
19	9.59 749	9.63 449	0.36 551	9.96 300	41
20	9.59 778	9.63 484	0.36 516	9.96 294	**40**
21	9.59 808	9.63 519	0.36 481	9.96 289	39
22	9.59 837	9.63 553	0.36 447	9.96 284	38
23	9.59 866	9.63 588	0.36 412	9.96 278	37
24	9.59 895	9.63 623	0.36 377	9.96 273	36
25	9.59 924	9.63 657	0.36 343	9.96 267	35
26	9.59 954	9.63 692	0.36 308	9.96 262	34
27	9.59 983	9.63 726	0.36 274	9.96 256	33
28	9.60 012	9.63 761	0.36 239	9.96 251	32
29	9.60 041	9.63 796	0.36 204	9.96 245	31
30	9.60 070	9.63 830	0.36 170	9.96 240	**30**
31	9.60 099	9.63 86$\bar{5}$	0.36 135	9.96 234	29
32	9.60 128	9.63 899	0.36 101	9.96 229	28
33	9.60 157	9.63 934	0.36 066	9.96 223	27
34	9.60 186	9.63 968	0.36 032	9.96 218	26
35	9.60 21$\bar{5}$	9.64 003	0.35 997	9.96 212	25
36	9.60 244	9.64 037	0.35 963	9.96 207	24
37	9.60 273	9.64 072	0.35 928	9.96 201	23
38	9.60 302	9.64 106	0.35 894	9.96 196	22
39	9.60 331	9.64 140	0.35 860	9.96 190	21
40	9.60 359	9.64 17$\bar{5}$	0.35 825	9.96 18$\bar{5}$	**20**
41	9.60 388	9.64 209	0.35 791	9.96 179	19
42	9.60 417	9.64 243	0.35 757	9.96 174	18
43	9.60 446	9.64 278	0.35 722	9.96 168	17
44	9.60 474	9.64 312	0.35 688	9.96 162	16
45	9.60 503	9.64 346	0.35 654	9.96 157	15
46	9.60 532	9.64 381	0.35 619	9.96 151	14
47	9.60 561	9.64 41$\bar{5}$	0.35 585	9.96 146	13
48	9.60 589	9.64 449	0.35 551	9.96 140	12
49	9.60 618	9.64 483	0.35 517	9.96 13$\bar{5}$	11
50	9.60 646	9.64 517	0.35 483	9.96 129	**10**
51	9.60 675	9.64 552	0.35 448	9.96 123	9
52	9.60 704	9.64 586	0.35 414	9.96 118	8
53	9.60 732	9.64 620	0.35 380	9.96 112	7
54	9.60 761	9.64 654	0.35 346	9.96 107	6
55	9.60 789	9.64 688	0.35 312	9.96 101	5
56	9.60 818	9.64 722	0.35 278	9.96 095	4
57	9.60 846	9.64 756	0.35 244	9.96 090	3
58	9.60 87$\bar{5}$	9.64 790	0.35 210	9.96 084	2
59	9.60 903	9.64 824	0.35 176	9.96 079	1
60	9.60 931	9.64 858	0.35 142	9.96 073	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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24°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.60 931	9.64 858	0.35 142	9.96 073	**60**
1	9.60 960	9.64 892	0.35 108	9.96 067	59
2	9.60 988	9.64 926	0.35 074	9.96 062	58
3	9.61 016	9.64 960	0.35 040	9.96 056	57
4	9.61 045̄	9.64 994	0.35 006	9.96 050	56
5	9.61 073	9.65 028	0.34 972	9.96 045̄	55
6	9.61 101	9.65 062	0.34 938	9.96 039	54
7	9.61 129	9.65 096	0.34 904	9.96 034	53
8	9.61 158	9.65 130	0.34 870	9.96 028	52
9	9.61 186	9.65 164	0.34 836	9.96 022	51
10	9.61 214	9.65 197	0.34 803	9.96 017	**50**
11	9.61 242	9.65 231	0.34 769	9.96 011	49
12	9.61 270	9.65 265	0.34 735̄	9.96 005	48
13	9.61 298	9.65 299	0.34 701	9.96 000	47
14	9.61 326	9.65 333	0.34 667	9.95 994	46
15	9.61 354	9.65 366	0.34 634	9.95 988	45
16	9.61 382	9.65 400	0.34 600	9.95 982	44
17	9.61 411	9.65 434	0.34 566	9.95 977	43
18	9.61 438	9.65 467	0.34 533	9.95 971	42
19	9.61 466	9.65 501	0.34 499	9.95 965	41
20	9.61 494	9.65 535̄	0.34 465	9.95 960	**40**
21	9.61 522	9.65 568	0.34 432	9.95 954	39
22	9.61 550	9.65 602	0.34 398	9.95 948	38
23	9.61 578	9.65 636	0.34 364	9.95 942	37
24	9.61 606	9.65 669	0.34 331	9.95 937	36
25	9.61 634	9.65 703	0.34 297	9.95 931	35
26	9.61 662	9.65 736	0.34 264	9.95 925	34
27	9.61 689	9.65 770	0.34 230	9.95 920	33
28	9.61 717	9.65 803	0.34 197	9.95 914	32
29	9.61 745̄	9.65 837	0.34 163	9.95 908	31
30	9.61 773	9.65 870	0.34 130	9.95 902	**30**
31	9.61 800	9.65 904	0.34 096	9.95 897	29
32	9.61 828	9.65 937	0.34 063	9.95 891	28
33	9.61 856	9.65 971	0.34 029	9.95 885̄	27
34	9.61 883	9.66 004	0.33 996	9.95 879	26
35	9.61 911	9.66 038	0.33 962	9.95 873	25
36	9.61 939	9.66 071	0.33 929	9.95 868	24
37	9.61 966	9.66 104	0.33 896	9.95 862	23
38	9.61 994	9.66 138	0.33 862	9.95 856	22
39	9.62 021	9.66 171	0.33 829	9.95 850	21
40	9.62 049	9.66 204	0.33 796	9.95 844	**20**
41	9.62 076	9.66 238	0.33 762	9.95 839	19
42	9.62 104	9.66 271	0.33 729	9.95 833	18
43	9.62 131	9.66 304	0.33 696	9.95 827	17
44	9.62 159	9.66 337	0.33 663	9.95 821	16
45	9.62 186	9.66 371	0.33 629	9.95 815	15
46	9.62 214	9.66 404	0.33 596	9.95 810	14
47	9.62 241	9.66 437	0.33 563	9.95 804	13
48	9.62 268	9.66 470	0.33 530	9.95 798	12
49	9.62 296	9.66 503	0.33 497	9.95 792	11
50	9.62 323	9.66 537	0.33 463	9.95 786	**10**
51	9.62 350	9.66 570	0.33 430	9.95 780	9
52	9.62 377	9.66 603	0.33 397	9.95 775̄	8
53	9.62 405̄	9.66 636	0.33 364	9.95 769	7
54	9.62 432	9.66 669	0.33 331	9.95 763	6
55	9.62 459	9.66 702	0.33 298	9.95 757	5
56	9.62 486	9.66 735	0.33 265̄	9.95 751	4
57	9.62 513	9.66 768	0.33 232	9.95 745	3
58	9.62 541	9.66 801	0.33 199	9.95 739	2
59	9.62 568	9.66 834	0.33 166	9.95 733	1
60	9.62 595̄	9.66 867	0.33 133	9.95 728	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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25°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.62 595	9.66 867	0.33 133	9.95 728	**60**
1	9.62 622	9.66 900	0.33 100	9.95 722	59
2	9.62 649	9.66 933	0.33 067	9.95 716	58
3	9.62 676	9.66 966	0.33 034	9.95 710	57
4	9.62 703	9.66 999	0.33 001	9.95 704	56
5	9.62 730	9.67 032	0.32 968	9.95 698	55
6	9.62 757	9.67 065	0.32 935	9.95 692	54
7	9.62 784	9.67 098	0.32 902	9.95 686	53
8	9.62 811	9.67 131	0.32 869	9.95 680	52
9	9.62 838	9.67 163	0.32 837	9.95 674	51
10	9.62 865	9.67 196	0.32 804	9.95 668	**50**
11	9.62 892	9.67 229	0.32 771	9.95 663	49
12	9.62 918	9.67 262	0.32 738	9.95 657	48
13	9.62 945	9.67 295	0.32 705	9.95 651	47
14	9.62 972	9.67 327	0.32 673	9.95 645	46
15	9.62 999	9.67 360	0.32 640	9.95 639	45
16	9.63 026	9.67 393	0.32 607	9.95 633	44
17	9.63 052	9.67 426	0.32 574	9.95 627	43
18	9.63 079	9.67 458	0.32 542	9.95 621	42
19	9.63 106	9.67 491	0.32 509	9.95 615	41
20	9.63 133	9.67 524	0.32 476	9.95 609	**40**
21	9.63 159	9.67 556	0.32 444	9.95 603	39
22	9.63 186	9.67 589	0.32 411	9.95 597	38
23	9.63 213	9.67 622	0.32 378	9.95 591	37
24	9.63 239	9.67 654	0.32 346	9.95 585	36
25	9.63 266	9.67 687	0.32 313	9.95 579	35
26	9.63 292	9.67 719	0.32 281	9.95 573	34
27	9.63 319	9.67 752	0.32 248	9.95 567	33
28	9.63 345	9.67 785	0.32 215	9.95 561	32
29	9.63 372	9.67 817	0.32 183	9.95 555	31
30	9.63 398	9.67 850	0.32 150	9.95 549	**30**
31	9.63 425	9.67 882	0.32 118	9.95 543	29
32	9.63 451	9.67 915	0.32 085	9.95 537	28
33	9.63 478	9.67 947	0.32 053	9.95 531	27
34	9.63 504	9.67 980	0.32 020	9.95 525	26
35	9.63 531	9.68 012	0.31 988	9.95 519	25
36	9.63 557	9.68 044	0.31 956	9.95 513	24
37	9.63 583	9.68 077	0.31 923	9.95 507	23
38	9.63 610	9.68 109	0.31 891	9.95 500	22
39	9.63 636	9.68 142	0.31 858	9.95 494	21
40	9.63 662	9.68 174	0.31 826	9.95 488	**20**
41	9.63 689	9.68 206	0.31 794	9.95 482	19
42	9.63 715	9.68 239	0.31 761	9.95 476	18
43	9.63 741	9.68 271	0.31 729	9.95 470	17
44	9.63 767	9.68 303	0.31 697	9.95 464	16
45	9.63 794	9.68 336	0.31 664	9.95 458	15
46	9.63 820	9.68 368	0.31 632	9.95 452	14
47	9.63 846	9.68 400	0.31 600	9.95 446	13
48	9.63 872	9.68 432	0.31 568	9.95 440	12
49	9.63 898	9.68 465	0.31 535	9.95 434	11
50	9.63 924	9.68 497	0.31 503	9.95 427	**10**
51	9.63 950	9.68 529	0.31 471	9.95 421	9
52	9.63 976	9.68 561	0.31 439	9.95 415	8
53	9.64 002	9.68 593	0.31 407	9.95 409	7
54	9.64 028	9.68 626	0.31 374	9.95 403	6
55	9.64 054	9.68 658	0.31 342	9.95 397	5
56	9.64 080	9.68 690	0.31 310	9.95 391	4
57	9.64 106	9.68 722	0.31 278	9.95 384	3
58	9.64 132	9.68 754	0.31 246	9.95 378	2
59	9.64 158	9.68 786	0.31 214	9.95 372	1
60	9.64 184	9.68 818	0.31 182	9.95 366	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

64°

26°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.64 184	9.68 818	0.31 182	9.95 366	**60**
1	9.64 210	9.68 850	0.31 15̄0	9.95 360	59
2	9.64 236	9.68 882	0.31 118	9.95 354	58
3	9.64 262	9.68 914	0.31 086	9.95 348	57
4	9.64 288	9.68 946	0.31 054	9.95 341	56
5	9.64 313	9.68 978	0.31 022	9.95 335	55
6	9.64 339	9.69 010	0.30 990	9.95 329	54
7	9.64 365	9.69 042	0.30 958	9.95 323	53
8	9.64 391	9.69 074	0.30 926	9.95 317	52
9	9.64 417	9.69 106	0.30 894	9.95 310	51
10	9.64 442	9.69 138	0.30 862	9.95 304	**50**
11	9.64 468	9.69 170	0.30 830	9.95 298	49
12	9.64 494	9.69 202	0.30 798	9.95 292	48
13	9.64 519	9.69 234	0.30 766	9.95 286	47
14	9.64 545̄	9.69 266	0.30 734	9.95 279	46
15	9.64 571	9.69 298	0.30 702	9.95 273	45
16	9.64 596	9.69 329	0.30 671	9.95 267	44
17	9.64 622	9.69 361	0.30 639	9.95 261	43
18	9.64 647	9.69 393	0.30 607	9.95 254	42
19	9.64 673	9.69 425̄	0.30 575	9.95 248	41
20	9.64 698	9.69 457	0.30 543	9.95 242	**40**
21	9.64 724	9.69 488	0.30 512	9.95 236	39
22	9.64 749	9.69 520	0.30 480	9.95 229	38
23	9.64 775̄	9.69 552	0.30 448	9.95 223	37
24	9.64 800	9.69 584	0.30 416	9.95 217	36
25	9.64 826	9.69 615	0.30 385̄	9.95 211	35
26	9.64 851	9.69 647	0.30 353	9.95 204	34
27	9.64 877	9.69 679	0.30 321	9.95 198	33
28	9.64 902	9.69 710	0.30 290	9.95 192	32
29	9.64 927	9.69 742	0.30 258	9.95 185	31
30	9.64 953	9.69 774	0.30 226	9.95 179	**30**
31	9.64 978	9.69 805	0.30 195̄	9.95 173	29
32	9.65 003	9.69 837	0.30 163	9.95 167	28
33	9.65 029	9.69 868	0.30 132	9.95 160	27
34	9.65 054	9.69 900	0.30 100	9.95 154	26
35	9.65 079	9.69 932	0.30 068	9.95 148	25
36	9.65 104	9.69 963	0.30 037	9.95 141	24
37	9.65 130	9.69 995̄	0.30 005	9.95 135̄	23
38	9.65 155̄	9.70 026	0.29 974	9.95 129	22
39	9.65 180	9.70 058	0.29 942	9.95 122	21
40	9.65 205	9.70 089	0.29 911	9.95 116	**20**
41	9.65 230	9.70 121	0.29 879	9.95 110	19
42	9.65 255	9.70 152	0.29 848	9.95 103	18
43	9.65 281	9.70 184	0.29 816	9.95 097	17
44	9.65 306	9.70 215	0.29 785̄	9.95 090	16
45	9.65 331	9.70 247	0.29 753	9.95 084	15
46	9.65 356	9.70 278	0.29 722	9.95 078	14
47	9.65 381	9.70 309	0.29 691	9.95 071	13
48	9.65 406	9.70 341	0.29 659	9.95 065̄	12
49	9.65 431	9.70 372	0.29 628	9.95 059	11
50	9.65 456	9.70 404	0.29 596	9.95 052	**10**
51	9.65 481	9.70 435̄	0.29 565	9.95 046	9
52	9.65 506	9.70 466	0.29 534	9.95 039	8
53	9.65 531	9.70 498	0.29 502	9.95 033	7
54	9.65 556	9.70 529	0.29 471	9.95 027	6
55	9.65 580	9.70 560	0.29 440	9.95 020	5
56	9.65 605	9.70 592	0.29 408	9.95 014	4
57	9.65 630	9.70 623	0.29 377	9.95 007	3
58	9.65 655	9.70 654	0.29 346	9.95 001	2
59	9.65 680	9.70 685	0.29 315̄	9.94 995̄	1
60	9.65 705̄	9.70 717	0.29 283	9.94 988	**0**
	L. Cos.	**L. Cot.**	**L. Tan.**	**L. Sin.**	**′**

63°

27°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.65 705	9.70 717	0.29 283	9.94 988	**60**
1	9.65 729	9.70 748	0.29 252	9.94 982	59
2	9.65 754	9.70 779	0.29 221	9.94 975	58
3	9.65 779	9.70 810	0.29 190	9.94 969	57
4	9.65 804	9.70 841	0.29 159	9.94 962	56
5	9.65 828	9.70 873	0.29 127	9.94 956	55
6	9.65 853	9.70 904	0.29 096	9.94 949	54
7	9.65 878	9.70 935	0.29 065	9.94 943	53
8	9.65 902	9.70 966	0.29 034	9.94 936	52
9	9.65 927	9.70 997	0.29 003	9.94 930	51
10	9.65 952	9.71 028	0.28 972	9.94 923	**50**
11	9.65 976	9.71 059	0.28 941	9.94 917	49
12	9.66 001	9.71 090	0.28 910	9.94 911	48
13	9.66 025	9.71 121	0.28 879	9.94 904	47
14	9.66 050	9.71 153	0.28 847	9.94 898	46
15	9.66 075	9.71 184	0.28 816	9.94 891	45
16	9.66 099	9.71 215	0.28 785	9.94 885	44
17	9.66 124	9.71 246	0.28 754	9.94 878	43
18	9.66 148	9.71 277	0.28 723	9.94 871	42
19	9.66 173	9.71 308	0.28 692	9.94 865	41
20	9.66 197	9.71 339	0.28 661	9.94 858	**40**
21	9.66 221	9.71 370	0.28 630	9.94 852	39
22	9.66 246	9.71 401	0.28 599	9.94 845	38
23	9.66 270	9.71 431	0.28 569	9.94 839	37
24	9.66 295	9.71 462	0.28 538	9.94 832	36
25	9.66 319	9.71 493	0.28 507	9.94 826	35
26	9.66 343	9.71 524	0.28 476	9.94 819	34
27	9.66 368	9.71 555	0.28 445	9.94 813	33
28	9.66 392	9.71 586	0.28 414	9.94 806	32
29	9.66 416	9.71 617	0.28 383	9.94 799	31
30	9.66 441	9.71 648	0.28 352	9.94 793	**30**
31	9.66 465	9.71 679	0.28 321	9.94 786	29
32	9.66 489	9.71 709	0.28 291	9.94 780	28
33	9.66 513	9.71 740	0.28 260	9.94 773	27
34	9.66 537	9.71 771	0.28 229	9.94 767	26
35	9.66 562	9.71 802	0.28 198	9.94 760	25
36	9.66 586	9.71 833	0.28 167	9.94 753	24
37	9.66 610	9.71 863	0.28 137	9.94 747	23
38	9.66 634	9.71 894	0.28 106	9.94 740	22
39	9.66 658	9.71 925	0.28 075	9.94 734	21
40	9.66 682	9.71 955	0.28 045	9.94 727	**20**
41	9.66 706	9.71 986	0.28 014	9.94 720	19
42	9.66 731	9.72 017	0.27 983	9.94 714	18
43	9.66 755	9.72 048	0.27 952	9.94 707	17
44	9.66 779	9.72 078	0.27 922	9.94 700	16
45	9.66 803	9.72 109	0.27 891	9.94 694	15
46	9.66 827	9.72 140	0.27 860	9.94 687	14
47	9.66 851	9.72 170	0.27 830	9.94 680	13
48	9.66 875	9.72 201	0.27 799	9.94 674	12
49	9.66 899	9.72 231	0.27 769	9.94 667	11
50	9.66 922	9.72 262	0.27 738	9.94 660	**10**
51	9.66 946	9.72 293	0.27 707	9.94 654	9
52	9.66 970	9.72 323	0.27 677	9.94 647	8
53	9.66 994	9.72 354	0.27 646	9.94 640	7
54	9.67 018	9.72 384	0.27 616	9.94 634	6
55	9.67 042	9.72 415	0.27 585	9.94 627	5
56	9.67 066	9.72 445	0.27 555	9.94 620	4
57	9.67 090	9.72 476	0.27 524	9.94 614	3
58	9.67 113	9.72 506	0.27 494	9.94 607	2
59	9.67 137	9.72 537	0.27 463	9.94 600	1
60	9.67 161	9.72 567	0.27 433	9.94 593	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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28°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.67 161	9.72 567	0.27 433	9.94 593	**60**
1	9.67 185	9.72 598	0.27 402	9.94 587	59
2	9.67 208	9.72 628	0.27 372	9.94 580	58
3	9.67 232	9.72 659	0.27 341	9.94 573	57
4	9.67 256	9.72 689	0.27 311	9.94 567	56
5	9.67 280	9.72 720	0.27 280	9.94 560	55
6	9.67 303	9.72 750	0.27 250	9.94 553	54
7	9.67 327	9.72 780	0.27 220	9.94 546	53
8	9.67 350	9.72 811	0.27 189	9.94 540	52
9	9.67 374	9.72 841	0.27 159	9.94 533	51
10	9.67 398	9.72 872	0.27 128	9.94 526	**50**
11	9.67 421	9.72 902	0.27 098	9.94 519	49
12	9.67 445	9.72 932	0.27 068	9.94 513	48
13	9.67 468	9.72 963	0.27 037	9.94 506	47
14	9.67 492	9.72 993	0.27 007	9.94 499	46
15	9.67 515	9.73 023	0.26 977	9.94 492	45
16	9.67 539	9.73 054	0.26 946	9.94 485	44
17	9.67 562	9.73 084	0.26 916	9.94 479	43
18	9.67 586	9.73 114	0.26 886	9.94 472	42
19	9.67 609	9.73 144	0.26 856	9.94 465	41
20	9.67 633	9.73 175	0.26 825	9.94 458	**40**
21	9.67 656	9.73 205	0.26 795	9.94 451	39
22	9.67 680	9.73 235	0.26 765	9.94 445	38
23	9.67 703	9.73 265	0.26 735	9.94 438	37
24	9.67 726	9.73 295	0.26 705	9.94 431	36
25	9.67 750	9.73 326	0.26 674	9.94 424	35
26	9.67 773	9.73 356	0.26 644	9.94 417	34
27	9.67 796	9.73 386	0.26 614	9.94 410	33
28	9.67 820	9.73 416	0.26 584	9.94 404	32
29	9.67 843	9.73 446	0.26 554	9.94 397	31
30	9.67 866	9.73 476	0.26 524	9.94 390	**30**
31	9.67 890	9.73 507	0.26 493	9.94 383	29
32	9.67 913	9.73 537	0.26 463	9.94 376	28
33	9.67 936	9.73 567	0.26 433	9.94 369	27
34	9.67 959	9.73 597	0.26 403	9.94 362	26
35	9.67 982	9.73 627	0.26 373	9.94 355	25
36	9.68 006	9.73 657	0.26 343	9.94 349	24
37	9.68 029	9.73 687	0.26 313	9.94 342	23
38	9.68 052	9.73 717	0.26 283	9.94 335	22
39	9.68 075	9.73 747	0.26 253	9.94 328	21
40	9.68 098	9.73 777	0.26 223	9.94 321	**20**
41	9.68 121	9.73 807	0.26 193	9.94 314	19
42	9.68 144	9.78 837	0.26 163	9.94 307	18
43	9.68 167	9.73 867	0.26 133	9.94 300	17
44	9.68 190	9.73 897	0.26 103	9.94 293	16
45	9.68 213	9.73 927	0.26 073	9.94 286	15
46	9.68 237	9.73 957	0.26 043	9.94 279	14
47	9.68 260	9.73 987	0.26 013	9.94 273	13
48	9.68 283	9.74 017	0.25 983	9.94 266	12
49	9.68 305	9.74 047	0.25 953	9.94 259	11
50	9.68 328	9.74 077	0.25 923	9.94 252	**10**
51	9.68 351	9.74 107	0.25 893	9.94 245	9
52	9.68 374	9.74 137	0.25 863	9.94 238	8
53	9.68 397	9.74 166	0.25 834	9.94 231	7
54	9.68 420	9.74 196	0.25 804	9.94 224	6
55	9.68 443	9.74 226	0.25 774	9.94 217	5
56	9.68 466	9.74 256	0.25 744	9.94 210	4
57	9.68 489	9.74 286	0.25 714	9.94 203	3
58	9.68 512	9.74 316	0.25 684	9.94 196	2
59	9.68 534	9.74 345	0.25 655	9.94 189	1
60	9.68 557	9.74 375	0.25 625	9.94 182	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

61°

29°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.68 557	9.74 375	0.25 62$\bar{5}$	9.94 182	**60**
1	9.68 580	9.74 40$\bar{5}$	0.25 595	9.94 17$\bar{5}$	59
2	9.68 603	9.74 43$\bar{5}$	0.25 565	9.94 168	58
3	9.68 625	9.74 46$\bar{5}$	0.25 535	9.94 161	57
4	9.68 648	9.74 494	0.25 506	9.94 154	56
5	9.68 671	9.74 524	0.25 476	9.94 147	55
6	9.68 694	9.74 554	0.25 446	9.94 140	54
7	9.68 716	9.74 583	0.25 417	9.94 133	53
8	9.68 739	9.74 613	0.25 387	9.94 126	52
9	9.68 762	9.74 643	0.25 357	9.94 119	51
10	9.68 784	9.74 673	0.25 327	9.94 112	**50**
11	9.68 807	9.74 702	0.25 298	9.94 10$\bar{5}$	49
12	9.68 829	9.74 732	0.25 268	9.94 098	48
13	9.68 852	9.74 762	0.25 238	9.94 090	47
14	9.68 87$\bar{5}$	9.74 791	0.25 209	9.94 083	46
15	9.68 897	9.74 821	0.25 179	9.94 076	45
16	9.68 920	9.74 851	0.25 149	9.94 069	44
17	9.68 942	9.74 880	0.25 120	9.94 062	43
18	9.68 96$\bar{5}$	9.74 910	0.25 090	9.94 055	42
19	9.68 987	9.74 939	0.25 061	9.94 048	41
20	9.69 010	9.74 969	0.25 031	9.94 041	**40**
21	9.69 032	9.74 998	0.25 002	9.94 034	39
22	9.69 05$\bar{5}$	9.75 028	0.24 972	9.94 027	38
23	9.69 077	9.75 058	0.24 942	9.94 020	37
24	9.69 100	9.75 087	0.24 913	9.94 012	36
25	9.69 122	9.75 117	0.24 883	9.94 005	35
26	9.69 144	9.75 146	0.24 854	9.93 998	34
27	9.69 167	9.75 176	0.24 824	9.93 991	33
28	9.69 189	9.75 205	0.24 79$\bar{5}$	9.93 984	32
29	9.69 212	9.75 23$\bar{5}$	0.24 765	9.93 977	31
30	9.69 234	9.75 264	0.24 736	9.93 970	**30**
31	9.69 256	9.75 294	0.24 706	9.93 963	29
32	9.69 279	9.75 323	0.24 677	9.93 955	28
33	9.69 301	9.75 353	0.24 647	9.93 948	27
34	9.69 323	9.75 382	0.24 618	9.93 941	26
35	9.69 345	9.75 411	0.24 589	9.93 934	25
36	9.69 368	9.75 441	0.24 559	9.93 927	24
37	9.69 390	9.75 470	0.24 530	9.93 920	23
38	9.69 412	9.75 $\bar{5}$00	0.24 500	9.93 912	22
39	9.69 434	9.75 529	0.24 471	9.93 905	21
40	9.69 456	9.75 558	0.24 442	9.93 898	**20**
41	9.69 479	9.75 588	0.24 412	9.93 891	19
42	9.69 501	9.75 617	0.24 383	9.93 884	18
43	9.69 523	9.75 647	0.24 353	9.93 876	17
44	9.69 545	9.75 676	0.24 324	9.93 869	16
45	9.69 567	9.75 705	0.24 29$\bar{5}$	9.93 862	15
46	9.69 589	9.75 73$\bar{5}$	0.24 265	9.93 85$\bar{5}$	14
47	9.69 611	9.75 764	0.24 236	9.93 847	13
48	9.69 633	9.75 793	0.24 207	9.93 840	12
49	9.69 655	9.75 822	0.24 178	9.93 833	11
50	9.69 677	9.75 852	0.24 148	9.93 826	**10**
51	9.69 699	9.75 881	0.24 119	9.93 819	9
52	9.69 721	9.75 910	0.24 090	9.93 811	8
53	9.69 743	9.75 939	0.24 061	9.93 804	7
54	9.69 765	9.75 969	0.24 031	9.93 797	6
55	9.69 787	9.75 998	0.24 002	9.93 789	5
56	9.69 809	9.76 027	0.23 973	9.93 782	4
57	9.69 831	9.76 056	0.23 944	9.93 77$\bar{5}$	3
58	9.69 853	9.76 086	0.23 914	9.93 768	2
59	9.69 875	9.76 11$\bar{5}$	0.23 885	9.93 760	1
60	9.69 897	9.76 144	0.23 856	9.93 753	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

60°

30°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.69 897	9.76 144	0.23 856	9.93 753	**60**
1	9.69 919	9.76 173	0.23 827	9.93 746	59
2	9.69 941	9.76 202	0.23 798	9.93 738	58
3	9.69 963	9.76 231	0.23 769	9.93 731	57
4	9.69 984	9.76 261	0.23 739	9.93 724	56
5	9.70 006	9.76 290	0.23 710	9.93 717	55
6	9.70 028	9.76 319	0.23 681	9.93 709	54
7	9.70 05̄0	9.76 348	0.23 652	9.93 702	53
8	9.70 072	9.76 377	0.23 623	9.93 695̄	52
9	9.70 093	9.76 406	0.23 594	9.93 687	51
10	9.70 115	9.76 435	0.23 565̄	9.93 680	**50**
11	9.70 137	9.76 464	0.23 536	9.93 673	49
12	9.70 159	9.76 493	0.23 507	9.93 665	48
13	9.70 180	9.76 522	0.23 478	9.93 658	47
14	9.70 202	9.76 551	0.23 449	9.93 650	46
15	9.70 224	9.76 580	0.23 420	9.93 643	45
16	9.70 245	9.76 609	0.23 391	9.93 636	44
17	9.70 267	9.76 639	0.23 361	9.93 628	43
18	9.70 288	9.76 668	0.23 332	9.93 621	42
19	9.70 310	9.76 697	0.23 303	9.93 614	41
20	9.70 332	9.76 725	0.23 275̄	9.93 606	**40**
21	9.70 353	9.76 754	0.23 246	9.93 599	39
22	9.70 375̄	9.76 783	0.23 217	9.93 591	38
23	9.70 396	9.76 812	0.23 188	9.93 584	37
24	9.70 418	9.76 841	0.23 159	9.93 577	36
25	9.70 439	9.76 870	0.23 130	9.93 569	35
26	9.70 461	9.76 899	0.23 101	9.93 562	34
27	9.70 482	9.76 928	0.23 072	9.93 554	33
28	9.70 504	9.76 957	0.23 043	9.93 547	32
29	9.70 525	9.76 986	0.23 014	9.93 539	31
30	9.70 547	9.77 015̄	0.22 985	9.93 532	**30**
31	9.70 568	9.77 044	0.22 956	9.93 525̄	29
32	9.70 590	9.77 073	0.22 927	9.93 517	28
33	9.70 611	9.77 101	0.22 899	9.93 510	27
34	9.70 633	9.77 130	0.22 870	9.93 502	26
35	9.70 654	9.77 159	0.22 841	9.93 495̄	25
36	9.70 675	9.77 188	0.22 812	9.93 487	24
37	9.70 697	9.77 217	0.22 783	9.93 480	23
38	9.70 718	9.77 246	0.22 754	9.93 472	22
39	9.70 739	9.77 274	0.22 726	9.93 465̄	21
40	9.70 761	9.77 303	0.22 697	9.93 457	**20**
41	9.70 782	9.77 332	0.22 668	9.93 45̄0	19
42	9.70 803	9.77 361	0.22 639	9.93 442	18
43	9.70 824	9.77 390	0.22 610	9.93 435̄	17
44	9.70 846	9.77 418	0.22 582	9.93 427	16
45	9.70 867	9.77 447	0.22 553	9.93 420	15
46	9.70 888	9.77 476	0.22 524	9.93 412	14
47	9.70 909	9.77 505̄	0.22 495	9.93 405̄	13
48	9.70 931	9.77 533	0.22 467	9.93 397	12
49	9.70 952	9.77 562	0.22 438	9.93 390	11
50	9.70 973	9.77 591	0.22 409	9.93 382	**10**
51	9.70 994	9.77 619	0.22 381	9.93 375̄	9
52	9.71 015	9.77 648	0.22 352	9.93 367	8
53	9.71 036	9.77 677	0.22 323	9.93 360	7
54	9.71 058	9.77 706	0.22 294	9.93 352	6
55	9.71 079	9.77 734	0.22 266	9.93 344	5
56	9.71 100	9.77 763	0.22 237	9.93 337	4
57	9.71 121	9.77 791	0.22 209	9.93 329	3
58	9.71 142	9.77 820	0.22 180	9.93 322	2
59	9.71 163	9.77 849	0.22 151	9.93 314	1
60	9.71 184	9.77 877	0.22 123	9.93 307	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

59°

31°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.71 184	9.77 877	0.22 123	9.93 307	**60**
1	9.71 20$\bar{5}$	9.77 906	0.22 094	9.93 299	59
2	9.71 226	9.77 93$\bar{5}$	0.22 065	9.93 291	58
3	9.71 247	9.77 963	0.22 037	9.93 284	57
4	9.71 268	9.77 992	0.22 008	9.93 276	56
5	9.71 289	9.78 020	0.21 980	9.93 269	55
6	9.71 310	9.78 049	0.21 951	9.93 261	54
7	9.71 331	9.78 077	0.21 923	9.93 253	53
8	9.71 352	9.78 106	0.21 894	9.93 246	52
9	9.71 373	9.78 13$\bar{5}$	0.21 865	9.93 238	51
10	9.71 393	9.78 163	0.21 837	9.93 230	**50**
11	9.71 414	9.78 192	0.21 808	9.93 223	49
12	9.71 435	9.78 220	0.21 780	9.93 215	48
13	9.71 456	9.78 249	0.21 751	9.93 207	47
14	9.71 477	9.78 277	0.21 723	9.93 200	46
15	9.71 498	9.78 306	0.21 694	9.93 192	45
16	9.71 519	9.78 334	0.21 666	9.93 184	44
17	9.71 539	9.78 363	0.21 637	9.93 177	43
18	9.71 560	9.78 391	0.21 609	9.93 169	42
19	9.71 581	9.78 419	0.21 581	9.93 161	41
20	9.71 602	9.78 448	0.21 552	9.93 154	**40**
21	9.71 622	9.78 476	0.21 524	9.93 146	39
22	9.71 643	9.78 50$\bar{5}$	0.21 495	9.93 138	38
23	9.71 664	9.78 533	0.21 467	9.93 131	37
24	9.71 68$\bar{5}$	9.78 562	0.21 438	9.93 123	36
25	9.71 705	9.78 590	0.21 410	9.93 115	35
26	9.71 726	9.78 618	0.21 382	9.93 108	34
27	9.71 747	9.78 647	0.21 353	9.93 100	33
28	9.71 767	9.78 675	0.21 32$\bar{5}$	9.93 092	32
29	9.71 788	9.78 704	0.21 296	9.93 084	31
30	9.71 809	9.78 732	0.21 268	9.93 077	**30**
31	9.71 829	9.78 760	0.21 240	9.93 069	29
32	9.71 8$\bar{5}$0	9.78 789	0.21 211	9.93 061	28
33	9.71 870	9.78 817	0.21 183	9.93 053	27
34	9.71 891	9.78 845	0.21 15$\bar{5}$	9.93 046	26
35	9.71 911	9.78 874	0.21 126	9.93 038	25
36	9.71 932	9.78 902	0.21 098	9.93 030	24
37	9.71 952	9.78 930	0.21 070	9.93 022	23
38	9.71 973	9.78 959	0.21 041	9.93 014	22
39	9.71 994	9.78 987	0.21 013	9.93 007	21
40	9.72 014	9.79 015	0.20 98$\bar{5}$	9.92 999	**20**
41	9.72 034	9.79 043	0.20 957	9.92 991	19
42	9.72 05$\bar{5}$	9.79 072	0.20 928	9.92 983	18
43	9.72 075	9.79 100	0.20 900	9.92 976	17
44	9.72 096	9.79 128	0.20 872	9.92 968	16
45	9.72 116	9.79 156	0.20 844	9.92 960	15
46	9.72 137	9.79 18$\bar{5}$	0.20 815	9.92 952	14
47	9.72 157	9.79 213	0.20 787	9.92 944	13
48	9.72 177	9.79 241	0.20 759	9.92 936	12
49	9.72 198	9.79 269	0.20 731	9.92 929	11
50	9.72 218	9.79 297	0.20 703	9.92 921	**10**
51	9.72 238	9.79 326	0.20 674	9.92 913	9
52	9.72 259	9.79 354	0.20 646	9.92 905	8
53	9.72 279	9.79 382	0.20 618	9.92 897	7
54	9.72 299	9.79 410	0.20 590	9.92 889	6
55	9.72 320	9.79 438	0.20 562	9.92 881	5
56	9.72 340	9.79 466	0.20 534	9.92 874	4
57	9.72 360	9.79 49$\bar{5}$	0.20 505	9.92 866	3
58	9.72 381	9.79 523	0.20 477	9.92 858	2
59	9.72 401	9.79 551	0.20 449	9.92 8$\bar{5}$0	1
60	9.72 421	9.79 579	0.20 421	9.92 842	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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32°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.72 421	9.79 579	0.20 421	9.92 842	60
1	9.72 441	9.79 607	0.20 393	9.92 834	59
2	9.72 461	9.79 635	0.20 365	9.92 826	58
3	9.72 482	9.79 663	0.20 337	9.92 818	57
4	9.72 502	9.79 691	0.20 309	9.92 810	56
5	9.72 522	9.79 719	0.20 281	9.92 803	55
6	9.72 542	9.79 747	0.20 253	9.92 795	54
7	9.72 562	9.79 776	0.20 224	9.92 787	53
8	9.72 582	9.79 804	0.20 196	9.92 779	52
9	9.72 602	9.79 832	0.20 168	9.92 771	51
10	9.72 622	9.79 860	0.20 140	9.92 763	50
11	9.72 643	9.79 888	0.20 112	9.92 755	49
12	9.72 663	9.79 916	0.20 084	9.92 747	48
13	9.72 683	9.79 944	0.20 056	9.92 739	47
14	9.72 703	9.79 972	0.20 028	9.92 731	46
15	9.72 723	9.80 000	0.20 000	9.92 723	45
16	9.72 743	9.80 028	0.19 972	9.92 715	44
17	9.72 763	9.80 056	0.19 944	9.92 707	43
18	9.72 783	9.80 084	0.19 916	9.92 699	42
19	9.72 803	9.80 112	0.19 888	9.92 691	41
20	9.72 823	9.80 140	0.19 860	9.92 683	40
21	9.72 843	9.80 168	0.19 832	9.92 675	39
22	9.72 863	9.80 195	0.19 805	9.92 667	38
23	9.72 883	9.80 223	0.19 777	9.92 659	37
24	9.72 902	9.80 251	0.19 749	9.92 651	36
25	9.72 922	9.80 279	0.19 721	9.92 643	35
26	9.72 942	9.80 307	0.19 693	9.92 635	34
27	9.72 962	9.80 335	0.19 665	9.92 627	33
28	9.72 982	9.80 363	0.19 637	9.92 619	32
29	9.73 002	9.80 391	0.19 609	9.92 611	31
30	9.73 022	9.80 419	0.19 581	9.92 603	30
31	9.73 041	9.80 447	0.19 553	9.92 595	29
32	9.73 061	9.80 474	0.19 526	9.92 587	28
33	9.73 081	9.80 502	0.19 498	9.92 579	27
34	9.73 101	9.80 530	0.19 470	9.92 571	26
35	9.73 121	9.80 558	0.19 442	9.92 563	25
36	9.73 140	9.80 586	0.19 414	9.92 555	24
37	9.73 160	9.80 614	0.19 386	9.92 546	23
38	9.73 180	9.80 642	0.19 358	9.92 538	22
39	9.73 200	9.80 669	0.19 331	9.92 530	21
40	9.73 219	9.80 697	0.19 303	9.92 522	20
41	9.73 239	9.80 725	0.19 275	9.92 514	19
42	9.73 259	9.80 753	0.19 247	9.92 506	18
43	9.73 278	9.80 781	0.19 219	9.92 498	17
44	9.73 298	9.80 808	0.19 192	9.92 490	16
45	9.73 318	9.80 836	0.19 164	9.92 482	15
46	9.73 337	9.80 864	0.19 136	9.92 473	14
47	9.73 357	9.80 892	0.19 108	9.92 465	13
48	9.73 377	9.80 919	0.19 081	9.92 457	12
49	9.73 396	9.80 947	0.19 053	9.92 449	11
50	9.73 416	9.80 975	0.19 025	9.92 441	10
51	9.73 435	9.81 003	0.18 997	9.92 433	9
52	9.73 455	9.81 030	0.18 970	9.92 425	8
53	9.73 474	9.81 058	0.18 942	9.92 416	7
54	9.73 494	9.81 086	0.18 914	9.92 408	6
55	9.73 513	9.81 113	0.18 887	9.92 400	5
56	9.73 533	9.81 141	0.18 859	9.92 392	4
57	9.73 552	9.81 169	0.18 831	9.92 384	3
58	9.73 572	9.81 196	0.18 804	9.92 376	2
59	9.73 591	9.81 224	0.18 776	9.92 367	1
60	9.73 611	9.81 252	0.18 748	9.92 359	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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33°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.73 611	9.81 252	0.18 748	9.92 359	60
1	9.73 630	9.81 279	0.18 721	9.92 351	59
2	9.73 650	9.81 307	0.18 693	9.92 343	58
3	9.73 669	9.81 335	0.18 665	9.92 335	57
4	9.73 689	9.81 362	0.18 638	9.92 326	56
5	9.73 708	9.81 390	0.18 610	9.92 318	55
6	9.73 727	9.81 418	0.18 582	9.92 310	54
7	9.73 747	9.81 445	0.18 555	9.92 302	53
8	9.73 766	9.81 473	0.18 527	9.92 293	52
9	9.73 785	9.81 500	0.18 500	9.92 285	51
10	9.73 805	9.81 528	0.18 472	9.92 277	50
11	9.73 824	9.81 556	0.18 444	9.92 269	49
12	9.73 843	9.81 583	0.18 417	9.92 260	48
13	9.73 863	9.81 611	0.18 389	9.92 252	47
14	9.73 882	9.81 638	0.18 362	9.92 244	46
15	9.73 901	9.81 666	0.18 334	9.92 235	45
16	9.73 921	9.81 693	0.18 307	9.92 227	44
17	9.73 940	9.81 721	0.18 279	9.92 219	43
18	9.73 959	9.81 748	0.18 252	9.92 211	42
19	9.73 978	9.81 776	0.18 224	9.92 202	41
20	9.73 997	9.81 803	0.18 197	9.92 194	40
21	9.74 017	9.81 831	0.18 169	9.92 186	39
22	9.74 036	9.81 858	0.18 142	9.92 177	38
23	9.74 055	9.81 886	0.18 114	9.92 169	37
24	9.74 074	9.81 913	0.18 087	9.92 161	36
25	9.74 093	9.81 941	0.18 059	9.92 152	35
26	9.74 113	9.81 968	0.18 032	9.92 144	34
27	9.74 132	9.81 996	0.18 004	9.92 136	33
28	9.74 151	9.82 023	0.17 977	9.92 127	32
29	9.74 170	9.82 051	0.17 949	9.92 119	31
30	9.74 189	9.82 078	0.17 922	9.92 111	30
31	9.74 208	9.82 106	0.17 894	9.92 102	29
32	9.74 227	9.82 133	0.17 867	9.92 094	28
33	9.74 246	9.82 161	0.17 839	9.92 086	27
34	9.74 265	9.82 188	0.17 812	9.92 077	26
35	9.74 284	9.82 215	0.17 785	9.92 069	25
36	9.74 303	9.82 243	0.17 757	9.92 060	24
37	9.74 322	9.82 270	0.17 730	9.92 052	23
38	9.74 341	9.82 298	0.17 702	9.92 044	22
39	9.74 360	9.82 325	0.17 675	9.92 035	21
40	9.74 379	9.82 352	0.17 648	9.92 027	20
41	9.74 398	9.82 380	0.17 620	9.92 018	19
42	9.74 417	9.82 407	0.17 593	9.92 010	18
43	9.74 436	9.82 435	0.17 565	9.92 002	17
44	9.74 455	9.82 462	0.17 538	9.91 993	16
45	9.74 474	9.82 489	0.17 511	9.91 985	15
46	9.74 493	9.82 517	0.17 483	9.91 976	14
47	9.74 512	9.82 544	0.17 456	9.91 968	13
48	9.74 531	9.82 571	0.17 429	9.91 959	12
49	9.74 549	9.82 599	0.17 401	9.91 951	11
50	9.74 568	9.82 626	0.17 374	9.91 942	10
51	9.74 587	9.82 653	0.17 347	9.91 934	9
52	9.74 606	9.82 681	0.17 319	9.91 925	8
53	9.74 625	9.82 708	0.17 292	9.91 917	7
54	9.74 644	9.82 735	0.17 265	9.91 908	6
55	9.74 662	9.82 762	0.17 238	9.91 900	5
56	9.74 681	9.82 790	0.17 210	9.91 891	4
57	9.74 700	9.82 817	0.18 183	9.91 883	3
58	9.74 719	9.82 844	0.17 156	9.91 874	2
59	9.74 737	9.82 871	0.17 129	9.91 866	1
60	9.74 756	9.82 899	0.17 101	9.91 857	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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34°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.74 756	9.82 899	0.17 101	9.91 857	**60**
1	9.74 77$\bar{5}$	9.82 926	0.17 074	9.91 849	59
2	9.74 794	9.82 953	0.17 047	9.91 840	58
3	9.74 812	9.82 980	0.17 020	9.91 832	57
4	9.74 831	9.83 008	0.16 992	9.91 823	56
5	9.74 8$\bar{5}$0	9.83 03$\bar{5}$	0.16 965	9.91 81$\bar{5}$	55
6	9.74 868	9.83 062	0.16 938	9.91 806	54
7	9.74 887	9.83 089	0.16 911	9.91 798	53
8	9.74 906	9.83 117	0.16 883	9.91 789	52
9	9.74 924	9.83 144	0.16 856	9.91 781	51
10	9.74 943	9.83 171	0.16 829	9.91 772	**50**
11	9.74 961	9.83 198	0.16 802	9.91 763	49
12	9.74 980	9.83 225	0.16 77$\bar{5}$	9.91 75$\bar{5}$	48
13	9.74 999	9.83 252	0.16 748	9.91 746	47
14	9.75 017	9.83 280	0.16 720	9.91 738	46
15	9.75 036	9.83 307	0.16 693	9.91 729	45
16	9.75 054	9.83 334	0.16 666	9.91 720	44
17	9.75 073	9.83 361	0.16 639	9.91 712	43
18	9.75 091	9.83 388	0.16 612	9.91 703	42
19	9.75 110	9.83 415	0.16 58$\bar{5}$	9.91 69$\bar{5}$	41
20	9.75 128	9.83 442	0.16 558	9.91 686	**40**
21	9.75 147	9.83 470	0.16 530	9.91 677	39
22	9.75 165	9.83 497	0.16 503	9.91 669	38
23	9.75 184	9.83 524	0.16 476	9.91 660	37
24	9.75 202	9.83 551	0.16 449	9.91 651	36
25	9.75 221	9.83 578	0.16 422	9.91 643	35
26	9.75 239	9.83 605	0.16 39$\bar{5}$	9.91 634	34
27	9.75 258	9.83 632	0.16 368	9.91 625	33
28	9.75 276	9.83 659	0.16 341	9.91 617	32
29	9.75 294	9.83 686	0.16 314	9.91 608	31
30	9.75 313	9.83 713	0.16 287	9.91 599	**30**
31	9.75 331	9.83 740	0.16 260	9.91 591	29
32	9.75 3$\bar{5}$0	9.83 768	0.16 232	9.91 582	28
33	9.75 368	9.83 79$\bar{5}$	0.16 205	9.91 573	27
34	9.75 386	9.83 822	0.16 178	9.91 56$\bar{5}$	26
35	9.75 40$\bar{5}$	9.83 849	0.16 151	9.91 556	25
36	9.75 423	9.83 876	0.16 124	9.91 547	24
37	9.75 441	9.83 903	0.16 097	9.91 538	23
38	9.75 459	9.83 930	0.16 070	9.91 530	22
39	9.75 478	9.83 957	0.16 043	9.91 521	21
40	9.75 496	9.83 984	0.16 016	9.91 512	**20**
41	9.75 514	9.84 011	0.15 989	9.91 504	19
42	9.75 533	9.84 038	0.15 962	9.91 49$\bar{5}$	18
43	9.75 551	9.84 06$\bar{5}$	0.15 935	9.91 486	17
44	9.75 569	9.84 092	0.15 908	9.91 477	16
45	9.75 587	9.84 119	0.15 881	9.91 469	15
46	9.75 605	9.84 146	0.15 854	9.91 460	14
47	9.75 624	9.84 173	0.15 827	9.91 451	13
48	9.75 642	9.84 200	0.15 800	9.91 442	12
49	9.75 660	9.84 227	0.15 773	9.91 433	11
50	9.75 678	9.84 254	0.15 746	9.91 42$\bar{5}$	**10**
51	9.75 696	9.84 280	0.15 720	9.91 416	9
52	9.75 714	9.84 307	0.15 693	9.91 407	8
53	9.75 733	9.84 334	0.15 666	9.91 398	7
54	9.75 751	9.84 361	0.15 639	9.91 389	6
55	9.75 769	9.84 388	0.15 612	9.91 381	5
56	9.75 787	9.84 415	0.15 58$\bar{5}$	9.91 372	4
57	9.75 80$\bar{5}$	9.84 442	0.15 558	9.91 363	3
58	9.75 823	9.84 469	0.15 531	9.91 354	2
59	9.75 841	9.84 496	0.15 504	9.91 345	1
60	9.75 859	9.84 523	0.15 477	9.91 336	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

55°

35°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.75 859	9.84 523	0.15 477	9.91 336	**60**
1	9.75 877	9.84 55̄0	0.15 450	9.91 328	59
2	9.75 895	9.84 576	0.15 424	9.91 319	58
3	9.75 913	9.84 603	0.15 397	9.91 310	57
4	9.75 931	9.84 630	0.15 370	9.91 301	56
5	9.75 949	9.84 657	0.15 343	9.91 292	55
6	9.75 967	9.84 684	0.15 316	9.91 283	54
7	9.75 985	9.84 711	0.15 289	9.91 274	53
8	9.76 003	9.84 738	0.15 262	9.91 266	52
9	9.76 021	9.84 764	0.15 236	9.91 257	51
10	9.76 039	9.84 791	0.15 209	9.91 248	**50**
11	9.76 057	9.84 818	0.15 182	9.91 239	49
12	9.76 075̄	9.84 845̄	0.15 155	9.91 230	48
13	9.76 093	9.84 872	0.15 128	9.91 221	47
14	9.76 111	9.84 899	0.15 101	9.91 212	46
15	9.76 129	9.84 925	0.15 075̄	9.91 203	45
16	9.76 146	9.84 952	0.15 048	9.91 194	44
17	9.76 164	9.84 979	0.15 021	9.91 185	43
18	9.76 182	9.85 006	0.14 994	9.91 176	42
19	9.76 200	9.85 033	0.14 967	9.91 167	41
20	9.76 218	9.85 059	0.14 941	9.91 158	**40**
21	9.76 236	9.85 086	0.14 914	9.91 149	39
22	9.76 253	9.85 113	0.14 887	9.91 141	38
23	9.76 271	9.85 140	0.14 860	9.91 132	37
24	9.76 289	9.85 166	0.14 834	9.91 123	36
25	9.76 307	9.85 193	0.14 807	9.91 114	35
26	9.76 324	9.85 220	0.14 780	9.91 105̄	34
27	9.76 342	9.85 247	0.14 753	9.91 096	33
28	9.76 360	9.85 273	0.14 727	9.91 087	32
29	9.76 378	9.85 300	0.14 700	9.91 078	31
30	9.76 395	9.85 327	0.14 673	9.91 069	**30**
31	9.76 413	9.85 354	0.14 646	9.91 060	29
32	9.76 431	9.85 380	0.14 620	9.91 051	28
33	9.76 448	9.85 407	0.14 593	9.91 042	27
34	9.76 466	9.85 434	0.14 566	9.91 033	26
35	9.76 484	9.85 460	0.14 540	9.91 023	25
36	9.76 501	9.85 487	0.14 513	9.91 014	24
37	9.76 519	9.85 514	0.14 486	9.91 005	23
38	9.76 537	9.85 540	0.14 460	9.90 996	22
39	9.76 554	9.85 567	0.14 433	9.90 987	21
40	9.76 572	9.85 594	0.14 406	9.90 978	**20**
41	9.76 590	9.85 620	0.14 380	9.90 969	19
42	9.76 607	9.85 647	0.14 353	9.90 960	18
43	9.76 625̄	9.85 674	0.14 326	9.90 951	17
44	9.76 642	9.85 700	0.14 300	9.90 942	16
45	9.76 660	9.85 727	0.14 273	9.90 933	15
46	9.76 677	9.85 754	0.14 246	9.90 924	14
47	9.76 695̄	9.85 780	0.14 220	9.90 915̄	13
48	9.76 712	9.85 807	0.14 193	9.90 906	12
49	9.76 730	9.85 834	0.14 166	9.90 896	11
50	9.76 747	9.85 860	0.14 140	9.90 887	**10**
51	9.76 765̄	9.85 887	0.14 113	9.90 878	9
52	9.76 782	9.85 913	0.14 087	9.90 869	8
53	9.76 800	9.85 940	0.14 060	9.90 860	7
54	9.76 817	9.85 967	0.14 033	9.90 851	6
55	9.76 835̄	9.85 993	0.14 007	9.90 842	5
56	9.76 852	9.86 020	0.13 980	9.90 832	4
57	9.76 870	9.86 046	0.13 954	9.90 823	3
58	9.76 887	9.86 073	0.13 927	9.90 814	2
59	9.76 904	9.86 100	0.13 900	9.90 805̄	1
60	9.76 922	9.86 126	0.13 874	9.90 796	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

54°

36°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.76 922	9.86 126	0.13 874	9.90 796	60
1	9.76 939	9.86 153	0.13 847	9.90 787	59
2	9.76 957	9.86 179	0.13 821	9.90 777	58
3	9.76 974	9.86 206	0.13 794	9.90 768	57
4	9.76 991	9.86 232	0.13 768	9.90 759	56
5	9.77 009	9.86 259	0.13 741	9.90 750	55
6	9.77 026	9.86 285	0.13 715	9.90 741	54
7	9.77 043	9.86 312	0.13 688	9.90 731	53
8	9.77 061	9.86 338	0.13 662	9.90 722	52
9	9.77 078	9.86 365	0.13 635	9.90 713	51
10	9.77 095	9.86 392	0.13 608	9.90 704	50
11	9.77 112	9.86 418	0.13 582	9.90 694	49
12	9.77 130	9.86 445	0.13 555	9.90 685	48
13	9.77 147	9.86 471	0.13 529	9.90 676	47
14	9.77 164	9.86 498	0.13 502	9.90 667	46
15	9.77 181	9.86 524	0.13 476	9.90 657	45
16	9.77 199	9.86 551	0.13 449	9.90 648	44
17	9.77 216	9.86 577	0.13 423	9.90 639	43
18	9.77 233	9.86 603	0.13 397	9.90 630	42
19	9.77 250	9.86 630	0.13 370	9.90 620	41
20	9.77 268	9.86 656	0.13 344	9.90 611	40
21	9.77 285	9.86 683	0.13 317	9.90 602	39
22	9.77 302	9.86 709	0.13 291	9.90 592	38
23	9.77 319	9.86 736	0.13 264	9.90 583	37
24	9.77 336	9.86 762	0.13 238	9.90 574	36
25	9.77 353	9.86 789	0.13 211	9.90 565	35
26	9.77 370	9.86 815	0.13 185	9.90 555	34
27	9.77 387	9.86 842	0.13 158	9.90 546	33
28	9.77 405	9.86 868	0.13 132	9.90 537	32
29	9.77 422	9.86 894	0.13 106	9.90 527	31
30	9.77 439	9.86 921	0.13 079	9.90 518	30
31	9.77 456	9.86 947	0.13 053	9.90 509	29
32	9.77 473	9.86 974	0.13 026	9.90 499	28
33	9.77 490	9.87 000	0.13 000	9.90 490	27
34	9.77 507	9.87 027	0.12 973	9.90 480	26
35	9.77 524	9.87 053	0.12 947	9.90 471	25
36	9.77 541	9.87 079	0.12 921	9.90 462	24
37	9.77 558	9.87 106	0.12 894	9.90 452	23
38	9.77 575	9.87 132	0.12 868	9.90 443	22
39	9.77 592	9.87 158	0.12 842	9.90 434	21
40	9.77 609	9.87 185	0.12 815	9.90 424	20
41	9.77 626	9.87 211	0.12 789	9.90 415	19
42	9.77 643	9.87 238	0.12 762	9.90 405	18
43	9.77 660	9.87 264	0.12 736	9.90 396	17
44	9.77 677	9.87 290	0.12 710	9.90 386	16
45	9.77 694	9.87 317	0.12 683	9.90 377	15
46	9.77 711	9.87 343	0.12 657	9.90 368	14
47	9.77 728	9.87 369	0.12 631	9.90 358	13
48	9.77 744	9.87 396	0.12 604	9.90 349	12
49	9.77 761	9.87 422	0.12 578	9.90 339	11
50	9.77 778	9.87 448	0.12 552	9.90 330	10
51	9.77 795	9.87 475	0.12 525	9.90 320	9
52	9.77 812	9.87 501	0.12 499	9.90 311	8
53	9.77 829	9.87 527	0.12 473	9.90 301	7
54	9.77 846	9.87 554	0.12 446	9.90 292	6
55	9.77 862	9.87 580	0.12 420	9.90 282	5
56	9.77 879	9.87 606	0.12 394	9.90 273	4
57	9.77 896	9.87 633	0.12 367	9.90 263	3
58	9.77 913	9.87 659	0.12 341	9.90 254	2
59	9.77 930	9.87 685	0.12 315	9.90 244	1
60	9.77 946	9.87 711	0.12 289	9.90 235	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

53°

37°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.77 946	9.87 711	0.12 289	9.90 235	**60**
1	9.77 963	9.87 738	0.12 262	9.90 225	59
2	9.77 980	9.87 764	0.12 236	9.90 216	58
3	9.77 997	9.87 790	0.12 210	9.90 206	57
4	9.78 013	9.87 817	0.12 183	9.90 197	56
5	9.78 030	9.87 843	0.12 157	9.90 187	55
6	9.78 047	9.87 869	0.12 131	9.90 178	54
7	9.78 063	9.87 895	0.12 105	9.90 168	53
8	9.78 080	9.87 922	0.12 078	9.90 159	52
9	9.78 097	9.87 948	0.12 052	9.90 149	51
10	9.78 113	9.87 974	0.12 026	9.90 139	**50**
11	9.78 130	9.88 000	0.12 000	9.90 130	49
12	9.78 147	9.88 027	0.11 973	9.90 120	48
13	9.78 163	9.88 053	0.11 947	9.90 111	47
14	9.78 180	9.88 079	0.11 921	9.90 101	46
15	9.78 197	9.88 105	0.11 895	9.90 091	45
16	9.78 213	9.88 131	0.11 869	9.90 082	44
17	9.78 230	9.88 158	0.11 842	9.90 072	43
18	9.78 246	9.88 184	0.11 816	9.90 063	42
19	9.78 263	9.88 210	0.11 790	9.90 053	41
20	9.78 280	9.88 236	0.11 764	9.90 043	**40**
21	9.78 296	9.88 262	0.11 738	9.90 034	39
22	9.78 313	9.88 289	0.11 711	9.90 024	38
23	9.78 329	9.88 315	0.11 685	9.90 014	37
24	9.78 346	9.88 341	0.11 659	9.90 005	36
25	9.78 362	9.88 367	0.11 633	9.89 995	35
26	9.78 379	9.88 393	0.11 607	9.89 985	34
27	9.78 395	9.88 420	0.11 580	9.89 976	33
28	9.78 412	9.88 446	0.11 554	9.89 966	32
29	9.78 428	9.88 472	0.11 528	9.89 956	31
30	9.78 445	9.88 498	0.11.502	9.89 947	**30**
31	9.78 461	9.88 524	0.11 476	9.89 937	29
32	9.78 478	9.88 550	0.11 450	9.89 927	28
33	9.78 494	9.88 577	0.11 423	9.89 918	27
34	9.78 510	9.88 603	0.11 397	9.89 908	26
35	9.78 527	9.88 629	0.11 371	9.89 898	25
36	9.78 543	9.88 655	0.11 345	9.89 888	24
37	9.78 560	9.88 681	0.11 319	9.89 879	23
38	9.78 576	9.88 707	0.11 293	9.89 869	22
39	9.78 592	9.88 733	0.11 267	9.89 859	21
40	9.78 609	9.88 759	0.11 241	9.89 849	**20**
41	9.78 625	9.88 786	0.11 214	9.89 840	19
42	9.78 642	9.88 812	0.11 188	9.89 830	18
43	9.78 658	9.88 838	0.11 162	9.89 820	17
44	9.78 674	9.88 864	0.11 136	9.89 810	16
45	9.78 691	9.88 890	0.11 110	9.89 801	15
46	9.78 707	9.88 916	0.11 084	9.89 791	14
47	9.78 723	9.88 942	0.11 058	9.89 781	13
48	9.78 739	9.88 968	0.11 032	9.89 771	12
49	9.78 756	9.88 994	0.11 006	9.89 761	11
50	9.78 772	9.89 020	0.10 980	9.89 752	**10**
51	9.78 788	9.89 046	0.10 954	9.89 742	9
52	9.78 805	9.89 073	0.10 927	9.89 732	8
53	9.78 821	9.89 099	0.10 901	9.89 722	7
54	9.78 837	9.89 125	0.10 875	9.89 712	6
55	9.78 853	9.89 151	0.10 849	9.89 702	5
56	9.78 869	9.89 177	0.10 823	9.89 693	4
57	9.78 886	9.89 203	0.10 797	9.89 683	3
58	9.78 902	9.89 229	0.10 771	9.89 673	2
59	9.78 918	9.89 255	0.10 745	9.89 663	1
60	9.78 934	9.89 281	0.10 719	9.89 653	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

52°

38°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.78 934	9.89 281	0.10 719	9.89 653	60
1	9.78 950	9.89 307	0.10 693	9.89 643	59
2	9.78 967	9.89 333	0.10 667	9.89 633	58
3	9.78 983	9.89 359	0.10 641	9.89 624	57
4	9.78 999	9.89 385	0.10 61$\bar{5}$	9.89 614	56
5	9.79 01$\bar{5}$	9.89 411	0.10 589	9.89 604	55
6	9.79 031	9.89 437	0.10 563	9.89 594	54
7	9.79 047	9.89 463	0.10 537	9.89 584	53
8	9.79 063	9.89 489	0.10 511	9.89 574	52
9	9.79 079	9.89 515	0.10 48$\bar{5}$	9.89 564	51
10	9.79 095	9.89 541	0.10 459	9.89 554	50
11	9.79 111	9.89 567	0.10 433	9.89 544	49
12	9.79 128	9.89 593	0.10 407	9.89 534	48
13	9.79 144	9.89 619	0.10 381	9.89 524	47
14	9.79 160	9.89 645	0.10 35$\bar{5}$	9.89 514	46
15	9.79 176	9.89 671	0.10 329	9.89 504	45
16	9.79 192	9.89 697	0.10 303	9.89 49$\bar{5}$	44
17	9.79 208	9.89 723	0.10 277	9.89 48$\bar{5}$	43
18	9.79 224	9.89 749	0.10 251	9.89 47$\bar{5}$	42
19	9.79 240	9.89 775	0.10 22$\bar{5}$	9.89 46$\bar{5}$	41
20	9.79 256	9.89 801	0.10 199	9.89 45$\bar{5}$	40
21	9.79 272	9.89 827	0.10 173	9.89 44$\bar{5}$	39
22	9.79 288	9.89 853	0.10 147	9.89 43$\bar{5}$	38
23	9.79 304	9.89 879	0.10 121	9.89 42$\bar{5}$	37
24	9.79 319	9.89 90$\bar{5}$	0.10 095	9.89 41$\bar{5}$	36
25	9.79 335	9.89 931	0.10 069	9.89 40$\bar{5}$	35
26	9.79 351	9.89 957	0.10 043	9.89 39$\bar{5}$	34
27	9.79 367	9.89 983	0.10 017	9.89 38$\bar{5}$	33
28	9.79 383	9.90 009	0.09 991	9.89 37$\bar{5}$	32
29	9.79 399	9.90 03$\bar{5}$	0.09 965	9.89 364	31
30	9.79 41$\bar{5}$	9.90 061	0.09 939	9.89 354	30
31	9.79 431	9.90 086	0.09 914	9.89 344	29
32	9.79 447	9.90 112	0.09 888	9.89 334	28
33	9.79 463	9.90 138	0.09 862	9.89 324	27
34	9.79 478	9.90 164	0.09 836	9.89 314	26
35	9.79 494	9.90 190	0.09 810	9.89 304	25
36	9.79 510	9.90 216	0.09 784	9.89 294	24
37	9.79 526	9.90 242	0.09 758	9.89 284	23
38	9.79 542	9.90 268	0.09 732	9.89 274	22
39	9.79 558	9.90 294	0.09 706	9.89 264	21
40	9.79 573	9.90 320	0.09 680	9.89 254	20
41	9.79 589	9.90 346	0.09 654	9.89 244	19
42	9.79 60$\bar{5}$	9.90 371	0.09 629	9.89 233	18
43	9.79 621	9.90 397	0.09 603	9.89 223	17
44	9.79 636	9.90 423	0.09 577	9.89 213	16
45	9.79 652	9.90 449	0.09 551	9.89 203	15
46	9.79 668	9.90 47$\bar{5}$	0.09 525	9.89 193	14
47	9.79 684	9.90 501	0.09 499	9.89 183	13
48	9.79 699	9.90 527	0.09 473	9.89 173	12
49	9.79 715	9.90 553	0.09 447	9.89 162	11
50	9.79 731	9.90 578	0.09 422	9.89 152	10
51	9.79 746	9.90 604	0.09 396	9.89 142	9
52	9.79 762	9.90 630	0.09 370	9.89 132	8
53	9.79 778	9.90 656	0.09 344	9.89 122	7
54	9.79 793	9.90 682	0.09 318	9.89 112	6
55	9.79 809	9.90 708	0.09 292	9.89 101	5
56	9.79 82$\bar{5}$	9.90 734	0.09 266	9.89 091	4
57	9.79 840	9.90 759	0.09 241	9.89 081	3
58	9.79 856	9.90 785	0.09 21$\bar{5}$	9.89 071	2
59	9.79 872	9.90 811	0.09 189	9.89 060	1
60	9.79 887	9.90 837	0.09 163	9.89 050	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

51°

39°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.79 887	9.90 837	0.09 163	9.89 050	**60**
1	9.79 903	9.90 863	0.09 137	9.89 040	59
2	9.79 918	9.90 889	0.09 111	9.89 030	58
3	9.79 934	9.90 914	0.09 086	9.89 020	57
4	9.79 95̄0	9.90 940	0.09 060	9.89 009	56
5	9.79 965	9.90 966	0.09 034	9.88 999	55
6	9.79 981	9.90 992	0.09 008	9.88 989	54
7	9.79 996	9.91 018	0.08 982	9.88 978	53
8	9.80 012	9.91 043	0.08 957	9.88 968	52
9	9.80 027	9.91 069	0.08 931	9.88 958	51
10	9.80 043	9.91 095	0.08 905̄	9.88 948	**50**
11	9.80 058	9.91 121	0.08 879	9.88 937	49
12	9.80 074	9.91 147	0.08 853	9.88 927	48
13	9.80 089	9.91 172	0.08 828	9.88 917	47
14	9.80 105̄	9.91 198	0.08 802	9.88 906	46
15	9.80 120	9.91 224	0.08 776	9.88 896	45
16	9.80 136	9.91 25̄0	0.08 750	9.88 886	44
17	9.80 151	9.91 276	0.08 724	9.88 875	43
18	9.80 166	9.91 301	0.08 699	9.88 865	42
19	9.80 182	9.91 327	0.08 673	9.88 855̄	41
20	9.80 197	9.91 353	0.08 647	9.88 844	**40**
21	9.80 213	9.91 379	0.08 621	9.88 834	39
22	9.80 228	9.91 404	0.08 596	9.88 824	38
23	9.80 244	9.91 430	0.08 570	9.88 813	37
24	9.80 259	9.91 456	0.08 544	9.88 803	36
25	9.80 274	9.91 482	0.08 518	9.88 793	35
26	9.80 290	9.91 507	0.08 493	9.88 782	34
27	9.80 305	9.91 533	0.08 467	9.88 772	33
28	9.80 320	9.91 559	0.08 441	9.88 761	32
29	9.80 336	9.91 585̄	0.08 415	9.88 751	31
30	9.80 351	9.91 610	0.08 390	9.88 741	**30**
31	9.80 366	9.91 636	0.08 364	9.88 730	29
32	9.80 382	9.91 662	0.08 338	9.88 720	28
33	9.80 397	9.91 688	0.08 312	9.88 709	27
34	9.80 412	9.91 713	0.08 287	9.88 699	26
35	9.80 428	9.91 739	0.08 261	9.88 688	25
36	9.80 443	9.91 765̄	0.08 235	9.88 678	24
37	9.80 458	9.91 791	0.08 209	9.88 668	23
38	9.80 473	9.91 816	0.08 184	9.88 657	22
39	9.80 489	9.91 842	0.08 158	9.88 647	21
40	9.80 504	9.91 868	0.08 132	9.88 636	**20**
41	9.80 519	9.91 893	0.08 107	9.88 626	19
42	9.80 534	9.91 919	0.08 081	9.88 615	18
43	9.80 55̄0	9.91 945̄	0.08 055	9.88 605̄	17
44	9.80 565̄	9.91 971	0.08 029	9.88 594	16
45	9.80 580	9.91 996	0.08 004	9.88 584	15
46	9.80 595	9.92 022	0.07 978	9.88 573	14
47	9.80 610	9.92 048	0.07 952	9.88 563	13
48	9.80 625	9.92 073	0.07 927	9.88 552	12
49	9.80 641	9.92 099	0.07 901	9.88 542	11
50	9.80 656	9.92 125	0.07 875	9.88 531	**10**
51	9.80 671	9.92 150	0.07 85̄0	9.88 521	9
52	9.80 686	9.92 176	0.07 824	9.88 510	8
53	9.80 701	9.92 202	0.07 798	9.88 499	7
54	9.80 716	9.92 227	0.07 773	9.88 489	6
55	9.80 731	9.92 253	0.07 747	9.88 478	5
56	9.80 746	9.92 279	0.07 721	9.88 468	4
57	9.80 762	9.92 304	0.07 696	9.88 457	3
58	9.80 777	9.92 330	0.07 670	9.88 447	2
59	9.80 792	9.92 356	0.07 644	9.88 436	1
60	9.80 807	9.92 381	0.07 619	9.88 425	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

50°

40°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.80 807	9.92 381	0.07 619	9.88 425	**60**
1	9.80 822	9.92 407	0.07 593	9.88 415	59
2	9.80 837	9.92 433	0.07 567	9.88 404	58
3	9.80 852	9.92 458	0.07 542	9.88 394	57
4	9.80 867	9.92 484	0.07 516	9.88 383	56
5	9.80 882	9.92 510	0.07 490	9.88 372	55
6	9.80 897	9.92 535	0.07 465	9.88 362	54
7	9.80 912	9.92 561	0.07 439	9.88 351	53
8	9.80 927	9.92 587	0.07 413	9.88 340	52
9	9.80 942	9.92 612	0.07 388	9.88 330	51
10	9.80 957	9.92 638	0.07 362	9.88 319	**50**
11	9.80 972	9.92 663	0.07 337	9.88 308	49
12	9.80 987	9.92 689	0.07 311	9.88 298	48
13	9.81 002	9.92 715	0.07 285	9.88 287	47
14	9.81 017	9.92 740	0.07 260	9.88 276	46
15	9.81 032	9.92 766	0.07 234	9.88 266	45
16	9.81 047	9.92 792	0.07 208	9.88 255	44
17	9.81 061	9.92 817	0.07 183	9.88 244	43
18	9.81 076	9.92 843	0.07 157	9.88 234	42
19	9.81 091	9.92 868	0.07 132	9.88 223	41
20	9.81 106	9.92 894	0.07 106	9.88 212	**40**
21	9.81 121	9.92 920	0.07 080	9.88 201	39
22	9.81 136	9.92 945	0.07 055	9.88 191	38
23	9.81 151	9.92 971	0.07 029	9.88 180	37
24	9.81 166	9.92 996	0.07 004	9.88 169	36
25	9.81 180	9.93 022	0.06 978	9.88 158	35
26	9.81 195	9.93 048	0.06 952	9.88 148	34
27	9.81 210	9.93 073	0.06 927	9.88 137	33
28	9.81 225	9.93 099	0.06 901	9.88 126	32
29	9.81 240	9.93 124	0.06 876	9.88 115	31
30	9.81 254	9.93 150	0.06 850	9.88 105	**30**
31	9.81 269	9.93 175	0.06 825	9.88 094	29
32	9.81 284	9.93 201	0.06 799	9.88 083	28
33	9.81 299	9.93 227	0.06 773	9.88 072	27
34	9.81 314	9.93 252	0.06 748	9.88 061	26
35	9.81 328	9.93 278	0.06 722	9.88 051	25
36	9.81 343	9.93 303	0.06 697	9.88 040	24
37	9.81 358	9.93 329	0.06 671	9.88 029	23
38	9.81 372	9.93 354	0.06 646	9.88 018	22
39	9.81 387	9.93 380	0.06 620	9.88 007	21
40	9.81 402	9.93 406	0.06 594	9.87 996	**20**
41	9.81 417	9.93 431	0.06 569	9.87 985	19
42	9.81 431	9.93 457	0.06 543	9.87 975	18
43	9.81 446	9.93 482	0.06 518	9.87 964	17
44	9.81 461	9.93 508	0.06 492	9.87 953	16
45	9.81 475	9.93 533	0.06 467	9.87 942	15
46	9.81 490	9.93 559	0.06 441	9.87 931	14
47	9.81 505	9.93 584	0.06 416	9.87 920	13
48	9.81 519	9.93 610	0.06 390	9.87 909	12
49	9.81 534	9.93 636	0.06 364	9.87 898	11
50	9.81 549	9.93 661	0.06 339	9.87 887	**10**
51	9.81 563	9.93 687	0.06 313	9.87 877	9
52	9.81 578	9.93 712	0.06 288	9.87 866	8
53	9.81 592	9.93 738	0.06 262	9.87 855	7
54	9.81 607	9.93 763	0.06 237	9.87 844	6
55	9.81 622	9.93 789	0.06 211	9.87 833	5
56	9.81 636	9.93 814	0.06 186	9.87 822	4
57	9.81 651	9.93 840	0.06 160	9.87 811	3
58	9.81 665	9.93 865	0.06 135	9.87 800	2
59	9.81 680	9.93 891	0.06 109	9.87 789	1
60	9.81 694	9.93 916	0.06 084	9.87 778	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

49°

41°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.81 694	9.93 916	0.06 084	9.87 778	**60**
1	9.81 709	9.93 942	0.06 058	9.87 767	**59**
2	9.81 723	9.93 967	0.06 033	9.87 756	**58**
3	9.81 738	9.93 993	0.06 007	9.87 745	**57**
4	9.81 752	9.94 018	0.05 982	9.87 734	**56**
5	9.81 767	9.94 044	0.05 956	9.87 723	**55**
6	9.81 781	9.94 069	0.05 931	9.87 712	**54**
7	9.81 796	9.94 095̄	0.05 905	9.87 701	**53**
8	9.81 810	9.94 120	0.05 880	9.87 690	**52**
9	9.81 825̄	9.94 146	0.05 854	9.87 679	**51**
10	9.81 839	9.94 171	0.05 829	9.87 668	**50**
11	9.81 854	9.94 197	0.05 803	9.87 657	**49**
12	9.81 868	9.94 222	0.05 778	9.87 646	**48**
13	9.81 882	9.94 248	0.05 752	9.87 635̄	**47**
14	9.81 897	9.94 273	0.05 727	9.87 624	**46**
15	9.81 911	9.94 299	0.05 701	9.87 613	**45**
16	9.81 926	9.94 324	0.05 676	9.87 601	**44**
17	9.81 940	9.94 35̄0	0.05 650	9.87 590	**43**
18	9.81 955̄	9.94 375	0.05 625̄	9.87 579	**42**
19	9.81 969	9.94 401	0.05 599	9.87 568	**41**
20	9.81 983	9.94 426	0.05 574	9.87 557	**40**
21	9.81 998	9.94 452	0.05 548	9.87 546	**39**
22	9.82 012	9.94 477	0.05 523	9.87 535̄	**38**
23	9.82 026	9.94 503	0.05 497	9.87 524	**37**
24	9.82 041	9.94 528	0.05 472	9.87 513	**36**
25	9.82 055̄	9.94 554	0.05 446	9.87 501	**35**
26	9.82 069	9.94 579	0.05 421	9.87 490	**34**
27	9.82 084	9.94 604	0.05 396	9.87 479	**33**
28	9.82 098	9.94 630	0.05 370	9.87 468	**32**
29	9.82 112	9.94 655	0.05 345̄	9.87 457	**31**
30	9.82 126	9.94 681	0.05 319	9.87 446	**30**
31	9.82 141	9.94 706	0.05 294	9.87 434	**29**
32	9.82 155̄	9.94 732	0.05 268	9.87 423	**28**
33	9.82 169	9.94 757	0.05 243	9.87 412	**27**
34	9.82 184	9.94 783	0.05 217	9.87 401	**26**
35	9.82 198	9.94 808	0.05 192	9.87 390	**25**
36	9.82 212	9.94 834	0.05 166	9.87 378	**24**
37	9.82 226	9.94 859	0.05 141	9.87 367	**23**
38	9.82 240	9.94 884	0.05 116	9.87 356	**22**
39	9.82 255̄	9.94 910	0.05 090	9.87 345̄	**21**
40	9.82 269	9.94 935	0.05 065̄	9.87 334	**20**
41	9.82 283	9.94 961	0.05 039	9.87 322	**19**
42	9.82 297	9.94 986	0.05 014	9.87 311	**18**
43	9.82 311	9.95 012	0.04 988	9.87 300	**17**
44	9.82 326	9.95 037	0.04 963	9.87 288	**16**
45	9.82 340	9.95 062	0.04 938	9.87 277	**15**
46	9.82 354	9.95 088	0.04 912	9.87 266	**14**
47	9.82 368	9.95 113	0.04 887	9.87 255	**13**
48	9.82 382	9.95 139	0.04 861	9.87 243	**12**
49	9.82 396	9.95 164	0.04 836	9.87 232	**11**
50	9.82 410	9.95 190	0.04 810	9.87 221	**10**
51	9.82 424	9.95 215	0.04 785̄	9.87 209	**9**
52	9.82 439	9.95 240	0.04 760	9.87 198	**8**
53	9.82 453	9.95 266	0.04 734	9.87 187	**7**
54	9.82 467	9.95 291	0.04 709	9.87 175	**6**
55	9.82 481	9.95 317	0.04 683	9.87 164	**5**
56	9.82 495̄	9.95 342	0.04 658	9.87 153	**4**
57	9.82 509	9.95 368	0.04 632	9.87 141	**3**
58	9.82 523	9.95 393	0.04 607	9.87 130	**2**
59	9.82 537	9.95 418	0.04 582	9.87 119	**1**
60	9.82 551	9.95 444	0.04 556	9.87 107	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

48°

42°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.82 551	9.95 444	0.04 556	9.87 107	**60**
1	9.82 565	9.95 469	0.04 531	9.87 096	59
2	9.82 579	9.95 49$\bar{5}$	0.04 505	9.87 08$\bar{5}$	58
3	9.82 593	9.95 520	0.04 480	9.87 073	57
4	9.82 607	9.95 545	0.04 45$\bar{5}$	9.87 062	56
5	9.82 621	9.95 571	0.04 429	9.87 050	55
6	9.82 635	9.95 596	0.04 404	9.87 039	54
7	9.82 649	9.95 622	0.04 378	9.87 028	53
8	9.82 663	9.95 647	0.04 353	9.87 016	52
9	9.82 677	9.95 672	0.04 328	9.87 00$\bar{5}$	51
10	9.82 691	9.95 698	0.04 302	9.86 993	**50**
11	9.82 70$\bar{5}$	9.95 723	0.04 277	9.86 982	49
12	9.82 719	9.95 748	0.04 252	9.86 970	48
13	9.82 733	9.95 774	0.04 226	9.86 959	47
14	9.82 747	9.95 799	0.04 201	9.86 947	46
15	9.82 761	9.95 82$\bar{5}$	0.04 175	9.86 936	45
16	9.82 77$\bar{5}$	9.95 850	0.04 1$\bar{5}$0	9.86 924	44
17	9.82 788	9.95 875	0.04 12$\bar{5}$	9.86 913	43
18	9.82 802	9.95 901	0.04 099	9.86 902	42
19	9.82 816	9.95 926	0.04 074	9.86 890	41
20	9.82 830	9.95 952	0.04 048	9.86 879	**40**
21	9.82 844	9.95 977	0.04 023	9.86 867	39
22	9.82 858	9.96 002	0.03 998	9.86 855	38
23	9.82 872	9.96 028	0.03 972	9.86 844	37
24	9.82 885	9.96 053	0.03 947	9.86 832	36
25	9.82 899	9.96 078	0.03 922	9.86 821	35
26	9.82 913	9.96 104	0.03 896	9.86 809	34
27	9.82 927	9.96 129	0.03 871	9.86 798	33
28	9.82 941	9.96 15$\bar{5}$	0.03 845	9.86 786	32
29	9.82 95$\bar{5}$	9.96 180	0.03 820	9.86 77$\bar{5}$	31
30	9.82 968	9.96 205	0.03 79$\bar{5}$	9.86 763	**30**
31	9.82 982	9.96 231	0.03 769	9.86 752	29
32	9.82 996	9.96 256	0.03 744	9.86 740	28
33	9.83 010	9.96 281	0.03 719	9.86 728	27
34	9.83 023	9.96 307	0.03 693	9.86 717	26
35	9.83 037	9.96 332	0.03 668	9.86 705	25
36	9.83 051	9.96 357	0.03 643	9.86 694	24
37	9.83 06$\bar{5}$	9.96 383	0.03 617	9.86 682	23
38	9.83 078	9.96 408	0.03 592	9.86 670	22
39	9.83 092	9.96 433	0.03 567	9.86 659	21
40	9.83 106	9.96 459	0.03 541	9.86 647	**20**
41	9.83 120	9.96 484	0.03 516	9.86 635	19
42	9.83 133	9.96 510	0.03 490	9.86 624	18
43	9.83 147	9.96 53$\bar{5}$	0.03 465	9.86 612	17
44	9.83 161	9.96 560	0.03 440	9.86 600	16
45	9.83 174	9.96 586	0.03 414	9.86 589	15
46	9.83 188	9.96 611	0.03 389	9.86 577	14
47	9.83 202	9.96 636	0.03 364	9.86 565	13
48	9.83 215	9.96 662	0.03 338	9.86 554	12
49	9.83 229	9.96 687	0.03 313	9.86 542	11
50	9.83 242	9.96 712	0.03 288	9.86 530	**10**
51	9.83 256	9.96 738	0.03 262	9.86 518	9
52	9.83 270	9.96 763	0.03 237	9.86 507	8
53	9.83 283	9.96 788	0.03 212	9.86 495	7
54	9.83 297	9.96 814	0.03 186	9.86 483	6
55	9.83 310	9.96 839	0.03 161	9.86 472	5
56	9.83 324	9.96 864	0.03 136	9.86 460	4
57	9.83 338	9.96 890	0.03 110	9.86 448	3
58	9.83 351	9.96 91$\bar{5}$	0.03 085	9.86 436	2
59	9.83 36$\bar{5}$	9.96 940	0.03 060	9.86 42$\bar{5}$	1
60	9.83 378	9.96 966	0.03 034	9.86 413	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

47°

43°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.83 378	9.96 966	0.03 034	9.86 413	**60**
1	9.83 392	9.96 991	0.03 009	9.86 401	59
2	9.83 405	9.97 016	0.02 984	9.86 389	58
3	9.83 419	9.97 042	0.02 958	9.86 377	57
4	9.83 432	9.97 067	0.02 933	9.86 366	56
5	9.83 446	9.97 092	0.02 908	9.86 354	55
6	9.83 459	9.97 118	0.02 882	9.86 342	54
7	9.83 473	9.97 143	0.02 857	9.86 330	53
8	9.83 486	9.97 168	0.02 832	9.86 318	52
9	9.83 5̄00	9.97 193	0.02 807	9.86 306	51
10	9.83 513	9.97 219	0.02 781	9.86 295̄	**50**
11	9.83 527	9.97 244	0.02 756	9.86 283	49
12	9.83 540	9.97 269	0.02 731	9.86 271	48
13	9.83 554	9.97 295̄	0.02 705	9.86 259	47
14	9.83 567	9.97 320	0.02 680	9.86 247	46
15	9.83 581	9.97 345	0.02 655̄	9.86 235	45
16	9.83 594	9.97 371	0.02 629	9.86 223	44
17	9.83 608	9.97 396	0.02 604	9.86 211	43
18	9.83 621	9.97 421	0.02 579	9.86 200	42
19	9.83 634	9.97 447	0.02 553	9.86 188	41
20	9.83 648	9.97 472	0.02 528	9.86 176	**40**
21	9.83 661	9.97 497	0.02 503	9.86 164	39
22	9.83 674	9.97 523	0.02 477	9.86 152	38
23	9.83 688	9.97 548	0.02 452	9.86 140	37
24	9.83 701	9.97 573	0.02 427	9.86 128	36
25	9.83 715̄	9.97 598	0.02 402	9.86 116	35
26	9.83 728	9.97 624	0.02 376	9.86 104	34
27	9.83 741	9.97 649	0.02 351	9.86 092	33
28	9.83 755̄	9.97 674	0.02 326	9.86 080	32
29	9.83 768	9.97 700	0.02 300	9.86 068	31
30	9.83 781	9.97 725̄	0.02 275	9.86 056	**30**
31	9.83 795̄	9.97 750	0.02 25̄0	9.86 044	29
32	9.83 808	9.97 776	0.02 224	9.86 032	28
33	9.83 821	9.97 801	0.02 199	9.86 020	27
34	9.83 834	9.97 826	0.02 174	9.86 008	26
35	9.83 848	9.97 851	0.02 149	9.85 996	25
36	9.83 861	9.97 877	0.02 123	9.85 984	24
37	9.83 874	9.97 902	0.02 098	9.85 972	23
38	9.83 887	9.97 927	0.02 073	9.85 960	22
39	9.83 901	9.97 953	0.02 047	9.85 948	21
40	9.83 914	9.97 978	0.02 022	9.85 936	**20**
41	9.83 927	9.98 003	0.01 997	9.85 924	19
42	9.83 940	9.98 029	0.01 971	9.85 912	18
43	9.83 954	9.98 054	0.01 946	9.85 900	17
44	9.83 967	9.98 079	0.01 921	9.85 888	16
45	9.83 980	9.98 104	0.01 896	9.85 876	15
46	9.83 993	9.98 130	0.01 870	9.85 864	14
47	9.84 006	9.98 155	0.01 845̄	9.85 851	13
48	9.84 020	9.98 180	0.01 820	9.85 839	12
49	9.84 033	9.98 206	0.01 794	9.85 827	11
50	9.84 046	9.98 231	0.01 769	9.85 815	**10**
51	9.84 059	9.98 256	0.01 744	9.85 803	9
52	9.84 072	9.98 281	0.01 719	9.85 791	8
53	9.84 085	9.98 307	0.01 693	9.85 779	7
54	9.84 098	9.98 332	0.01 668	9.85 766	6
55	9.84 112	9.98 357	0.01 643	9.85 754	5
56	9.84 125̄	9.98 383	0.01 617	9.85 742	4
57	9.84 138	9.98 408	0.01 592	9.85 730	3
58	9.84 151	9.98 433	0.01 567	9.85 718	2
59	9.84 164	9.98 458	0.01 542	9.85 706	1
60	9.84 177	9.98 484	0.01 516	9.85 693	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

46°

44°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.84 177	9.98 484	0.01 516	9.85 693	60
1	9.84 190	9.98 509	0.01 491	9.85 681	59
2	9.84 203	9.98 534	0.01 466	9.85 669	58
3	9.84 216	9.98 560	0.01 440	9.85 657	57
4	9.84 229	9.98 585	0.01 415	9.85 645	56
5	9.84 242	9.98 610	0.01 390	9.85 632	55
6	9.84 255	9.98 635	0.01 365	9.85 620	54
7	9.84 269	9.98 661	0.01 339	9.85 608	53
8	9.84 282	9.98 686	0.01 314	9.85 596	52
9	9.84 295	9.98 711	0.01 289	9.85 583	51
10	9.84 308	9.98 737	0.01 263	9.85 571	50
11	9.84 321	9.98 762	0.01 238	9.85 559	49
12	9.84 334	9.98 787	0.01 213	9.85 547	48
13	9.84 347	9.98 812	0.01 188	9.85 534	47
14	9.84 360	9.98 838	0.01 162	9.85 522	46
15	9.84 373	9.98 863	0.01 137	9.85 510	45
16	9.84 385	9.98 888	0.01 112	9.85 497	44
17	9.84 398	9.98 913	0.01 087	9.85 485	43
18	9.84 411	9.98 939	0.01 061	9.85 473	42
19	9.84 424	9.98 964	0.01 036	9.85 460	41
20	9.84 437	9.98 989	0.01 011	9.85 448	40
21	9.84 450	9.99 015	0.00 985	9.85 436	39
22	9.84 463	9.99 040	0.00 960	9.85 423	38
23	9.84 476	9.99 065	0.00 935	9.85 411	37
24	9.84 489	9.99 090	0.00 910	9.85 399	36
25	9.84 502	9.99 116	0.00 884	9.85 386	35
26	9.84 515	9.99 141	0.00 859	9.85 374	34
27	9.84 528	9.99 166	0.00 834	9.85 361	33
28	9.84 540	9.99 191	0.00 809	9.85 349	32
29	9.84 553	9.99 217	0.00 783	9.85 337	31
30	9.84 566	9.99 242	0.00 758	9.85 324	30
31	9.84 579	9.99 267	0.00 733	9.85 312	29
32	9.84 592	9.99 293	0.00 707	9.85 299	28
33	9.84 605	9.99 318	0.00 682	9.85 287	27
34	9.84 618	9.99 343	0.00 657	9.85 274	26
35	9.84 630	9.99 368	0.00 632	9.85 262	25
36	9.84 643	9.99 394	0.00 606	9.85 250	24
37	9.84 656	9.99 419	0.00 581	9.85 237	23
38	9.84 669	9.99 444	0.00 556	9.85 225	22
39	9.84 682	9.99 469	0.00 531	9.85 212	21
40	9.84 694	9.99 495	0.00 505	9.85 200	20
41	9.84 707	9.99 520	0.00 480	9.85 187	19
42	9.84 720	9.99 545	0.00 455	9.85 175	18
43	9.84 733	9.99 570	0.00 430	9.85 162	17
44	9.84 745	9.99 596	0.00 404	9.85 150	16
45	9.84 758	9.99 621	0.00 379	9.85 137	15
46	9.84 771	9.99 646	0.00 354	9.85 125	14
47	9.84 784	9.99 672	0.00 328	9.85 112	13
48	9.84 796	9.99 697	0.00 303	9.85 100	12
49	9.84 809	9.99 722	0.00 278	9.85 087	11
50	9.84 822	9.99 747	0.00 253	9.85 074	10
51	9.84 835	9.99 773	0.00 227	9.85 062	9
52	9.84 847	9.99 798	0.00 202	9.85 049	8
53	9.84 860	9.99 823	0.00 177	9.85 037	7
54	9.84 873	9.99 848	0.00 152	9.85 024	6
55	9.84 885	9.99 874	0.00 126	9.85 012	5
56	9.84 898	9.99 899	0.00 101	9.84 999	4
57	9.84 911	9.99 924	0.00 076	9.84 986	3
58	9.84 923	9.99 949	0.00 051	9.84 974	2
59	9.84 936	9.99 975	0.00 025	9.84 961	1
60	9.84 949	10.00 000	0.00 000	9.84 949	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

45°

TABLE III

S AND T TABLE

to be used when the angle is between 0° and 2°
or between 88° and 90°.

FORMULAS

$\log \sin x = \log x \text{ (in seconds)} + S$

$\log \tan x = \log x \text{ (in seconds)} + T$

Angle	x''	log sin x	S	log tan x	T
0° 00′	0	$-\infty$	4.68557	$-\infty$	4.68557
0° 03′	180	6.94085	4.68557	6.94085	4.68557
0° 04′	240	7.06579	4.68557	7.06579	4.68558
0° 28′	1680	7.91088	4.68557	7.91089	4.68558
0° 29′	1740	7.92612	4.68557	7.92613	4.68559
0° 40′	2400	8.06578	4.68557	8.06581	4.68559
0° 41′	2460	8.07650	4.68556	8.07653	4.68560
0° 49′	2940	8.15391	4.68556	8.15395	4.68560
0° 50′	3000	8.16268	4.68556	8.16273	4.68561
0° 56′	3360	8.21189	4.68556	8.21195	4.68561
0° 57′	3420	8.21958	4.68555	8.21964	4.68561
0° 58′	3480	8.22713	4.68555	8.22720	4.68562
1° 03′	3780	8.26304	4.68555	8.26312	4.68562
1° 04′	3840	8.26988	4.68555	8.26996	4.68563
1° 09′	4140	8.30255	4.68555	8.30263	4.68563
1° 10′	4200	8.30879	4.68554	8.30888	4.68563
1° 11′	4260	8.31495	4.68554	8.31505	4.68564
1° 15′	4500	8.33875	4.68554	8.33886	4.68564
1° 16′	4560	8.34450	4.68554	8.34461	4.68565
1° 20′	4800	8.36678	4.68554	8.36689	4.68565
1° 21′	4860	8.37217	4.68553	8.37229	4.68566
1° 25′	5100	8.39310	4.68553	8.39323	4.68566
1° 26′	5160	8.39818	4.68553	8.39832	4.68567
1° 30′	5400	8.41792	4.68553	8.41807	4.68567
1° 31′	5460	8.42272	4.68552	8.42287	4.68568
1° 34′	5640	8.43680	4.68552	8.43696	4.68568
1° 35′	5700	8.44139	4.68552	8.44156	4.68569
1° 38′	5880	8.45489	4.68552	8.45507	4.68569
1° 39′	5940	8.45930	4.68551	8.45948	4.68569
1° 40′	6000	8.46366	4.68551	8.46385	4.68570
1° 43′	6180	8.47650	4.68551	8.47669	4.68570
1° 44′	6240	8.48069	4.68551	8.48089	4.68571
1° 46′	6360	8.48896	4.68551	8.48917	4.68571
1° 47′	6420	8.49304	4.68550	8.49325	4.68572
1° 50′	6600	8.50504	4.68550	8.50527	4.68572
1° 51′	6660	8.50897	4.68550	8.50920	4.68573
1° 54′	6840	8.52055	4.68550	8.52079	4.68573
1° 55′	6900	8.52434	4.68549	8.52459	4.68574
1° 57′	7020	8.53183	4.68549	8.53208	4.68574
1° 58′	7080	8.53552	4.68549	8.53578	4.68575
2° 00′	7200	8.54282	4.68549	8.54308	4.68575

′	0°		1°		2°		3°		4°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0000	1.000	0175	9998	0349	9994	0523	9986	0698	9976	60
1	0003	1.000	0177	9998	0352	9994	0526	9986	0700	9975	59
2	0006	1.000	0180	9998	0355	9994	0529	9986	0703	9975	58
3	0009	1.000	0183	9998	0358	9994	0532	9986	0706	9975	57
4	0012	1.000	0186	9998	0361	9993	0535	9986	0709	9975	56
5	0015	1.000	0189	9998	0364	9993	0538	9986	0712	9975	55
6	0017	1.000	0192	9998	0366	9993	0541	9985	0715	9974	54
7	0020	1.000	0195	9998	0369	9993	0544	9985	0718	9974	53
8	0023	1.000	0198	9998	0372	9993	0547	9985	0721	9974	52
9	0026	1.000	0201	9998	0375	9993	0550	9985	0724	9974	51
10	0029	1.000	0204	9998	0378	9993	0552	9985	0727	9974	50
11	0032	1.000	0207	9998	0381	9993	0555	9985	0729	9973	49
12	0035	1.000	0209	9998	0384	9993	0558	9984	0732	9973	48
13	0038	1.000	0212	9998	0387	9993	0561	9984	0735	9973	47
14	0041	1.000	0215	9998	0390	9992	0564	9984	0738	9973	46
15	0044	1.000	0218	9998	0393	9992	0567	9984	0741	9973	45
16	0047	1.000	0221	9998	0396	9992	0570	9984	0744	9972	44
17	0049	1.000	0224	9997	0398	9992	0573	9984	0747	9972	43
18	0052	1.000	0227	9997	0401	9992	0576	9983	0750	9972	42
19	0055	1.000	0230	9997	0404	9992	0579	9983	0753	9972	41
20	0058	1.000	0233	9997	0407	9992	0581	9983	0756	9971	40
21	0061	1.000	0236	9997	0410	9992	0584	9983	0758	9971	39
22	0064	1.000	0239	9997	0413	9991	0587	9983	0761	9971	38
23	0067	1.000	0241	9997	0416	9991	0590	9983	0764	9971	37
24	0070	1.000	0244	9997	0419	9991	0593	9982	0767	9971	36
25	0073	1.000	0247	9997	0422	9991	0596	9982	0770	9970	35
26	0076	1.000	0250	9997	0425	9991	0599	9982	0773	9970	34
27	0079	1.000	0253	9997	0427	9991	0602	9982	0776	9970	33
28	0081	1.000	0256	9997	0430	9991	0605	9982	0779	9970	32
29	0084	1.000	0259	9997	0433	9991	0608	9982	0782	9969	31
30	0087	1.000	0262	9997	0436	9990	0610	9981	0785	9969	30
31	0090	1.000	0265	9996	0439	9990	0613	9981	0787	9969	29
32	0093	1.000	0268	9996	0442	9990	0616	9981	0790	9969	28
33	0096	1.000	0270	9996	0445	9990	0619	9981	0793	9968	27
34	0099	1.000	0273	9996	0448	9990	0622	9981	0796	9968	26
35	0102	9999	0276	9996	0451	9990	0625	9980	0799	9968	25
36	0105	9999	0279	9996	0454	9990	0628	9980	0802	9968	24
37	0108	9999	0282	9996	0457	9990	0631	9980	0805	9968	23
38	0111	9999	0285	9996	0459	9989	0634	9980	0808	9967	22
39	0113	9999	0288	9996	0462	9989	0637	9980	0811	9967	21
40	0116	9999	0291	9996	0465	9989	0640	9980	0814	9967	20
41	0119	9999	0294	9996	0468	9989	0642	9979	0816	9967	19
42	0122	9999	0297	9996	0471	9989	0645	9979	0819	9966	18
43	0125	9999	0300	9996	0474	9989	0648	9979	0822	9966	17
44	0128	9999	0302	9995	0477	9989	0651	9979	0825	9966	16
45	0131	9999	0305	9995	0480	9988	0654	9979	0828	9966	15
46	0134	9999	0308	9995	0483	9988	0657	9978	0831	9965	14
47	0137	9999	0311	9995	0486	9988	0660	9978	0834	9965	13
48	0140	9999	0314	9995	0488	9988	0663	9978	0837	9965	12
49	0143	9999	0317	9995	0491	9988	0666	9978	0840	9965	11
50	0145	9999	0320	9995	0494	9988	0669	9978	0843	9964	10
51	0148	9999	0323	9995	0497	9988	0671	9977	0845	9964	9
52	0151	9999	0326	9995	0500	9987	0674	9977	0848	9964	8
53	0154	9999	0329	9995	0503	9987	0677	9977	0851	9964	7
54	0157	9999	0332	9995	0506	9987	0680	9977	0854	9963	6
55	0160	9999	0334	9994	0509	9987	0683	9977	0857	9963	5
56	0163	9999	0337	9994	0512	9987	0686	9976	0860	9963	4
57	0166	9999	0340	9994	0515	9987	0689	9976	0863	9963	3
58	0169	9999	0343	9994	0518	9987	0692	9976	0866	9962	2
59	0172	9999	0346	9994	0520	9986	0695	9976	0869	9962	1
60	0175	9999	0349	9994	0523	9986	0698	9976	0872	9962	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	89°		88°		87°		86°		85°		′

′	5°		6°		7°		8°		9°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	0872	9962	1045	9945	1219	9925	1392	9903	1564	9877	**60**
1	0874	9962	1048	9945	1222	9925	1395	9902	1567	9876	59
2	0877	9961	1051	9945	1224	9925	1397	9902	1570	9876	58
3	0880	9961	1054	9944	1227	9924	1400	9901	1573	9876	57
4	0883	9961	1057	9944	1230	9924	1403	9901	1576	9875	56
5	0886	9961	1060	9944	1233	9924	1406	9901	1579	9875	**55**
6	0889	9960	1063	9943	1236	9923	1409	9900	1582	9874	54
7	0892	9960	1066	9943	1239	9923	1412	9900	1584	9874	53
8	0895	9960	1068	9943	1241	9923	1415	9899	1587	9873	52
9	0898	9960	1071	9942	1245	9922	1418	9899	1590	9873	51
10	0901	9959	1074	9942	1248	9922	1421	9899	1593	9872	**50**
11	0903	9959	1077	9942	1250	9922	1423	9898	1596	9872	49
12	0906	9959	1080	9942	1253	9921	1426	9898	1599	9871	48
13	0909	9959	1083	9941	1256	9921	1429	9897	1602	9871	47
14	0912	9958	1086	9941	1259	9920	1432	9897	1605	9870	46
15	0915	9958	1089	9941	1262	9920	1435	9897	1607	9870	**45**
16	0918	9958	1092	9940	1265	9920	1438	9896	1610	9869	44
17	0921	9958	1094	9940	1268	9919	1441	9896	1613	9869	43
18	0924	9957	1097	9940	1271	9919	1444	9895	1616	9869	42
19	0927	9957	1100	9939	1274	9919	1446	9895	1619	9868	41
20	0929	9957	1103	9939	1276	9918	1449	9894	1622	9868	**40**
21	0932	9956	1106	9939	1279	9918	1452	9894	1625	9867	39
22	0935	9956	1109	9938	1282	9917	1455	9894	1628	9867	38
23	0938	9956	1112	9938	1285	9917	1458	9893	1630	9866	37
24	0941	9956	1115	9938	1288	9917	1461	9893	1633	9866	36
25	0944	9955	1118	9937	1291	9916	1464	9892	1636	9865	**35**
26	0947	9955	1120	9937	1294	9916	1467	9892	1639	9865	34
27	0950	9955	1123	9937	1297	9916	1469	9891	1642	9864	33
28	0953	9955	1126	9936	1299	9915	1472	9891	1645	9864	32
29	0956	9954	1129	9936	1302	9915	1475	9891	1648	9863	31
30	0958	9954	1132	9936	1305	9914	1478	9890	1650	9863	**30**
31	0961	9954	1135	9935	1308	9914	1481	9890	1653	9862	29
32	0964	9953	1138	9935	1311	9914	1484	9889	1656	9862	28
33	0967	9953	1141	9935	1314	9913	1487	9889	1659	9861	27
34	0970	9953	1144	9934	1317	9913	1490	9888	1662	9861	26
35	0973	9953	1146	9934	1320	9913	1492	9888	1665	9860	**25**
36	0976	9952	1149	9934	1323	9912	1495	9888	1668	9860	24
37	0979	9952	1152	9933	1325	9912	1498	9887	1671	9859	23
38	0982	9952	1155	9933	1328	9911	1501	9887	1673	9859	22
39	0985	9951	1158	9933	1331	9911	1504	9886	1676	9859	21
40	0987	9951	1161	9932	1334	9911	1507	9886	1679	9858	**20**
41	0990	9951	1164	9932	1337	9910	1510	9885	1682	9858	19
42	0993	9951	1167	9932	1340	9910	1513	9885	1685	9857	18
43	0996	9950	1170	9931	1343	9909	1515	9884	1688	9857	17
44	0999	9950	1172	9931	1346	9909	1518	9884	1691	9856	16
45	1002	9950	1175	9931	1349	9909	1521	9884	1693	9856	**15**
46	1005	9949	1178	9930	1351	9908	1524	9883	1696	9855	14
47	1008	9949	1181	9930	1354	9908	1527	9883	1699	9855	13
48	1011	9949	1184	9930	1357	9907	1530	9882	1702	9854	12
49	1013	9949	1187	9929	1360	9907	1533	9882	1705	9854	11
50	1016	9948	1190	9929	1363	9907	1536	9881	1708	9853	**10**
51	1019	9948	1193	9929	1366	9906	1538	9881	1711	9853	9
52	1022	9948	1196	9928	1369	9906	1541	9880	1714	9852	8
53	1025	9947	1198	9928	1372	9905	1544	9880	1716	9852	7
54	1028	9947	1201	9928	1374	9905	1547	9880	1719	9851	6
55	1031	9947	1204	9927	1377	9905	1550	9879	1722	9851	**5**
56	1034	9946	1207	9927	1380	9904	1553	9879	1725	9850	4
57	1037	9946	1210	9927	1383	9904	1556	9878	1728	9850	3
58	1039	9946	1213	9926	1386	9903	1559	9878	1731	9849	2
59	1042	9946	1216	9926	1389	9903	1561	9877	1734	9849	1
60	1045	9945	1219	9925	1392	9903	1564	9877	1736	9848	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	84°		83°		82°		81°		80°		′

′	10°		11°		12°		13°		14°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	1736	9848	1908	9816	2079	9781	2250	9744	2419	9703	60
1	1739	9848	1911	9816	2082	9781	2252	9743	2422	9702	59
2	1742	9847	1914	9815	2085	9780	2255	9742	2425	9702	58
3	1745	9847	1917	9815	2088	9780	2258	9742	2428	9701	57
4	1748	9846	1920	9814	2090	9779	2261	9741	2431	9700	56
5	1751	9846	1922	9813	2093	9778	2264	9740	2433	9699	55
6	1754	9845	1925	9813	2096	9778	2267	9740	2436	9699	54
7	1757	9845	1928	9812	2099	9777	2269	9739	2439	9698	53
8	1759	9844	1931	9812	2102	9777	2272	9738	2442	9697	52
9	1762	9843	1934	9811	2105	9776	2275	9738	2445	9697	51
10	1765	9843	1937	9811	2108	9775	2278	9737	2447	9696	50
11	1768	9842	1939	9810	2110	9775	2281	9736	2450	9695	49
12	1771	9842	1942	9810	2113	9774	2284	9736	2453	9694	48
13	1774	9841	1945	9809	2116	9774	2286	9735	2456	9694	47
14	1777	9841	1948	9808	2119	9773	2289	9734	2459	9693	46
15	1779	9840	1951	9808	2122	9772	2292	9734	2462	9692	45
16	1782	9840	1954	9807	2125	9772	2295	9733	2464	9692	44
17	1785	9839	1957	9807	2127	9771	2298	9732	2467	9691	43
18	1788	9839	1959	9806	2130	9770	2300	9732	2470	9690	42
19	1791	9838	1962	9806	2133	9770	2303	9731	2473	9689	41
20	1794	9838	1965	9805	2136	9769	2306	9730	2476	9689	40
21	1797	9837	1968	9804	2139	9769	2309	9730	2478	9688	39
22	1799	9837	1971	9804	2142	9768	2312	9729	2481	9687	38
23	1802	9836	1974	9803	2145	9767	2315	9728	2484	9687	37
24	1805	9836	1977	9803	2147	9767	2317	9728	2487	9686	36
25	1808	9835	1979	9802	2150	9766	2320	9727	2490	9685	35
26	1811	9835	1982	9802	2153	9765	2323	9726	2493	9684	34
27	1814	9834	1985	9801	2156	9765	2326	9726	2495	9684	33
28	1817	9834	1988	9800	2159	9764	2329	9725	2498	9683	32
29	1819	9833	1991	9800	2162	9764	2332	9724	2501	9682	31
30	1822	9833	1994	9799	2164	9763	2334	9724	2504	9681	30
31	1825	9832	1997	9799	2167	9762	2337	9723	2507	9681	29
32	1828	9831	1999	9798	2170	9762	2340	9722	2509	9680	28
33	1831	9831	2002	9798	2173	9761	2343	9722	2512	9679	27
34	1834	9830	2005	9797	2176	9760	2346	9721	2515	9679	26
35	1837	9830	2008	9796	2179	9760	2349	9720	2518	9678	25
36	1840	9829	2011	9796	2181	9759	2351	9720	2521	9677	24
37	1842	9829	2014	9795	2184	9759	2354	9719	2524	9676	23
38	1845	9828	2016	9795	2187	9758	2357	9718	2526	9676	22
39	1848	9828	2019	9794	2190	9757	2360	9718	2529	9675	21
40	1851	9827	2022	9793	2193	9757	2363	9717	2532	9674	20
41	1854	9827	2025	9793	2196	9756	2366	9716	2535	9673	19
42	1857	9826	2028	9792	2198	9755	2368	9715	2538	9673	18
43	1860	9826	2031	9792	2201	9755	2371	9715	2540	9672	17
44	1862	9825	2034	9791	2204	9754	2374	9714	2543	9671	16
45	1865	9825	2036	9790	2207	9753	2377	9713	2546	9670	15
46	1868	9824	2039	9790	2210	9753	2380	9713	2549	9670	14
47	1871	9823	2042	9789	2213	9752	2383	9712	2552	9669	13
48	1874	9823	2045	9789	2215	9751	2385	9711	2554	9668	12
49	1877	9822	2048	9788	2218	9751	2388	9711	2557	9667	11
50	1880	9822	2051	9787	2221	9750	2391	9710	2560	9667	10
51	1882	9821	2054	9787	2224	9750	2394	9709	2563	9666	9
52	1885	9821	2056	9786	2227	9749	2397	9709	2566	9665	8
53	1888	9820	2059	9786	2230	9748	2399	9708	2569	9665	7
54	1891	9820	2062	9785	2233	9748	2402	9707	2571	9664	6
55	1894	9819	2065	9784	2235	9747	2405	9706	2574	9663	5
56	1897	9818	2068	9784	2238	9746	2408	9706	2577	9662	4
57	1900	9818	2071	9783	2241	9746	2411	9705	2580	9662	3
58	1902	9817	2073	9783	2244	9745	2414	9704	2583	9661	2
59	1905	9817	2076	9782	2247	9744	2416	9704	2585	9660	1
60	1908	9816	2079	9781	2250	9744	2419	9703	2588	9659	0
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′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
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1	2591	9659	2759	9612	2926	9562	3093	9510	3258	9454	59
2	2594	9658	2762	9611	2929	9561	3096	9509	3261	9453	58
3	2597	9657	2765	9610	2932	9560	3098	9508	3264	9452	57
4	2599	9656	2768	9609	2935	9560	3101	9507	3267	9451	56
5	2602	9655	2770	9609	2938	9559	3104	9506	3269	9450	55
6	2605	9655	2773	9608	2940	9558	3107	9505	3272	9449	54
7	2608	9654	2776	9607	2943	9557	3110	9504	3275	9449	53
8	2611	9653	2779	9606	2946	9556	3112	9503	3278	9448	52
9	2613	9652	2782	9605	2949	9555	3115	9502	3280	9447	51
10	2616	9652	2784	9605	2952	9555	3118	9502	3283	9446	50
11	2619	9651	2787	9604	2954	9554	3121	9501	3286	9445	49
12	2622	9650	2790	9603	2957	9553	3123	9500	3289	9444	48
13	2625	9649	2793	9602	2960	9552	3126	9499	3291	9443	47
14	2628	9649	2795	9601	2963	9551	3129	9498	3294	9442	46
15	2630	9648	2798	9600	2965	9550	3132	9497	3297	9441	45
16	2633	9647	2801	9600	2968	9549	3134	9496	3300	9440	44
17	2636	9646	2804	9599	2971	9548	3137	9495	3302	9439	43
18	2639	9646	2807	9598	2974	9548	3140	9494	3305	9438	42
19	2642	9645	2809	9597	2977	9547	3143	9493	3308	9437	41
20	2644	9644	2812	9596	2979	9546	3145	9492	3311	9436	40
21	2647	9643	2815	9596	2982	9545	3148	9492	3313	9435	39
22	2650	9642	2818	9595	2985	9544	3151	9491	3316	9434	38
23	2653	9642	2821	9594	2988	9543	3154	9490	3319	9433	37
24	2656	9641	2823	9593	2990	9542	3156	9489	3322	9432	36
25	2658	9640	2826	9592	2993	9542	3159	9488	3324	9431	35
26	2661	9639	2829	9591	2996	9541	3162	9487	3327	9430	34
27	2664	9639	2832	9591	2999	9540	3165	9486	3330	9429	33
28	2667	9638	2835	9590	3002	9539	3168	9485	3333	9428	32
29	2670	9637	2837	9589	3004	9538	3170	9484	3335	9427	31
30	2672	9636	2840	9588	3007	9537	3173	9483	3338	9426	30
31	2675	9636	2843	9587	3010	9536	3176	9482	3341	9425	29
32	2678	9635	2846	9587	3013	9535	3179	9481	3344	9424	28
33	2681	9634	2849	9586	3015	9535	3181	9480	3346	9423	27
34	2684	9633	2851	9585	3018	9534	3184	9480	3349	9423	26
35	2686	9632	2854	9584	3021	9533	3187	9479	3352	9422	25
36	2689	9632	2857	9583	3024	9532	3190	9478	3355	9421	24
37	2692	9631	2860	9582	3026	9531	3192	9477	3357	9420	23
38	2695	9630	2862	9582	3029	9530	3195	9476	3360	9419	22
39	2698	9629	2865	9581	3032	9529	3198	9475	3363	9418	21
40	2700	9628	2868	9580	3035	9528	3201	9474	3365	9417	20
41	2703	9628	2871	9579	3038	9527	3203	9473	3368	9416	19
42	2706	9627	2874	9578	3040	9527	3206	9472	3371	9415	18
43	2709	9626	2876	9577	3043	9526	3209	9471	3374	9414	17
44	2712	9625	2879	9577	3046	9525	3212	9470	3376	9413	16
45	2714	9625	2882	9576	3049	9524	3214	9469	3379	9412	15
46	2717	9624	2885	9575	3051	9523	3217	9468	3382	9411	14
47	2720	9623	2888	9574	3054	9522	3220	9467	3385	9410	13
48	2723	9622	2890	9573	3057	9521	3223	9466	3387	9409	12
49	2726	9621	2893	9572	3060	9520	3225	9466	3390	9408	11
50	2728	9621	2896	9572	3062	9520	3228	9465	3393	9407	10
51	2731	9620	2899	9571	3065	9519	3231	9464	3396	9406	9
52	2734	9619	2901	9570	3068	9518	3234	9463	3398	9405	8
53	2737	9618	2904	9569	3071	9517	3236	9462	3401	9404	7
54	2740	9617	2907	9568	3074	9516	3239	9461	3404	9403	6
55	2742	9617	2910	9567	3076	9515	3242	9460	3407	9402	5
56	2745	9616	2913	9566	3079	9514	3245	9459	3409	9401	4
57	2748	9615	2915	9566	3082	9513	3247	9458	3412	9400	3
58	2751	9614	2918	9565	3085	9512	3250	9457	3415	9399	2
59	2754	9613	2921	9564	3087	9511	3253	9456	3417	9398	1
60	2756	9613	2924	9563	3090	9511	3256	9455	3420	9397	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	3420	9397	3584	9336	3746	9272	3907	9205	4067	9135	**60**
1	3423	9396	3586	9335	3749	9271	3910	9204	4070	9134	59
2	3426	9395	3589	9334	3751	9270	3913	9203	4073	9133	58
3	3428	9394	3592	9333	3754	9269	3915	9202	4075	9132	57
4	3431	9393	3595	9332	3757	9267	3918	9200	4078	9131	56
5	3434	9392	3597	9331	3760	9266	3921	9199	4081	9130	**55**
6	3437	9391	3600	9330	3762	9265	3923	9198	4083	9128	54
7	3439	9390	3603	9328	3765	9264	3926	9197	4086	9127	53
8	3442	9389	3605	9327	3768	9263	3929	9196	4089	9126	52
9	3445	9388	3608	9326	3770	9262	3931	9195	4091	9125	51
10	3448	9387	3611	9325	3773	9261	3934	9194	4094	9124	**50**
11	3450	9386	3614	9324	3776	9260	3937	9192	4097	9122	49
12	3453	9385	3616	9323	3778	9259	3939	9191	4099	9121	48
13	3456	9384	3619	9322	3781	9258	3942	9190	4102	9120	47
14	3458	9383	3622	9321	3784	9257	3945	9189	4105	9119	46
15	3461	9382	3624	9320	3786	9255	3947	9188	4107	9118	**45**
16	3464	9381	3627	9319	3789	9254	3950	9187	4110	9116	44
17	3467	9380	3630	9318	3792	9253	3953	9186	4112	9115	43
18	3469	9379	3633	9317	3795	9252	3955	9184	4115	9114	42
19	3472	9378	3635	9316	3797	9251	3958	9183	4118	9113	41
20	3475	9377	3638	9315	3800	9250	3961	9182	4120	9112	**40**
21	3478	9376	3641	9314	3803	9249	3963	9181	4123	9110	39
22	3480	9375	3643	9313	3805	9248	3966	9180	4126	9109	38
23	3483	9374	3646	9312	3808	9247	3969	9179	4128	9108	37
24	3486	9373	3649	9311	3811	9245	3971	9178	4131	9107	36
25	3488	9372	3651	9309	3813	9244	3974	9176	4134	9106	**35**
26	3491	9371	3654	9308	3816	9243	3977	9175	4136	9104	34
27	3494	9370	3657	9307	3819	9242	3979	9174	4139	9103	33
28	3497	9369	3660	9306	3821	9241	3982	9173	4142	9102	32
29	3499	9368	3662	9305	3824	9240	3985	9172	4144	9101	31
30	3502	9367	3665	9304	3827	9239	3987	9171	4147	9100	**30**
31	3505	9366	3668	9303	3830	9238	3990	9169	4150	9098	29
32	3508	9365	3670	9302	3832	9237	3993	9168	4152	9097	28
33	3510	9364	3673	9301	3835	9235	3995	9167	4155	9096	27
34	3513	9363	3676	9300	3838	9234	3998	9166	4158	9095	26
35	3516	9362	3679	9299	3840	9233	4001	9165	4160	9094	**25**
36	3518	9361	3681	9298	3843	9232	4003	9164	4163	9092	24
37	3521	9360	3684	9297	3846	9231	4006	9162	4165	9091	23
38	3524	9359	3687	9296	3848	9230	4009	9161	4168	9090	22
39	3527	9358	3689	9295	3851	9229	4011	9160	4171	9088	21
40	3529	9356	3692	9293	3854	9228	4014	9159	4173	9088	**20**
41	3532	9355	3695	9292	3856	9227	4017	9158	4176	9086	19
42	3535	9354	3697	9291	3859	9225	4019	9157	4179	9085	18
43	3537	9353	3700	9290	3862	9224	4022	9155	4181	9084	17
44	3540	9352	3703	9289	3864	9223	4025	9154	4184	9083	16
45	3543	9351	3706	9288	3867	9222	4027	9153	4187	9081	**15**
46	3546	9350	3708	9287	3870	9221	4030	9152	4189	9080	14
47	3548	9349	3711	9286	3872	9220	4033	9151	4192	9079	13
48	3551	9348	3714	9285	3875	9219	4035	9150	4195	9078	12
49	3554	9347	3716	9284	3878	9218	4038	9148	4197	9077	11
50	3557	9346	3719	9283	3881	9216	4041	9147	4200	9075	**10**
51	3559	9345	3722	9282	3883	9215	4043	9146	4202	9074	9
52	3562	9344	3724	9281	3886	9214	4046	9145	4205	9073	8
53	3565	9343	3727	9279	3889	9213	4049	9144	4208	9072	7
54	3567	9342	3730	9278	3891	9212	4051	9143	4210	9070	6
55	3570	9341	3733	9277	3894	9211	4054	9141	4213	9069	**5**
56	3573	9340	3735	9276	3897	9210	4057	9140	4216	9068	4
57	3576	9339	3738	9275	3899	9208	4059	9139	4218	9067	3
58	3578	9338	3741	9274	3902	9207	4062	9138	4221	9066	2
59	3581	9337	3743	9273	3905	9206	4065	9137	4224	9064	1
60	3584	9336	3746	9272	3907	9205	4067	9135	4226	9063	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
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1	4229	9062	4386	8987	4542	8909	4697	8828	4851	8745	59
2	4231	9061	4389	8985	4545	8907	4700	8827	4853	8743	58
3	4234	9059	4392	8984	4548	8906	4702	8825	4856	8742	57
4	4237	9058	4394	8983	4550	8905	4705	8824	4858	8741	56
5	4239	9057	4397	8982	4553	8903	4708	8823	4861	8739	55
6	4242	9056	4399	8980	4555	8902	4710	8821	4863	8738	54
7	4245	9054	4402	8979	4558	8901	4713	8820	4866	8736	53
8	4247	9053	4405	8978	4561	8899	4715	8819	4868	8735	52
9	4250	9052	4407	8976	4563	8898	4718	8817	4871	8733	51
10	4253	9051	4410	8975	4566	8897	4720	8816	4874	8732	50
11	4255	9050	4412	8974	4568	8895	4723	8814	4876	8731	49
12	4258	9048	4415	8973	4571	8894	4726	8813	4879	8729	48
13	4260	9047	4418	8971	4574	8893	4728	8812	4881	8728	47
14	4263	9046	4420	8970	4576	8892	4731	8810	4884	8726	46
15	4266	9045	4423	8969	4579	8890	4733	8809	4886	8725	45
16	4268	9043	4425	8967	4581	8889	4736	8808	4889	8724	44
17	4271	9042	4428	8966	4584	8888	4738	8806	4891	8722	43
18	4274	9041	4431	8965	4586	8886	4741	8805	4894	8721	42
19	4276	9040	4433	8964	4589	8885	4743	8803	4896	8719	41
20	4279	9038	4436	8962	4592	8884	4746	8802	4899	8718	40
21	4281	9037	4439	8961	4594	8882	4749	8801	4901	8716	39
22	4284	9036	4441	8960	4597	8881	4751	8799	4904	8715	38
23	4287	9035	4444	8958	4599	8879	4754	8798	4907	8714	37
24	4289	9033	4446	8957	4602	8878	4756	8796	4909	8712	36
25	4292	9032	4449	8956	4605	8877	4759	8795	4912	8711	35
26	4295	9031	4452	8955	4607	8875	4761	8794	4914	8709	34
27	4297	9030	4454	8953	4610	8874	4764	8792	4917	8708	33
28	4300	9028	4457	8952	4612	8873	4766	8791	4919	8706	32
29	4302	9027	4459	8951	4615	8871	4769	8790	4922	8705	31
30	4305	9026	4462	8949	4617	8870	4772	8788	4924	8704	30
31	4308	9025	4465	8948	4620	8869	4774	8787	4927	8702	29
32	4310	9023	4467	8947	4623	8867	4777	8785	4929	8701	28
33	4313	9022	4470	8945	4625	8866	4779	8784	4932	8699	27
34	4316	9021	4472	8944	4628	8865	4782	8783	4934	8698	26
35	4318	9020	4475	8943	4630	8863	4784	8781	4937	8696	25
36	4321	9018	4478	8942	4633	8862	4787	8780	4939	8695	24
37	4323	9017	4480	8940	4636	8861	4789	8778	4942	8694	23
38	4326	9016	4483	8939	4638	8859	4792	8777	4944	8692	22
39	4329	9015	4485	8938	4641	8858	4795	8776	4947	8691	21
40	4331	9013	4488	8936	4643	8857	4797	8774	4950	8689	20
41	4334	9012	4491	8935	4646	8855	4800	8773	4952	8688	19
42	4337	9011	4493	8934	4648	8854	4802	8771	4955	8686	18
43	4339	9010	4496	8932	4651	8853	4805	8770	4957	8685	17
44	4342	9008	4498	8931	4654	8851	4807	8769	4960	8683	16
45	4344	9007	4501	8930	4656	8850	4810	8767	4962	8682	15
46	4347	9006	4504	8928	4659	8849	4812	8766	4965	8681	14
47	4350	9004	4506	8927	4661	8847	4815	8764	4967	8679	13
48	4352	9003	4509	8926	4664	8846	4818	8763	4970	8678	12
49	4355	9002	4511	8925	4666	8844	4820	8762	4972	8676	11
50	4358	9001	4514	8923	4669	8843	4823	8760	4975	8675	10
51	4360	8999	4517	8922	4672	8842	4825	8759	4977	8673	9
52	4363	8998	4519	8921	4674	8840	4828	8757	4980	8672	8
53	4365	8997	4522	8919	4677	8839	4830	8756	4982	8670	7
54	4368	8996	4524	8918	4679	8838	4833	8755	4985	8669	6
55	4371	8994	4527	8917	4682	8836	4835	8753	4987	8668	5
56	4373	8993	4530	8915	4684	8835	4838	8752	4990	8666	4
57	4376	8992	4532	8914	4687	8834	4840	8750	4992	8665	3
58	4378	8990	4535	8913	4690	8832	4843	8749	4995	8663	2
59	4381	8989	4537	8911	4692	8831	4846	8748	4997	8662	1
60	4384	8988	4540	8910	4695	8829	4848	8746	5000	8660	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	5000	8660	5150	8572	5299	8480	5446	8387	5592	8290	**60**
1	5003	8659	5153	8570	5302	8479	5449	8385	5594	8289	59
2	5005	8657	5155	8569	5304	8477	5451	8384	5597	8287	58
3	5008	8656	5158	8567	5307	8476	5454	8382	5599	8285	57
4	5010	8654	5160	8566	5309	8474	5456	8380	5602	8284	56
5	5013	8653	5163	8564	5312	8473	5459	8379	5604	8282	**55**
6	5015	8652	5165	8563	5314	8471	5461	8377	5606	8281	54
7	5018	8650	5168	8561	5316	8470	5463	8376	5609	8279	53
8	5020	8649	5170	8560	5319	8468	5466	8374	5611	8277	52
9	5023	8647	5173	8558	5321	8467	5468	8372	5614	8276	51
10	5025	8646	5175	8557	5324	8465	5471	8371	5616	8274	**50**
11	5028	8644	5178	8555	5326	8463	5473	8369	5618	8272	49
12	5030	8643	5180	8554	5329	8462	5476	8368	5621	8271	48
13	5033	8641	5183	8552	5331	8460	5478	8366	5623	8269	47
14	5035	8640	5185	8551	5334	8459	5480	8364	5626	8268	46
15	5038	8638	5188	8549	5336	8457	5483	8363	5628	8266	**45**
16	5040	8637	5190	8548	5339	8456	5485	8361	5630	8264	44
17	5043	8635	5193	8546	5341	8454	5488	8360	5633	8263	43
18	5045	8634	5195	8545	5344	8453	5490	8358	5635	8261	42
19	5048	8632	5198	8543	5346	8451	5493	8356	5638	8259	41
20	5050	8631	5200	8542	5348	8450	5495	8355	5640	8258	**40**
21	5053	8630	5203	8540	5351	8448	5498	8353	5642	8256	39
22	5055	8628	5205	8539	5353	8446	5500	8352	5645	8254	38
23	5058	8627	5208	8537	5356	8445	5502	8350	5647	8253	37
24	5060	8625	5210	8536	5358	8443	5505	8348	5650	8251	36
25	5063	8624	5213	8534	5361	8442	5507	8347	5652	8249	**35**
26	5065	8622	5215	8532	5363	8440	5510	8345	5654	8248	34
27	5068	8621	5218	8531	5366	8439	5512	8344	5657	8246	33
28	5070	8619	5220	8529	5368	8437	5515	8342	5659	8245	32
29	5073	8618	5223	8528	5371	8435	5517	8340	5662	8243	31
30	5075	8616	5225	8526	5373	8434	5519	8339	5664	8241	**30**
31	5078	8615	5227	8525	5375	8432	5522	8337	5666	8240	29
32	5080	8613	5230	8523	5378	8431	5524	8336	5669	8238	28
33	5083	8612	5232	8522	5380	8429	5527	8334	5671	8236	27
34	5085	8610	5235	8520	5383	8428	5529	8332	5674	8235	26
35	5088	8609	5237	8519	5385	8426	5531	8331	5676	8233	**25**
36	5090	8607	5240	8517	5388	8425	5534	8329	5678	8231	24
37	5093	8606	5242	8516	5390	8423	5536	8328	5681	8230	23
38	5095	8604	5245	8514	5393	8421	5539	8326	5683	8228	22
39	5098	8603	5247	8513	5395	8420	5541	8324	5686	8226	21
40	5100	8601	5250	8511	5398	8418	5544	8323	5688	8225	**20**
41	5103	8600	5252	8510	5400	8417	5546	8321	5690	8223	19
42	5105	8599	5255	8508	5402	8415	5548	8320	5693	8221	18
43	5108	8597	5257	8507	5405	8414	5551	8318	5695	8220	17
44	5110	8596	5260	8505	5407	8412	5553	8316	5698	8218	16
45	5113	8594	5262	8504	5410	8410	5556	8315	5700	8216	**15**
46	5115	8593	5265	8502	5412	8409	5558	8313	5702	8215	14
47	5118	8591	5267	8500	5415	8407	5561	8311	5705	8213	13
48	5120	8590	5270	8499	5417	8406	5563	8310	5707	8211	12
49	5123	8588	5272	8497	5420	8404	5565	8308	5710	8210	11
50	5125	8587	5275	8496	5422	8403	5568	8307	5712	8208	**10**
51	5128	8585	5277	8494	5424	8401	5570	8305	5714	8207	9
52	5130	8584	5279	8493	5427	8399	5573	8303	5717	8205	8
53	5133	8582	5282	8491	5429	8398	5575	8302	5719	8203	7
54	5135	8581	5284	8490	5432	8396	5577	8300	5721	8202	6
55	5138	8579	5287	8488	5434	8395	5580	8299	5724	8200	**5**
56	5140	8578	5289	8487	5437	8393	5582	8297	5726	8198	4
57	5143	8576	5292	8485	5439	8391	5585	8295	5729	8197	3
58	5145	8575	5294	8484	5442	8390	5587	8294	5731	8195	2
59	5148	8573	5297	8482	5444	8388	5590	8292	5733	8193	1
60	5150	8572	5299	8480	5446	8387	5592	8290	5736	8192	**0**
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	sin	**cos**	**sin**	**cos**	**sin**	**cos**	**sin**	**cos**	**sin**	**cos**	
0	5736	8192	5878	8090	6018	7986	6157	7880	6293	7771	**60**
1	5738	8190	5880	8088	6020	7985	6159	7878	6295	7770	59
2	5741	8188	5883	8087	6023	7983	6161	7877	6298	7768	58
3	5743	8187	5885	8085	6025	7981	6163	7875	6300	7766	57
4	5745	8185	5887	8083	6027	7979	6166	7873	6302	7764	56
5	5748	8183	5890	8082	6030	7978	6168	7871	6305	7762	**55**
6	5750	8181	5892	8080	6032	7976	6170	7869	6307	7760	54
7	5752	8180	5894	8078	6034	7974	6173	7868	6309	7759	53
8	5755	8178	5897	8076	6037	7972	6175	7866	6311	7757	52
9	5757	8176	5899	8075	6039	7971	6177	7864	6314	7755	51
10	5760	8175	5901	8073	6041	7969	6180	7862	6316	7753	**50**
11	5762	8173	5904	8071	6044	7967	6182	7860	6318	7751	49
12	5764	8171	5906	8070	6046	7965	6184	7859	6320	7749	48
13	5767	8170	5908	8068	6048	7964	6186	7857	6323	7748	47
14	5769	8168	5911	8066	6051	7962	6189	7855	6325	7746	46
15	5771	8166	5913	8064	6053	7960	6191	7853	6327	7744	**45**
16	5774	8165	5915	8063	6055	7958	6193	7851	6329	7742	44
17	5776	8163	5918	8061	6058	7956	6196	7850	6332	7740	43
18	5779	8161	5920	8059	6060	7955	6198	7848	6334	7738	42
19	5781	8160	5922	8058	6062	7953	6200	7846	6336	7737	41
20	5783	8158	5925	8056	6065	7951	6202	7844	6338	7735	**40**
21	5786	8156	5927	8054	6067	7950	6205	7842	6341	7733	39
22	5788	8155	5930	8052	6069	7948	6207	7841	6343	7731	38
23	5790	8153	5932	8051	6071	7946	6209	7839	6345	7729	37
24	5793	8151	5934	8049	6074	7944	6211	7837	6347	7727	36
25	5795	8150	5937	8047	6076	7942	6214	7835	6350	7725	**35**
26	5798	8148	5939	8045	6078	7941	6216	7833	6352	7724	34
27	5800	8146	5941	8044	6081	7939	6218	7832	6354	7722	33
28	5802	8145	5944	8042	6083	7937	6221	7830	6356	7720	32
29	5805	8143	5946	8040	6085	7935	6223	7828	6359	7718	31
30	5807	8141	5948	8039	6088	7934	6225	7826	6361	7716	**30**
31	5809	8139	5951	8037	6090	7932	6227	7824	6363	7714	29
32	5812	8138	5953	8035	6092	7930	6230	7822	6365	7713	28
33	5814	8136	5955	8033	6095	7928	6232	7821	6368	7711	27
34	5816	8134	5958	8032	6097	7926	6234	7819	6370	7709	26
35	5819	8133	5960	8030	6099	7925	6237	7817	6372	7707	**25**
36	5821	8131	5962	8028	6101	7923	6239	7815	6374	7705	24
37	5824	8129	5965	8026	6104	7921	6241	7813	6376	7703	23
38	5826	8128	5967	8025	6106	7919	6243	7812	6379	7701	22
39	5828	8126	5969	8023	6108	7918	6246	7810	6381	7700	21
40	5831	8124	5972	8021	6111	7916	6248	7808	6383	7698	**20**
41	5833	8123	5974	8020	6113	7914	6250	7806	6385	7696	19
42	5835	8121	5976	8018	6115	7912	6252	7804	6388	7694	18
43	5838	8119	5979	8016	6118	7910	6255	7802	6390	7692	17
44	5840	8117	5981	8014	6120	7909	6257	7801	6392	7690	16
45	5842	8116	5983	8013	6122	7907	6259	7799	6394	7688	**15**
46	5845	8114	5986	8011	6124	7905	6262	7797	6397	7687	14
47	5847	8112	5988	8009	6127	7903	6264	7795	6399	7685	13
48	5850	8111	5990	8007	6129	7902	6266	7793	6401	7683	12
49	5852	8109	5993	8006	6131	7900	6268	7792	6403	7681	11
50	5854	8107	5995	8004	6134	7898	6271	7790	6406	7679	**10**
51	5857	8106	5997	8002	6136	7896	6273	7788	6408	7677	9
52	5859	8104	6000	8000	6138	7894	6275	7786	6410	7675	8
53	5861	8102	6002	7999	6141	7893	6277	7784	6412	7674	7
54	5864	8100	6004	7997	6143	7891	6280	7782	6414	7672	6
55	5866	8099	6007	7995	6145	7889	6282	7781	6417	7670	**5**
56	5868	8097	6009	7993	6147	7887	6284	7779	6419	7668	4
57	5871	8095	6011	7992	6150	7885	6286	7777	6421	7666	3
58	5873	8094	6014	7990	6152	7884	6289	7775	6423	7664	2
59	5875	8092	6016	7988	6154	7882	6291	7773	6426	7662	1
60	5878	8090	6018	7986	6157	7880	6293	7771	6428	7660	**0**
	cos	**sin**	**cos**	**sin**	**cos**	**sin**	**cos**	**sin**	**cos**	**sin**	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	
0	6428	7660	6561	7547	6691	7431	6820	7314	6947	7193	60
1	6430	7659	6563	7545	6693	7430	6822	7312	6949	7191	59
2	6432	7657	6565	7543	6696	7428	6824	7310	6951	7189	58
3	6435	7655	6567	7541	6698	7426	6826	7308	6953	7187	57
4	6437	7653	6569	7539	6700	7424	6828	7306	6955	7185	56
5	6439	7651	6572	7538	6702	7422	6831	7304	6957	7183	55
6	6441	7649	6574	7536	6704	7420	6833	7302	6959	7181	54
7	6443	7647	6576	7534	6706	7418	6835	7300	6961	7179	53
8	6446	7645	6578	7532	6709	7416	6837	7298	6963	7177	52
9	6448	7644	6580	7530	6711	7414	6839	7296	6965	7175	51
10	6450	7642	6583	7528	6713	7412	6841	7294	6967	7173	50
11	6452	7640	6585	7526	6715	7410	6843	7292	6970	7171	49
12	6455	7638	6587	7524	6717	7408	6845	7290	6972	7169	48
13	6457	7636	6589	7522	6719	7406	6848	7288	6974	7167	47
14	6459	7634	6591	7520	6722	7404	6850	7286	6976	7165	46
15	6461	7632	6593	7518	6724	7402	6852	7284	6978	7163	45
16	6463	7630	6596	7516	6726	7400	6854	7282	6980	7161	44
17	6466	7629	6598	7515	6728	7398	6856	7280	6982	7159	43
18	6468	7627	6600	7513	6730	7396	6858	7278	6984	7157	42
19	6470	7625	6602	7511	6732	7394	6860	7276	6986	7155	41
20	6472	7623	6604	7509	6734	7392	6862	7274	6988	7153	40
21	6475	7621	6607	7507	6737	7390	6865	7272	6990	7151	39
22	6477	7619	6609	7505	6739	7388	6867	7270	6992	7149	38
23	6479	7617	6611	7503	6741	7387	6869	7268	6995	7147	37
24	6481	7615	6613	7501	6743	7385	6871	7266	6997	7145	36
25	6483	7613	6615	7499	6745	7383	6873	7264	6999	7143	35
26	6486	7612	6617	7497	6747	7381	6875	7262	7001	7141	34
27	6488	7610	6620	7495	6749	7379	6877	7260	7003	7139	33
28	6490	7608	6622	7493	6752	7377	6879	7258	7005	7137	32
29	6492	7606	6624	7491	6754	7375	6881	7256	7007	7135	31
30	6494	7604	6626	7490	6756	7373	6884	7254	7009	7133	30
31	6497	7602	6628	7488	6758	7371	6886	7252	7011	7130	29
32	6499	7600	6631	7486	6760	7369	6888	7250	7013	7128	28
33	6501	7598	6633	7484	6762	7367	6890	7248	7015	7126	27
34	6503	7596	6635	7482	6764	7365	6892	7246	7017	7124	26
35	6506	7595	6637	7480	6767	7363	6894	7244	7019	7122	25
36	6508	7593	6639	7478	6769	7361	6896	7242	7022	7120	24
37	6510	7591	6641	7476	6771	7359	6898	7240	7024	7118	23
38	6512	7589	6644	7474	6773	7357	6900	7238	7026	7116	22
39	6514	7587	6646	7472	6775	7355	6903	7236	7028	7114	21
40	6517	7585	6648	7470	6777	7353	6905	7234	7030	7112	20
41	6519	7583	6650	7468	6779	7351	6907	7232	7032	7110	19
42	6521	7581	6652	7466	6782	7349	6909	7230	7034	7108	18
43	6523	7579	6654	7464	6784	7347	6911	7228	7036	7106	17
44	6525	7578	6657	7463	6786	7345	6913	7226	7038	7104	16
45	6528	7576	6659	7461	6788	7343	6915	7224	7040	7102	15
46	6530	7574	6661	7459	6790	7341	6917	7222	7042	7100	14
47	6532	7572	6663	7457	6792	7339	6919	7220	7044	7098	13
48	6534	7570	6665	7455	6794	7337	6921	7218	7046	7096	12
49	6536	7568	6667	7453	6797	7335	6924	7216	7048	7094	11
50	6539	7566	6670	7451	6799	7333	6926	7214	7050	7092	10
51	6541	7564	6672	7449	6801	7331	6928	7212	7053	7090	9
52	6543	7562	6674	7447	6803	7329	6930	7210	7055	7088	8
53	6545	7560	6676	7445	6805	7327	6932	7208	7057	7085	7
54	6547	7559	6678	7443	6807	7325	6934	7206	7059	7083	6
55	6550	7557	6680	7441	6809	7323	6936	7203	7061	7081	5
56	6552	7555	6683	7439	6811	7321	6938	7201	7063	7079	4
57	6554	7553	6685	7437	6814	7319	6940	7199	7065	7077	3
58	6556	7551	6687	7435	6816	7318	6942	7197	7067	7075	2
59	6558	7549	6689	7433	6818	7316	6944	7195	7069	7073	1
60	6561	7547	6691	7431	6820	7314	6947	7193	7071	7071	0
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	
′	49°		48°		47°		46°		45°		′

	0°		1°		2°		3°		4°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	0000	Infinite	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	**60**
1	0003	3437.75	0177	56.3506	0352	28.3994	0527	18.9755	0702	14.2411	59
2	0006	1718.87	0180	55.4415	0355	28.1664	0530	18.8711	0705	14.1821	58
3	0009	1145.92	0183	54.5613	0358	27.9372	0533	18.7678	0708	14.1235	57
4	0012	859.436	0186	53.7086	0361	27.7117	0536	18.6656	0711	14.0655	56
5	0015	687.549	0189	52.8821	0364	27.4899	0539	18.5645	0714	14.0079	**55**
6	0017	572.957	0192	52.0807	0367	27.2715	0542	18.4645	0717	13.9507	54
7	0020	491.106	0195	51.3032	0370	27.0566	0544	18.3655	0720	13.8940	53
8	0023	429.718	0198	50.5485	0373	26.8450	0547	18.2677	0723	13.8378	52
9	0026	381.971	0201	49.8157	0375	26.6367	0550	18.1708	0726	13.7821	51
10	0029	343.774	0204	49.1039	0378	26.4316	0553	18.0750	0729	13.7267	**50**
11	0032	312.521	0207	48.4121	0381	26.2296	0556	17.9802	0731	13.6719	49
12	0035	286.478	0209	47.7395	0384	26.0307	0559	17.8863	0734	13.6174	48
13	0038	264.441	0212	47.0853	0387	25.8348	0562	17.7934	0737	13.5634	47
14	0041	245.552	0215	46.4489	0390	25.6418	0565	17.7015	0740	13.5098	46
15	0044	229.182	0218	45.8294	0393	25.4517	0568	17.6106	0743	13.4566	**45**
16	0047	214.858	0221	45.2261	0396	25.2644	0571	17.5205	0746	13.4039	44
17	0049	202.219	0224	44.6386	0399	25.0798	0574	17.4314	0749	13.3515	43
18	0052	190.984	0227	44.0661	0402	24.8978	0577	17.3432	0752	13.2996	42
19	0055	180.932	0230	43.5081	0405	24.7185	0580	17.2558	0755	13.2480	41
20	0058	171.885	0233	42.9641	0407	24.5418	0582	17.1693	0758	13.1969	**40**
21	0061	163.700	0236	42.4335	0410	24.3675	0585	17.0837	0761	13.1461	39
22	0064	156.259	0239	41.9158	0413	24.1957	0588	16.9990	0764	13.0958	38
23	0067	149.465	0241	41.4106	0416	24.0263	0591	16.9150	0767	13.0458	37
24	0070	143.237	0244	40.9174	0419	23.8593	0594	16.8319	0769	12.9962	36
25	0073	137.507	0247	40.4358	0422	23.6945	0597	16.7496	0772	12.9469	**35**
26	0076	132.219	0250	39.9655	0425	23.5321	0600	16.6681	0775	12.8981	34
27	0079	127.321	0253	39.5059	0428	23.3718	0603	16.5874	0778	12.8496	33
28	0081	122.774	0256	39.0568	0431	23.2137	0606	16.5075	0781	12.8014	32
29	0084	118.540	0259	38.6177	0434	23.0577	0609	16.4283	0784	12.7536	31
30	0087	114.589	0262	38.1885	0437	22.9038	0612	16.3499	0787	12.7062	**30**
31	0090	110.892	0265	37.7686	0440	22.7519	0615	16.2722	0790	12.6591	29
32	0093	107.426	0268	37.3579	0442	22.6020	0617	16.1952	0793	12.6124	28
33	0096	104.171	0271	36.9560	0445	22.4541	0620	16.1190	0796	12.5660	27
34	0099	101.107	0274	36.5627	0448	22.3081	0623	16.0435	0799	12.5199	26
35	0102	98.2179	0276	36.1776	0451	22.1640	0626	15.9687	0802	12.4742	**25**
36	0105	95.4895	0279	35.8006	0454	22.0217	0629	15.8945	0805	12.4288	24
37	0108	92.9085	0282	35.4313	0457	21.8813	0632	15.8211	0808	12.3838	23
38	0111	90.4633	0285	35.0695	0460	21.7426	0635	15.7483	0810	12.3390	22
39	0113	88.1436	0288	34.7151	0463	21.6056	0638	15.6762	0813	12.2946	21
40	0116	85.9398	0291	34.3678	0466	21.4704	0641	15.6048	0816	12.2505	**20**
41	0119	83.8435	0294	34.0273	0469	21.3369	0644	15.5340	0819	12.2067	19
42	0122	81.8470	0297	33.6935	0472	21.2049	0647	15.4638	0822	12.1632	18
43	0125	79.9434	0300	33.3662	0475	21.0747	0650	15.3943	0825	12.1201	17
44	0128	78.1263	0303	33.0452	0477	20.9460	0653	15.3254	0828	12.0772	16
45	0131	76.3900	0306	32.7303	0480	20.8188	0655	15.2571	0831	12.0346	**15**
46	0134	74.7292	0308	32.4213	0483	20.6932	0658	15.1893	0834	11.9923	14
47	0137	73.1390	0311	32.1181	0486	20.5691	0661	15.1222	0837	11.9504	13
48	0140	71.6151	0314	31.8205	0489	20.4465	0664	15.0557	0840	11.9087	12
49	0143	70.1533	0317	31.5284	0492	20.3253	0667	14.9898	0843	11.8673	11
50	0146	68.7501	0320	31.2416	0495	20.2056	0670	14.9244	0846	11.8262	**10**
51	0148	67.4019	0323	30.9599	0498	20.0872	0673	14.8596	0849	11.7853	9
52	0151	66.1055	0326	30.6833	0501	19.9702	0676	14.7954	0851	11.7448	8
53	0154	64.8580	0329	30.4116	0504	19.8546	0679	14.7317	0854	11.7045	7
54	0157	63.6567	0332	30.1446	0507	19.7403	0682	14.6685	0857	11.6645	6
55	0160	62.4992	0335	29.8823	0509	19.6273	0685	14.6059	0860	11.6248	**5**
56	0163	61.3829	0338	29.6245	0512	19.5156	0688	14.5438	0863	11.5853	4
57	0166	60.3058	0340	29.3711	0515	19.4051	0690	14.4823	0866	11.5461	3
58	0169	59.2659	0343	29.1220	0518	19.2959	0693	14.4212	0869	11.5072	2
59	0172	58.2612	0346	28.8771	0521	19.1879	0696	14.3607	0872	11.4685	1
60	0175	57.2900	0349	28.6363	0524	19.0811	0699	14.3007	0875	11.4301	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	89°		88°		87°		86°		85°		′

′	5°		6°		7°		8°		9°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	0875	11.4301	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	**60**
1	0878	11.3919	1054	9.4878	1231	8.1248	1408	7.1004	1587	6.3019	59
2	0881	11.3540	1057	9.4614	1234	8.1054	1411	7.0855	1590	6.2901	58
3	0884	11.3163	1060	9.4352	1237	8.0860	1414	7.0706	1593	6.2783	57
4	0887	11.2789	1063	9.4090	1240	8.0667	1417	7.0558	1596	6.2666	56
5	0890	11.2417	1066	9.3831	1243	8.0476	1420	7.0410	1599	6.2549	**55**
6	0892	11.2048	1069	9.3572	1246	8.0285	1423	7.0264	1602	6.2432	54
7	0895	11.1681	1072	9.3315	1249	8.0095	1426	7.0117	1605	6.2316	53
8	0898	11.1316	1075	9.3060	1251	7.9906	1429	6.9972	1608	6.2200	52
9	0901	11.0954	1078	9.2806	1254	7.9718	1432	6.9827	1611	6.2085	51
10	0904	11.0594	1080	9.2553	1257	7.9530	1435	6.9682	1614	6.1970	**50**
11	0907	11.0237	1083	9.2302	1260	7.9344	1438	6.9538	1617	6.1856	49
12	0910	10.9882	1086	9.2052	1263	7.9158	1441	6.9395	1620	6.1742	48
13	0913	10.9529	1089	9.1803	1266	7.8973	1444	6.9252	1623	6.1628	47
14	0916	10.9178	1092	9.1555	1269	7.8789	1447	6.9110	1626	6.1515	46
15	0919	10.8829	1095	9.1309	1272	7.8606	1450	6.8969	1629	6.1402	**45**
16	0922	10.8483	1098	9.1065	1275	7.8424	1453	6.8828	1632	6.1290	44
17	0925	10.8139	1101	9.0821	1278	7.8243	1456	6.8687	1635	6.1178	43
18	0928	10.7797	1104	9.0579	1281	7.8062	1459	6.8548	1638	6.1066	42
19	0931	10.7457	1107	9.0338	1284	7.7883	1462	6.8408	1641	6.0955	41
20	0934	10.7119	1110	9.0098	1287	7.7704	1465	6.8269	1644	6.0844	**40**
21	0936	10.6783	1113	8.9860	1290	7.7525	1468	6.8131	1647	6.0734	39
22	0939	10.6450	1116	8.9623	1293	7.7348	1471	6.7994	1650	6.0624	38
23	0942	10.6118	1119	8.9387	1296	7.7171	1474	6.7856	1653	6.0514	37
24	0945	10.5789	1122	8.9152	1299	7.6996	1477	6.7720	1655	6.0405	36
25	0948	10.5462	1125	8.8919	1302	7.6821	1480	6.7584	1658	6.0296	**35**
26	0951	10.5136	1128	8.8686	1305	7.6647	1483	6.7448	1661	6.0188	34
27	0954	10.4813	1131	8.8455	1308	7.6473	1486	6.7313	1664	6.0080	33
28	0957	10.4491	1134	8.8225	1311	7.6301	1489	6.7179	1667	5.9972	32
29	0960	10.4172	1136	8.7996	1314	7.6129	1492	6.7045	1670	5.9865	31
30	0963	10.3854	1139	8.7769	1317	7.5958	1495	6.6912	1673	5.9758	**30**
31	0966	10.3538	1142	8.7542	1319	7.5787	1497	6.6779	1676	5.9651	29
32	0969	10.3224	1145	8.7317	1322	7.5618	1500	6.6646	1679	5.9545	28
33	0972	10.2913	1148	8.7093	1325	7.5449	1503	6.6514	1682	5.9439	27
34	0975	10.2602	1151	8.6870	1328	7.5281	1506	6.6383	1685	5.9333	26
35	0978	10.2294	1154	8.6648	1331	7.5113	1509	6.6252	1688	5.9228	**25**
36	0981	10.1988	1157	8.6427	1334	7.4947	1512	6.6122	1691	5.9124	24
37	0983	10.1683	1160	8.6208	1337	7.4781	1515	6.5992	1694	5.9019	23
38	0986	10.1381	1163	8.5989	1340	7.4615	1518	6.5863	1697	5.8915	22
39	0989	10.1080	1166	8.5772	1343	7.4451	1521	6.5734	1700	5.8811	21
40	0992	10.0780	1169	8.5555	1346	7.4287	1524	6.5606	1703	5.8708	**20**
41	0995	10.0483	1172	8.5340	1349	7.4124	1527	6.5478	1706	5.8605	19
42	0998	10.0187	1175	8.5126	1352	7.3962	1530	6.5350	1709	5.8502	18
43	1001	9.9893	1178	8.4913	1355	7.3800	1533	6.5223	1712	5.8400	17
44	1004	9.9601	1181	8.4701	1358	7.3639	1536	6.5097	1715	5.8298	16
45	1007	9.9310	1184	8.4490	1361	7.3479	1539	6.4971	1718	5.8197	**15**
46	1010	9.9021	1187	8.4280	1364	7.3319	1542	6.4846	1721	5.8095	14
47	1013	9.8734	1189	8.4071	1367	7.3160	1545	6.4721	1724	5.7994	13
48	1016	9.8448	1192	8.3863	1370	7.3002	1548	6.4596	1727	5.7894	12
49	1019	9.8164	1195	8.3656	1373	7.2844	1551	6.4472	1730	5.7794	11
50	1022	9.7882	1198	8.3450	1376	7.2687	1554	6.4348	1733	5.7694	**10**
51	1025	9.7601	1201	8.3245	1379	7.2531	1557	6.4225	1736	5.7594	9
52	1028	9.7322	1204	8.3041	1382	7.2375	1560	6.4103	1739	5.7495	8
53	1030	9.7044	1207	8.2838	1385	7.2220	1563	6.3980	1742	5.7396	7
54	1033	9.6768	1210	8.2636	1388	7.2066	1566	6.3859	1745	5.7297	6
55	1036	9.6499	1213	8.2434	1391	7.1912	1569	6.3737	1748	5.7199	**5**
56	1039	9.6220	1216	8.2234	1394	7.1759	1572	6.3617	1751	5.7101	4
57	1042	9.5949	1219	8.2035	1397	7.1607	1575	6.3496	1754	5.7004	3
58	1045	9.5679	1222	8.1837	1399	7.1455	1578	6.3376	1757	5.6906	2
59	1048	9.5411	1225	8.1640	1402	7.1304	1581	6.3257	1760	5.6809	1
60	1051	9.5144	1228	8.1443	1405	7.1154	1584	6.3138	1763	5.6713	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	tan	**cot**	**tan**	**cot**	**tan**	**cot**	**tan**	**cot**	**tan**	**cot**	
0	1763	5.6713	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	**60**
1	1766	5.6617	1947	5.1366	2129	4.6979	2312	4.3257	2496	4.0058	59
2	1769	5.6521	1950	5.1286	2132	4.6912	2315	4.3200	2499	4.0009	58
3	1772	5.6425	1953	5.1207	2135	4.6845	2318	4.3143	2503	3.9959	57
4	1775	5.6330	1956	5.1128	2138	4.6779	2321	4.3086	2506	3.9910	56
5	1778	5.6234	1959	5.1049	2141	4.6712	2324	4.3029	2509	3.9861	**55**
6	1781	5.6140	1962	5.0970	2144	4.6646	2327	4.2972	2512	3.9812	54
7	1784	5.6045	1965	5.0892	2147	4.6580	2330	4.2916	2515	3.9763	53
8	1787	5.5951	1968	5.0814	2150	4.6514	2333	4.2859	2518	3.9714	52
9	1790	5.5857	1971	5.0736	2153	4.6448	2336	4.2803	2521	3.9665	51
10	1793	5.5764	1974	5.0658	2156	4.6382	2339	4.2747	2524	3.9617	**50**
11	1796	5.5671	1977	5.0581	2159	4.6317	2342	4.2691	2527	3.9568	49
12	1799	5.5578	1980	5.0504	2162	4.6252	2345	4.2635	2530	3.9520	48
13	1802	5.5485	1983	5.0427	2165	4.6187	2349	4.2580	2533	3.9471	47
14	1805	5.5393	1986	5.0350	2168	4.6122	2352	4.2524	2537	3.9423	46
15	1808	5.5301	1989	5.0273	2171	4.6057	2355	4.2468	2540	3.9375	**45**
16	1811	5.5209	1992	5.0197	2174	4.5993	2358	4.2413	2543	3.9327	44
17	1814	5.5118	1995	5.0121	2177	4.5928	2361	4.2358	2546	3.9279	43
18	1817	5.5026	1998	5.0045	2180	4.5864	2364	4.2303	2549	3.9232	42
19	1820	5.4936	2001	4.9969	2183	4.5800	2367	4.2248	2552	3.9184	41
20	1823	5.4845	2004	4.9894	2186	4.5736	2370	4.2193	2555	3.9136	**40**
21	1826	5.4755	2007	4.9819	2189	4.5673	2373	4.2139	2558	3.9089	39
22	1829	5.4665	2010	4.9744	2193	4.5609	2376	4.2084	2561	3.9042	38
23	1832	5.4575	2013	4.9669	2196	4.5546	2379	4.2030	2564	3.8995	37
24	1835	5.4486	2016	4.9594	2199	4.5483	2382	4.1976	2568	3.8947	36
25	1838	5.4397	2019	4.9520	2202	4.5420	2385	4.1922	2571	3.8900	**35**
26	1841	5.4308	2022	4.9446	2205	4.5357	2388	4.1868	2574	3.8854	34
27	1844	5.4219	2025	4.9372	2208	4.5294	2392	4.1814	2577	3.8807	33
28	1847	5.4131	2028	4.9298	2211	4.5232	2395	4.1760	2580	3.8760	32
29	1850	5.4043	2031	4.9225	2214	4.5169	2398	4.1706	2583	3.8714	31
30	1853	5.3955	2035	4.9152	2217	4.5107	2401	4.1653	2586	3.8667	**30**
31	1856	5.3868	2038	4.9078	2220	4.5045	2404	4.1600	2589	3.8621	29
32	1859	5.3781	2041	4.9006	2223	4.4983	2407	4.1547	2592	3.8575	28
33	1862	5.3694	2044	4.8933	2226	4.4922	2410	4.1493	2595	3.8528	27
34	1865	5.3607	2047	4.8860	2229	4.4860	2413	4.1441	2599	3.8482	26
35	1868	5.3521	2050	4.8788	2232	4.4799	2416	4.1388	2602	3.8436	**25**
36	1871	5.3435	2053	4.8716	2235	4.4737	2419	4.1335	2605	3.8391	24
37	1874	5.3349	2056	4.8644	2238	4.4676	2422	4.1282	2608	3.8345	23
38	1877	5.3263	2059	4.8573	2241	4.4615	2425	4.1230	2611	3.8299	22
39	1880	5.3178	2062	4.8501	2244	4.4555	2428	4.1178	2614	3.8254	21
40	1883	5.3093	2065	4.8430	2247	4.4494	2432	4.1126	2617	3.8208	**20**
41	1887	5.3008	2068	4.8359	2251	4.4434	2435	4.1074	2620	3.8163	19
42	1890	5.2924	2071	4.8288	2254	4.4374	2438	4.1022	2623	3.8118	18
43	1893	5.2839	2074	4.8218	2257	4.4313	2441	4.0970	2627	3.8073	17
44	1896	5.2755	2077	4.8147	2260	4.4253	2444	4.0918	2630	3.8028	16
45	1899	5.2672	2080	4.8077	2263	4.4194	2447	4.0867	2633	3.7983	**15**
46	1902	5.2588	2083	4.8007	2266	4.4134	2450	4.0815	2636	3.7938	14
47	1905	5.2505	2086	4.7937	2269	4.4075	2453	4.0764	2639	3.7893	13
48	1908	5.2422	2089	4.7867	2272	4.4015	2456	4.0713	2642	3.7848	12
49	1911	5.2339	2092	4.7798	2275	4.3956	2459	4.0662	2645	3.7804	11
50	1914	5.2257	2095	4.7729	2278	4.3897	2462	4.0611	2648	3.7760	**10**
51	1917	5.2174	2098	4.7659	2281	4.3838	2465	4.0560	2651	3.7715	9
52	1920	5.2092	2101	4.7591	2284	4.3779	2469	4.0509	2655	3.7671	8
53	1923	5.2011	2104	4.7522	2287	4.3721	2472	4.0459	2658	3.7627	7
54	1926	5.1929	2107	4.7453	2290	4.3662	2475	4.0408	2661	3.7583	6
55	1929	5.1848	2110	4.7385	2293	4.3604	2478	4.0358	2664	3.7539	**5**
56	1932	5.1767	2113	4.7317	2296	4.3546	2481	4.0308	2667	3.7495	4
57	1935	5.1686	2116	4.7249	2299	4.3488	2484	4.0257	2670	3.7451	3
58	1938	5.1606	2119	4.7181	2303	4.3430	2487	4.0207	2673	3.7408	2
59	1941	5.1526	2123	4.7114	2306	4.3372	2490	4.0158	2676	3.7364	1
60	1944	5.1446	2126	4.7046	2309	4.3315	2493	4.0108	2679	3.7321	**0**
	cot	**tan**	**cot**	**tan**	**cot**	**tan**	**cot**	**tan**	**cot**	**tan**	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	2679	3.7321	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	60
1	2683	3.7277	2871	3.4836	3060	3.2675	3252	3.0746	3447	2.9015	59
2	2686	3.7234	2874	3.4798	3064	3.2641	3256	3.0716	3450	2.8987	58
3	2689	3.7191	2877	3.4760	3067	3.2607	3259	3.0686	3453	2.8960	57
4	2692	3.7148	2880	3.4722	3070	3.2573	3262	3.0655	3456	2.8933	56
5	2695	3.7105	2883	3.4684	3073	3.2539	3265	3.0625	3460	2.8905	55
6	2698	3.7062	2886	3.4646	3076	3.2506	3269	3.0595	3463	2.8878	54
7	2701	3.7019	2890	3.4608	3080	3.2472	3272	3.0565	3466	2.8851	53
8	2704	3.6976	2893	3.4570	3083	3.2438	3275	3.0535	3469	2.8824	52
9	2708	3.6933	2896	3.4533	3086	3.2405	3278	3.0505	3473	2.8797	51
10	2711	3.6891	2899	3.4495	3089	3.2371	3281	3.0475	3476	2.8770	50
11	2714	3.6848	2902	3.4458	3092	3.2338	3285	3.0445	3479	2.8743	49
12	2717	3.6806	2905	3.4420	3096	3.2305	3288	3.0415	3482	2.8716	48
13	2720	3.6764	2908	3.4383	3099	3.2272	3291	3.0385	3486	2.8689	47
14	2723	3.6722	2912	3.4346	3102	3.2238	3294	3.0356	3489	2.8662	46
15	2726	3.6680	2915	3.4308	3105	3.2205	3298	3.0326	3492	2.8636	45
16	2729	3.6638	2918	3.4271	3108	3.2172	3301	3.0296	3495	2.8609	44
17	2733	3.6596	2921	3.4234	3111	3.2139	3304	3.0267	3499	2.8582	43
18	2736	3.6554	2924	3.4197	3115	3.2106	3307	3.0237	3502	2.8556	42
19	2739	3.6512	2927	3.4160	3118	3.2073	3310	3.0208	3505	2.8529	41
20	2742	3.6470	2931	3.4124	3121	3.2041	3314	3.0178	3508	2.8502	40
21	2745	3.6429	2934	3.4087	3124	3.2008	3317	3.0149	3512	2.8476	39
22	2748	3.6387	2937	3.4050	3127	3.1975	3320	3.0120	3515	2.8449	38
23	2751	3.6346	2940	3.4014	3131	3.1943	3323	3.0090	3518	2.8423	37
24	2754	3.6305	2943	3.3977	3134	3.1910	3327	3.0061	3522	2.8397	36
25	2758	3.6264	2946	3.3941	3137	3.1878	3330	3.0032	3525	2.8370	35
26	2761	3.6222	2949	3.3904	3140	3.1845	3333	3.0003	3528	2.8344	34
27	2764	3.6181	2953	3.3868	3143	3.1813	3336	2.9974	3531	2.8318	33
28	2767	3.6140	2956	3.3832	3147	3.1780	3339	2.9945	3535	2.8291	32
29	2770	3.6100	2959	3.3796	3150	3.1748	3343	2.9916	3538	2.8265	31
30	2773	3.6059	2962	3.3759	3153	3.1716	3346	2.9887	3541	2.8239	30
31	2776	3.6018	2965	3.3723	3156	3.1684	3349	2.9858	3544	2.8213	29
32	2780	3.5978	2968	3.3687	3159	3.1652	3352	2.9829	3548	2.8187	28
33	2783	3.5937	2972	3.3652	3163	3.1620	3356	2.9800	3551	2.8161	27
34	2786	3.5897	2975	3.3616	3166	3.1588	3359	2.9772	3554	2.8135	26
35	2789	3.5856	2978	3.3580	3169	3.1556	3362	2.9743	3558	2.8109	25
36	2792	3.5816	2981	3.3544	3172	3.1524	3365	2.9714	3561	2.8083	24
37	2795	3.5776	2984	3.3509	3175	3.1492	3369	2.9686	3564	2.8057	23
38	2798	3.5736	2987	3.3473	3179	3.1460	3372	2.9657	3567	2.8032	22
39	2801	3.5696	2991	3.3438	3182	3.1429	3375	2.9629	3571	2.8006	21
40	2805	3.5656	2994	3.3402	3185	3.1397	3378	2.9600	3574	2.7980	20
41	2808	3.5616	2997	3.3367	3188	3.1366	3382	2.9572	3577	2.7955	19
42	2811	3.5576	3000	3.3332	3191	3.1334	3385	2.9544	3581	2.7929	18
43	2814	3.5536	3003	3.3297	3195	3.1303	3388	2.9515	3584	2.7903	17
44	2817	3.5497	3006	3.3261	3198	3.1271	3391	2.9487	3587	2.7878	16
45	2820	3.5457	3010	3.3226	3201	3.1240	3395	2.9459	3590	2.7852	15
46	2823	3.5418	3013	3.3191	3204	3.1209	3398	2.9431	3594	2.7827	14
47	2827	3.5379	3016	3.3156	3207	3.1178	3401	2.9403	3597	2.7801	13
48	2830	3.5339	3019	3.3122	3211	3.1146	3404	2.9375	3600	2.7776	12
49	2833	3.5300	3022	3.3087	3214	3.1115	3408	2.9347	3604	2.7751	11
50	2836	3.5261	3026	3.3052	3217	3.1084	3411	2.9319	3607	2.7725	10
51	2839	3.5222	3029	3.3017	3220	3.1053	3414	2.9291	3610	2.7700	9
52	2842	3.5183	3032	3.2983	3223	3.1022	3417	2.9263	3613	2.7675	8
53	2845	3.5144	3035	3.2948	3227	3.0991	3421	2.9235	3617	2.7650	7
54	2849	3.5105	3038	3.2914	3230	3.0961	3424	2.9208	3620	2.7625	6
55	2852	3.5067	3041	3.2880	3233	3.0930	3427	2.9180	3623	2.7600	5
56	2855	3.5028	3045	3.2845	3236	3.0899	3430	2.9152	3627	2.7575	4
57	2858	3.4989	3048	3.2811	3240	3.0868	3434	2.9125	3630	2.7550	3
58	2861	3.4951	3051	3.2777	3243	3.0838	3437	2.9097	3633	2.7525	2
59	2864	3.4912	3054	3.2743	3246	3.0807	3440	2.9070	3636	2.7500	1
60	2867	3.4874	3057	3.2709	3249	3.0777	3443	2.9042	3640	2.7475	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	3640	2.7475	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	60
1	3643	2.7450	3842	2.6028	4044	2.4730	4248	2.3539	4456	2.2443	59
2	3646	2.7425	3845	2.6006	4047	2.4709	4252	2.3520	4459	2.2425	58
3	3650	2.7400	3849	2.5983	4050	2.4689	4255	2.3501	4463	2.2408	57
4	3653	2.7376	3852	2.5961	4054	2.4668	4258	2.3483	4466	2.2390	56
5	3656	2.7351	3855	2.5938	4057	2.4648	4262	2.3464	4470	2.2373	55
6	3659	2.7326	3859	2.5916	4061	2.4627	4265	2.3445	4473	2.2355	54
7	3663	2.7302	3862	2.5893	4064	2.4606	4269	2.3426	4477	2.2338	53
8	3666	2.7277	3865	2.5871	4067	2.4586	4272	2.3407	4480	2.2320	52
9	3669	2.7253	3869	2.5848	4071	2.4566	4276	2.3388	4484	2.2303	51
10	3673	2.7228	3872	2.5826	4074	2.4545	4279	2.3369	4487	2.2286	50
11	3676	2.7204	3875	2.5804	4078	2.4525	4283	2.3351	4491	2.2268	49
12	3679	2.7179	3879	2.5782	4081	2.4504	4286	2.3332	4494	2.2251	48
13	3683	2.7155	3882	2.5759	4084	2.4484	4289	2.3313	4498	2.2234	47
14	3686	2.7130	3885	2.5737	4088	2.4464	4293	2.3294	4501	2.2216	46
15	3689	2.7108	3889	2.5715	4091	2.4443	4296	2.3276	4505	2.2199	45
16	3693	2.7082	3892	2.5693	4095	2.4423	4300	2.3257	4508	2.2182	44
17	3696	2.7058	3895	2.5671	4098	2.4403	4303	2.3238	4512	2.2165	43
18	3699	2.7034	3899	2.5649	4101	2.4383	4307	2.3220	4515	2.2148	42
19	3702	2.7009	3902	2.5627	4105	2.4362	4310	2.3201	4519	3.2130	41
20	3706	2.6985	3906	2.5605	4108	2.4342	4314	2.3183	4522	2.2113	40
21	3709	2.6061	3909	2.5533	4111	2.4322	4317	2.3164	4526	2.2096	39
22	3712	2.6937	3912	2.5561	4115	2.4302	4320	2.3146	4529	2.2079	38
23	3716	2.6913	3916	2.5539	4118	2.4282	4324	2.3127	4533	2.2062	37
24	3719	2.6889	3919	2.5517	4122	2.4262	4327	2.3109	4536	2.2045	36
25	3722	2.6865	3922	2.5495	4125	2.4242	4331	2.3090	4540	2.2028	35
26	3726	2.6841	3926	2.5473	4129	2.4222	4334	2.3072	4543	2.2011	34
27	3729	2.6818	3929	2.5452	4132	2.4202	4338	2.3053	4547	2.1994	33
28	3732	2.6794	3932	2.5430	4135	2.4182	4341	2.3035	4550	2.1977	32
29	3736	2.6770	3936	2.5408	4139	2.4162	4345	2.3017	4554	2.1960	31
30	3739	2.6746	3939	2.5386	4142	2.4142	4348	2.2998	4557	2.1943	30
31	3742	2.6723	3942	2.5365	4146	2.4122	4352	2.2980	4561	2.1926	29
32	3745	2.6699	3946	2.5343	4149	2.4102	4355	2.2962	4564	2.1909	28
33	3749	2.6675	3949	2.5322	4152	2.4083	4359	2.2944	4568	2.1892	27
34	3752	2.6652	3953	2.5300	4156	2.4063	4362	2.2925	4571	2.1876	26
35	3755	2.6628	3956	2.5279	4159	2.4043	4365	2.2907	4575	2.1859	25
36	3759	2.6605	3959	2.5257	4163	2.4023	4369	2.2889	4578	2.1842	24
37	3762	2.6581	3963	2.5236	4166	2.4004	4372	2.2871	4582	2.1825	23
38	3765	2.6558	3966	2.5214	4169	2.3984	4376	2.2853	4585	2.1808	22
39	3769	2.6534	3969	2.5193	4173	2.3964	4379	2.2835	4589	2.1792	21
40	3772	2.6511	3973	2.5172	4176	2.3945	4383	2.2817	4592	2.1775	20
41	3775	2.6488	3976	2.5150	4180	2.3925	4386	2.2799	4596	2.1758	19
42	3779	2.6464	3979	2.5129	4183	2.3906	4390	2.2781	4599	2.1742	18
43	3782	2.6441	3983	2.5108	4187	2.3886	4393	2.2763	4603	2.1725	17
44	3785	2.6418	3986	2.5086	4190	2.3867	4397	2.2745	4607	2.1708	16
45	3789	2.6395	3990	2.5065	4193	2.3847	4400	2.2727	4610	2.1692	15
46	3792	2.6371	3993	2.5044	4197	2.3828	4404	2.2709	4614	2.1675	14
47	3795	2.6348	3996	2.5023	4200	2.3808	4407	2.2691	4617	2.1659	13
48	3799	2.6325	4000	2.5002	4204	2.3789	4411	2.2673	4621	2.1642	12
49	3802	2.6302	4003	2.4981	4207	2.3770	4414	2.2655	4624	2.1625	11
50	3805	2.6279	4006	2.4960	4210	2.3750	4417	2.2637	4628	2.1609	10
51	3809	2.6256	4010	2.4939	4214	2.3731	4421	2.2620	4631	2.1592	9
52	3812	2.6233	4013	2.4918	4217	2.3712	4424	2.2602	4635	2.1576	8
53	3815	2.6210	4017	2.4897	4221	2.3693	4428	2.2584	4638	2.1560	7
54	3819	2.6187	4020	2.4876	4224	2.3673	4431	2.2566	4642	2.1543	6
55	3822	2.6165	4023	2.4855	4228	2.3654	4435	2.2549	4645	2.1527	5
56	3825	2.6142	4027	2.4834	4231	2.3635	4438	2.2531	4649	2.1510	4
57	3829	2.6119	4030	2.4813	4234	2.3616	4442	2.2513	4652	2.1494	3
58	3832	2.6096	4033	2.4792	4238	2.3597	4445	2.2496	4656	2.1478	2
59	3835	2.6074	4037	2.4772	4241	2.3578	4449	2.2478	4660	2.1461	1
60	3839	2.6051	4040	2.4751	4245	2.3559	4452	2.2460	4663	2.1445	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	69°		68°		67°		66°		65°		′

′	25°		26°		27°		28°		29°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	4663	2.1445	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	**60**
1	4667	2.1429	4881	2.0488	5099	1.9612	5321	1.8794	5547	1.8028	59
2	4670	2.1413	4885	2.0473	5103	1.9598	5325	1.8781	5551	1.8016	58
3	4674	2.1396	4888	2.0458	5106	1.9584	5328	1.8768	5555	1.8003	57
4	4677	2.1380	4892	2.0443	5110	1.9570	5332	1.8755	5558	1.7991	56
5	4681	2.1364	4895	2.0428	5114	1.9556	5336	1.8741	5562	1.7979	**55**
6	4684	2.1348	4899	2.0413	5117	1.9542	5340	1.8728	5566	1.7966	54
7	4688	2.1332	4903	2.0398	5121	1.9528	5343	1.8715	5570	1.7954	53
8	4691	2.1315	4906	2.0383	5125	1.9514	5347	1.8702	5574	1.7942	52
9	4695	2.1299	4910	2.0368	5128	1.9500	5351	1.8689	5577	1.7930	51
10	4699	2.1283	4913	2.0353	5132	1.9486	5354	1.8676	5581	1.7917	**50**
11	4702	2.1267	4917	2.0338	5136	1.9472	5358	1.8663	5585	1.7905	49
12	4706	2.1251	4921	2.0323	5139	1.9458	5362	1.8650	5589	1.7893	48
13	4709	2.1235	4924	2.0308	5143	1.9444	5366	1.8637	5593	1.7881	47
14	4713	2.1219	4928	2.0293	5147	1.9430	5369	1.8624	5596	1.7868	46
15	4716	2.1203	4931	2.0278	5150	1.9416	5373	1.8611	5600	1.7856	**45**
16	4720	2.1187	4935	2.0263	5154	1.9402	5377	1.8598	5604	1.7844	44
17	4723	2.1171	4939	2.0248	5158	1.9388	5381	1.8585	5608	1.7832	43
18	4727	2.1155	4942	2.0233	5161	1.9375	5384	1.8572	5612	1.7820	42
19	4731	2.1139	4946	2.0219	5165	1.9361	5388	1.8559	5616	1 7808	41
20	4734	2.1123	4950	2.0204	5169	1.9347	5392	1.8546	5619	1.7796	**40**
21	4738	2.1107	4953	2.0189	5172	1.9333	5396	1.8533	5623	1.7783	39
22	4741	2.1092	4957	2.0174	5176	1.9319	5399	1.8520	5627	1.7771	38
23	4745	2.1076	4960	2.0160	5180	1.9306	5403	1.8507	5631	1.7759	37
24	4748	2.1060	4964	2.0145	5184	1.9292	5407	1.8495	5635	1.7747	36
25	4752	2.1044	4968	2.0130	5187	1.9278	5411	1.8482	5639	1.7735	**35**
26	4755	2.1028	4971	2.0115	5191	1.9265	5415	1.8469	5642	1.7723	34
27	4759	2.1013	4975	2.0101	5195	1.9251	5418	1.8456	5646	1.7711	33
28	4763	2.0997	4979	2.0086	5198	1.9237	5422	1.8443	5650	1.7699	32
29	4766	2.0981	4982	2.0072	5202	1.9223	5426	1.8430	5654	1.7687	31
30	4770	2.0965	4986	2.0057	5206	1.9210	5430	1.8418	5658	1.7675	**30**
31	4773	2.0950	4989	2.0042	5209	1.9196	5433	1.8405	5662	1.7663	29
32	4777	2.0934	4993	2.0028	5213	1.9183	5437	1.8392	5665	1.7651	28
33	4780	2.0918	4997	2.0013	5217	1.9169	5441	1.8379	5669	1.7639	27
34	4784	2.0903	5000	1.9999	5220	1.9155	5445	1.8367	5673	1.7627	26
35	4788	2.0887	5004	1.9984	5224	1.9142	5448	1.8354	5677	1.7615	**25**
36	4791	2.0872	5008	1.9970	5228	1.9128	5452	1.8341	5681	1.7603	24
37	4795	2.0856	5011	1.9955	5232	1.9115	5456	1.8329	5685	1.7591	23
38	4798	2.0840	5015	1.9941	5235	1.9101	5460	1.8316	5688	1.7579	22
39	4802	2.0825	5019	1.9926	5239	1.9088	5464	1.8303	5692	1.7567	21
40	4806	2.0809	5022	1.9912	5243	1.9074	5467	1.8291	5696	1.7556	**20**
41	4809	2.0794	5026	1.9897	5246	1.9061	5471	1.8278	5700	1.7544	19
42	4813	2.0778	5029	1.9883	5250	1.9047	5475	1.8265	5704	1.7532	18
43	4816	2.0763	5033	1.9868	5254	1.9034	5479	1.8253	5708	1.7520	17
44	4820	2.0748	5037	1 9854	5258	1.9020	5482	1.8240	5712	1.7508	16
45	4823	2.0732	5040	1.9840	5261	1.9007	5486	1.8228	5715	1.7496	**15**
46	4827	2.0717	5044	1.9825	5265	1.8993	5490	1.8215	5719	1.7485	14
47	4831	2.0701	5048	1.9811	5269	1.8980	5494	1.8202	5723	1.7473	13
48	4834	2.0686	5051	1.9797	5272	1.8967	5498	1.8190	5727	1.7461	12
49	4838	2.0671	5055	1.9782	5276	1 8953	5501	1.8177	5731	1.7449	11
50	4841	2.0655	5059	1.9768	5280	1.8940	5505	1.8165	5735	1.7437	**10**
51	4845	2.0640	5062	1.9754	5284	1.8927	5509	1.8152	5739	1.7426	9
52	4849	2.0625	5066	1.9740	5287	1.8913	5513	1.8140	5743	1.7414	8
53	4852	2.0609	5070	1.9725	5291	1.8900	5517	1.8127	5746	1.7402	7
54	4856	2.0594	5073	1.9711	5295	1.8887	5520	1.8115	5750	1.7391	6
55	4859	2.0579	5077	1.9697	5298	1.8873	5524	1.8103	5754	1.7379	**5**
56	4863	2.0564	5081	1.9683	5302	1.8860	5528	1.8090	5758	1.7367	4
57	4867	2.0549	5084	1.9669	5306	1.8847	5532	1.8078	5762	1.7355	3
58	4870	2.0533	5088	1.9654	5310	1.8834	5535	1.8065	5766	1.7344	2
59	4874	2.0518	5092	1.9640	5313	1.8820	5539	1.8053	5770	1.7332	1
60	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	5774	1.7321	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	5774	1.7321	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	**60**
1	5777	1.7309	6013	1.6632	6253	1.5993	6498	1.5389	6749	1.4816	59
2	5781	1.7297	6017	1.6621	6257	1.5983	6502	1.5379	6754	1.4807	58
3	5785	1.7286	6020	1.6610	6261	1.5972	6506	1.5369	6758	1.4798	57
4	5789	1.7274	6024	1.6599	6265	1.5962	6511	1.5359	6762	1.4788	56
5	5793	1.7262	6028	1.6588	6269	1.5952	6515	1.5350	6766	1.4779	**55**
6	5797	1.7251	6032	1.6577	6273	1.5941	6519	1.5340	6771	1.4770	54
7	5801	1.7239	6036	1.6566	6277	1.5931	6523	1.5330	6775	1.4761	53
8	5805	1.7228	6040	1.6555	6281	1.5921	6527	1.5320	6779	1.4751	52
9	5808	1.7216	6044	1.6545	6285	1.5911	6531	1.5311	6783	1.4742	51
10	5812	1.7205	6048	1.6534	6289	1.5900	6536	1.5301	6787	1.4733	**50**
11	5816	1.7193	6052	1.6523	6293	1.5890	6540	1.5291	6792	1.4724	49
12	5820	1.7182	6056	1.6512	6297	1.5880	6544	1.5282	6796	1.4715	48
13	5824	1.7170	6060	1.6501	6301	1.5869	6548	1.5272	6800	1.4705	47
14	5828	1.7159	6064	1.6490	6305	1.5859	6552	1.5262	6805	1.4696	46
15	5832	1.7147	6068	1.6479	6310	1.5849	6556	1.5253	6809	1.4687	**45**
16	5836	1.7136	6072	1.6469	6314	1.5839	6560	1.5243	6813	1.4678	44
17	5840	1.7124	6076	1.6458	6318	1.5829	6565	1.5233	6817	1.4669	43
18	5844	1.7113	6080	1.6447	6322	1.5818	6569	1.5224	6822	1.4659	42
19	5847	1.7102	6084	1.6436	6326	1.5808	6573	1.5214	6826	1.4650	41
20	5851	1.7090	6088	1.6426	6330	1.5798	6577	1.5204	6830	1.4641	**40**
21	5855	1.7079	6002	1.6415	6334	1.5788	6581	1.5195	6834	1.4632	39
22	5859	1.7067	6096	1.6404	6338	1.5778	6585	1.5185	6839	1.4623	38
23	5863	1.7056	6100	1.6393	6342	1.5768	6590	1.5175	6843	1.4614	37
24	5867	1.7045	6104	1.6383	6346	1.5757	6594	1.5166	6847	1.4605	36
25	5871	1.7033	6108	1.6372	6350	1.5747	6598	1.5156	6851	1.4596	**35**
26	5875	1.7022	6112	1.6361	6354	1.5737	6602	1.5147	6856	1.4586	34
27	5879	1.7011	6116	1.6351	6358	1.5727	6606	1.5137	6860	1.4577	33
28	5883	1.6999	6120	1.6340	6363	1.5717	6610	1.5127	6864	1.4568	32
29	5887	1.6988	6124	1.6329	6367	1.5707	6615	1.5118	6869	1.4559	31
30	5890	1.6977	6128	1.6319	6371	1.5697	6619	1.5108	6873	1.4550	**30**
31	5894	1.6965	6132	1.6308	6375	1.5687	6623	1.5099	6877	1.4541	29
32	5898	1.6954	6136	1.6297	6379	1.5677	6627	1.5089	6881	1.4532	28
33	5902	1.6943	6140	1.6287	6383	1.5667	6631	1.5080	6886	1.4523	27
34	5906	1.6932	6144	1.6276	6387	1.5657	6636	1.5070	6890	1.4514	26
35	5910	1.6920	6148	1.6265	6391	1.5647	6640	1.5061	6894	1.4505	**25**
36	5914	1.6909	6152	1.6255	6395	1.5637	6644	1.5051	6899	1.4496	24
37	5918	1.6898	6156	1.6244	6399	1.5627	6648	1.5042	6903	1.4487	23
38	5922	1.6887	6160	1.6234	6403	1.5617	6652	1.5032	6907	1.4478	22
39	5926	1.6875	6164	1.6223	6408	1.5607	6657	1.5023	6911	1.4469	21
40	5930	1.6864	6168	1.6212	6412	1.5597	6661	1.5013	6916	1.4460	**20**
41	5934	1.6853	6172	1.6202	6416	1.5587	6665	1.5004	6920	1.4451	19
42	5938	1.6842	6176	1.6191	6420	1.5577	6669	1.4994	6924	1.4442	18
43	5942	1.6831	6180	1.6181	6424	1.5567	6673	1.4985	6929	1.4433	17
44	5945	1.6820	6184	1.6170	6428	1.5557	6678	1.4975	6933	1.4424	16
45	5949	1.6808	6188	1.6160	6432	1.5547	6682	1.4966	6937	1.4415	**15**
46	5953	1.6797	6192	1.6149	6436	1.5537	6686	1.4957	6942	1.4406	14
47	5957	1.6786	6196	1.6139	6440	1.5527	6690	1.4947	6946	1.4397	13
48	5961	1.6775	6200	1.6128	6445	1.5517	6694	1.4938	6950	1.4388	12
49	5965	1.6764	6204	1.6118	6449	1.5507	6699	1.4928	6954	1.4379	11
50	5969	1.6753	6208	1.6107	6453	1.5497	6703	1.4919	6959	1.4370	**10**
51	5973	1.6742	6212	1.6097	6457	1.5487	6707	1.4910	6963	1.4361	9
52	5977	1.6731	6216	1.6087	6461	1.5477	6711	1.4900	6967	1.4352	8
53	5981	1.6720	6220	1.6076	6465	1.5468	6716	1.4891	6972	1.4344	7
54	5985	1.6709	6224	1.6066	6469	1.5458	6720	1.4882	6976	1.4335	6
55	5989	1.6698	6228	1.6055	6473	1.5448	6724	1.4872	6980	1.4326	**5**
56	5993	1.6687	6233	1.6045	6478	1.5438	6728	1.4863	6985	1.4317	4
57	5997	1.6676	6237	1.6034	6482	1.5428	6732	1.4854	6989	1.4308	3
58	6001	1.6665	6241	1.6024	6486	1.5418	6737	1.4844	6993	1.4299	2
59	6005	1.6654	6245	1.6014	6490	1.5408	6741	1.4835	6998	1.4290	1
60	6009	1.6643	6249	1.6003	6494	1.5399	6745	1.4826	7002	1.4281	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	7002	1.4281	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	**60**
1	7006	1.4273	7270	1.3755	7540	1.3262	7818	1.2792	8103	1.2342	59
2	7011	1.4264	7274	1.3747	7545	1.3254	7822	1.2784	8107	1.2334	58
3	7015	1.4255	7279	1.3739	7549	1.3246	7827	1.2776	8112	1.2327	57
4	7019	1.4246	7283	1.3730	7554	1.3238	7832	1.2769	8117	1.2320	56
5	7024	1.4237	7288	1.3722	7558	1.3230	7836	1.2761	8122	1.2312	**55**
6	7028	1.4229	7292	1.3713	7563	1.3222	7841	1.2753	8127	1.2305	54
7	7032	1.4220	7297	1.3705	7568	1.3214	7846	1.2746	8132	1.2298	53
8	7037	1.4211	7301	1.3697	7572	1.3206	7850	1.2738	8136	1.2290	52
9	7041	1.4202	7306	1.3688	7577	1.3198	7855	1.2731	8141	1.2283	51
10	7046	1.4193	7310	1.3680	7581	1.3190	7860	1.2723	8146	1.2276	**50**
11	7050	1.4185	7314	1.3672	7586	1.3182	7865	1.2715	8151	1.2268	49
12	7054	1.4176	7319	1.3663	7590	1.3175	7869	1.2708	8156	1.2261	48
13	7059	1.4167	7323	1.3655	7595	1.3167	7874	1.2700	8161	1.2254	47
14	7063	1.4158	7328	1.3647	7600	1.3159	7879	1.2693	8165	1.2247	46
15	7067	1.4150	7332	1.3638	7604	1.3151	7883	1.2685	8170	1.2239	**45**
16	7072	1.4141	7337	1.3630	7609	1.3143	7888	1.2677	8175	1.2232	44
17	7076	1.4132	7341	1.3622	7613	1.3135	7893	1.2670	8180	1.2225	43
18	7080	1.4124	7346	1.3613	7618	1.3127	7898	1.2662	8185	1.2218	42
19	7085	1.4115	7350	1.3605	7623	1.3119	7902	1.2655	8190	1.2210	41
20	7089	1.4106	7355	1.3597	7627	1.3111	7907	1.2647	8195	1.2203	**40**
21	7094	1.4097	7359	1.3588	7632	1.3103	7912	1.2640	8199	1.2196	39
22	7098	1.4089	7364	1.3580	7636	1.3095	7916	1.2632	8204	1.2189	38
23	7102	1.4080	7368	1.3572	7641	1.3087	7921	1.2624	8209	1.2181	37
24	7107	1.4071	7373	1.3564	7646	1.3079	7926	1.2617	8214	1.2174	36
25	7111	1.4063	7377	1.3555	7650	1.3072	7931	1.2609	8219	1.2167	**35**
26	7115	1.4054	7382	1.3547	7655	1.3064	7935	1.2602	8224	1.2160	34
27	7120	1.4045	7386	1.3539	7659	1.3056	7940	1.2594	8229	1.2153	33
28	7124	1.4037	7391	1.3531	7664	1.3048	7945	1.2587	8234	1.2145	32
29	7129	1.4028	7395	1.3522	7669	1.3040	7950	1.2579	8238	1.2138	31
30	7133	1.4019	7400	1.3514	7673	1.3032	7954	1.2572	8243	1.2131	**30**
31	7137	1.4011	7404	1.3506	7678	1.3024	7959	1.2564	8248	1.2124	29
32	7142	1.4002	7409	1.3498	7683	1.3017	7964	1.2557	8253	1.2117	28
33	7146	1.3994	7413	1.3490	7687	1.3009	7969	1.2549	8258	1.2109	27
34	7151	1.3985	7418	1.3481	7692	1.3001	7973	1.2542	8263	1.2102	26
35	7155	1.3976	7422	1.3473	7696	1.2993	7978	1.2534	8268	1.2095	**25**
36	7159	1.3968	7427	1.3465	7701	1.2985	7983	1.2527	8273	1.2088	24
37	7164	1.3959	7431	1.3457	7706	1.2977	7988	1.2519	8278	1.2081	23
38	7168	1.3951	7436	1.3449	7710	1.2970	7992	1.2512	8283	1.2074	22
39	7173	1.3942	7440	1.3440	7715	1.2962	7997	1.2504	8287	1.2066	21
40	7177	1.3934	7445	1.3432	7720	1.2954	8002	1.2497	8292	1.2059	**20**
41	7181	1.3925	7449	1.3424	7724	1.2946	8007	1.2489	8297	1.2052	19
42	7186	1.3916	7454	1.3416	7729	1.2938	8012	1.2482	8302	1.2045	18
43	7190	1.3908	7458	1.3408	7734	1.2931	8016	1.2475	8307	1.2038	17
44	7195	1.3899	7463	1.3400	7738	1.2923	8021	1.2467	8312	1.2031	16
45	7199	1.3891	7467	1.3392	7743	1.2915	8026	1.2460	8317	1.2024	**15**
46	7203	1.3882	7472	1.3384	7747	1.2907	8031	1.2452	8322	1.2017	14
47	7208	1.3874	7476	1.3375	7752	1.2900	8035	1.2445	8327	1.2009	13
48	7212	1.3865	7481	1.3367	7757	1.2892	8040	1.2437	8332	1.2002	12
49	7217	1.3857	7485	1.3359	7761	1.2884	8045	1.2430	8337	1.1995	11
50	7221	1.3848	7490	1.3351	7766	1.2876	8050	1.2423	8342	1.1988	**10**
51	7226	1.3840	7495	1.3343	7771	1.2869	8055	1.2415	8346	1.1981	9
52	7230	1.3831	7499	1.3335	7775	1.2861	8059	1.2408	8351	1.1974	8
53	7234	1.3823	7504	1.3327	7780	1.2853	8064	1.2401	8356	1.1967	7
54	7239	1.3814	7508	1.3319	7785	1.2846	8069	1.2393	8361	1.1960	6
55	7243	1.3806	7513	1.3311	7789	1.2838	8074	1.2386	8366	1.1953	**5**
56	7248	1.3798	7517	1.3303	7794	1.2830	8079	1.2378	8371	1.1946	4
57	7252	1.3789	7522	1.3295	7799	1.2822	8083	1.2371	8376	1.1939	3
58	7257	1.3781	7526	1.3287	7803	1.2815	8088	1.2364	8381	1.1932	2
59	7261	1.3772	7531	1.3278	7808	1.2807	8093	1.2356	8386	1.1925	1
60	7265	1.3764	7536	1.3270	7813	1.2799	8098	1.2349	8391	1.1918	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	8391	1.1918	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	60
1	8396	1.1910	8698	1.1497	9009	1.1100	9331	1.0717	9663	1.0349	59
2	8401	1.1903	8703	1.1490	9015	1.1093	9336	1.0711	9668	1.0343	58
3	8406	1.1896	8708	1.1483	9020	1.1087	9341	1.0705	9674	1.0337	57
4	8411	1.1889	8713	1.1477	9025	1.1080	9347	1.0699	9679	1.0331	56
5	8416	1.1882	8718	1.1470	9030	1.1074	9352	1.0692	9685	1.0325	55
6	8421	1.1875	8724	1.1463	9036	1.1067	9358	1.0686	9691	1.0319	54
7	8426	1.1868	8729	1.1456	9041	1.1061	9363	1.0680	9696	1.0313	53
8	8431	1.1861	8734	1.1450	9046	1.1054	9369	1.0674	9702	1.0307	52
9	8436	1.1854	8739	1.1443	9052	1.1048	9374	1.0668	9708	1.0301	51
10	8441	1.1847	8744	1.1436	9057	1.1041	9380	1.0661	9713	1.0295	50
11	8446	1.1840	8749	1.1430	9062	1.1035	9385	1.0655	9719	1.0289	49
12	8451	1.1833	8754	1.1423	9067	1.1028	9391	1.0649	9725	1.0283	48
13	8456	1.1826	8759	1.1416	9073	1.1022	9396	1.0643	9730	1.0277	47
14	8461	1.1819	8765	1.1410	9078	1.1016	9402	1.0637	9736	1.0271	46
15	8466	1.1812	8770	1.1403	9083	1.1009	9407	1.0630	9742	1.0265	45
16	8471	1.1806	8775	1.1396	9089	1.1003	9413	1.0624	9747	1.0259	44
17	8476	1.1799	8780	1.1389	9094	1.0996	9418	1.0618	9753	1.0253	43
18	8481	1.1792	8785	1.1383	9099	1.0990	9424	1.0612	9759	1.0247	42
19	8486	1.1785	8790	1.1376	9105	1.0983	9429	1.0606	9764	1.0241	41
20	8491	1.1778	8796	1.1369	9110	1.0977	9435	1.0599	9770	1.0235	40
21	8496	1.1771	8801	1.1363	9115	1.0971	9440	1.0593	9776	1.0230	39
22	8501	1.1764	8806	1.1356	9121	1.0964	9446	1.0587	9781	1.0224	38
23	8506	1.1757	8811	1.1349	9126	1.0958	9451	1.0581	9787	1.0218	37
24	8511	1.1750	8816	1.1343	9131	1.0951	9457	1.0575	9793	1.0212	36
25	8516	1.1743	8821	1.1336	9137	1.0945	9462	1.0569	9798	1.0206	35
26	8521	1.1736	8827	1.1329	9142	1.0939	9468	1.0562	9804	1.0200	34
27	8526	1.1729	8832	1.1323	9147	1.0932	9473	1.0556	9810	1.0194	33
28	8531	1.1722	8837	1.1316	9153	1.0926	9479	1.0550	9816	1.0188	32
29	8536	1.1715	8842	1.1310	9158	1.0919	9484	1.0544	9821	1.0182	31
30	8541	1.1708	8847	1.1303	9163	1.0913	9490	1.0538	9827	1.0176	30
31	8546	1.1702	8852	1.1296	9169	1.0907	9495	1.0532	9833	1.0170	29
32	8551	1.1695	8858	1.1290	9174	1.0900	9501	1.0526	9838	1.0164	28
33	8556	1.1688	8863	1.1283	9179	1.0894	9506	1.0519	9844	1.0158	27
34	8561	1.1681	8868	1.1276	9185	1.0888	9512	1.0513	9850	1.0152	26
35	8566	1.1674	8873	1.1270	9190	1.0881	9517	1.0507	9856	1.0147	25
36	8571	1.1667	8878	1.1263	9195	1.0875	9523	1.0501	9861	1.0141	24
37	8576	1.1660	8884	1.1257	9201	1.0869	9528	1.0495	9867	1.0135	23
38	8581	1.1653	8889	1.1250	9206	1.0862	9534	1.0489	9873	1.0129	22
39	8586	1.1647	8894	1.1243	9212	1.0856	9540	1.0483	9879	1.0123	21
40	8591	1.1640	8899	1.1237	9217	1.0850	9545	1.0477	9884	1.0117	20
41	8596	1.1633	8904	1.1230	9222	1 0843	9551	1.0470	9890	1.0111	19
42	8601	1.1626	8910	1.1224	9228	1.0837	9556	1.0464	9896	1.0105	18
43	8606	1.1619	8915	1.1217	9233	1.0831	9562	1.0458	9902	1.0099	17
44	8611	1.1612	8920	1.1211	9239	1.0824	9567	1.0452	9907	1.0094	16
45	8617	1.1606	8925	1.1204	9244	1.0818	9573	1.0446	9913	1.0088	15
46	8622	1.1599	8931	1.1197	9249	1.0812	9578	1.0440	9919	1.0082	14
47	8627	1.1592	8936	1.1191	9255	1.0805	9584	1.0434	9925	1.0076	13
48	8632	1.1585	8941	1.1184	9260	1.0799	9590	1.0428	9930	1.0070	12
49	8637	1.1578	8946	1.1178	9266	1.0793	9595	1.0422	9936	1.0064	11
50	8642	1.1571	8952	1.1171	9271	1.0786	9601	1.0416	9942	1.0058	10
51	8647	1.1565	8957	1.1165	9276	1.0780	9606	1.0410	9948	1.0052	9
52	8652	1.1558	8962	1.1158	9282	1.0774	9612	1.0404	9954	1.0047	8
53	8657	1.1551	8967	1.1152	9287	1.0768	9618	1.0398	9959	1.0041	7
54	8662	1.1544	8972	1.1145	9293	1.0761	9623	1.0392	9965	1.0035	6
55	8667	1.1538	8978	1.1139	9298	1.0755	9629	1.0385	9971	1.0029	5
56	8672	1.1531	8983	1.1132	9303	1.0749	9634	1.0379	9977	1.0023	4
57	8678	1.1524	8988	1.1126	9309	1.0742	9640	1.0373	9983	1.0017	3
58	8683	1.1517	8994	1.1119	9314	1.0736	9646	1.0367	9988	1.0012	2
59	8688	1.1510	8999	1.1113	9320	1.0730	9651	1.0361	9994	1.0006	1
60	8693	1.1504	9004	1.1106	9325	1.0724	9657	1.0355	1.000	1.0000	0
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	49°		48°		47°		46°		45°		′

TABLE V.—CONSTANTS.

	Number.	Logarithm.
Radians in one degree	0.017 453 3	8.241 877 4
Radians in one minute	0.000 290 9	6.463 726 1
Radians in one second	0.000 004 8	4.685 574 9
sin 1″	0.000 004 8	4.685 574 9
tan 1″	0.000 004 8	4.685 574 9
Degrees in one radian	57.295 779 5	1.758 122 6
Minutes in one radian	3 437.746 77	3.536 273 9
Seconds in one radian	206 264.806	5.314 425 1
π (Ratio of circum. to diam.)	3.141 592 7	0.497 149 9
$\frac{1}{\pi}$	0.318 309 9	9.502 850 1
π^2	9.869 604 4	0.994 299 7
$\frac{1}{\pi^2}$	0.101 321 2	9.005 700 3
$\sqrt{\pi}$	1.772 453 9	0.248 574 9
$\frac{1}{\sqrt{\pi}}$	0.564 189 6	9.751 425 1
M (Modulus of common logarithms)	0.434 294 5	9.637 784 3
e (Napierian base)	2.718 281 8	0.434 294 5
$\frac{1}{e}$	0.367 879 4	9.565 705 5
e^2	7.389 005 6	0.868 589 0
$\frac{1}{e^2}$	0.135 335 3	9.131 411 0
$\sqrt{2}$	2	0.301 030 0
	1.414 213 6	0.150 515 0
$\sqrt{3}$	3	0.477 121 3
	1.732 050 8	0.238 560 6
$\sqrt{5}$	5	0.698 970 0
	2.236 068 0	0.369 485 0
1 meter	= 39.370 000 inches	1.595 165 4
	= 3.280 833 feet	0.515 984 2
1 kilometer	= 0.621 370 mile	9.793 350 3
1 foot	= 0.304 801 meter	9.484 015 8
1 mile	= 1.609 347 kilom.	0.206 649 7
1 mile	= 5280 feet	3.722 633 9
1 nautical mile	= 6080.290 feet	3.783 924 4
Feet per sec. in 1 mile per hour	1.466 667	0.166 331 5
Miles per hour in 1 foot per second	0.681 818	9.833 368 6
360°	= 21 600′	4.334 453 8
	= 1 296 000″	6.112 605 0

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